DIPLOMARBEIT

Algebraic Toric Varieties and Complete Intersections

ausgeführt am Institut für
Theoretische Physik
der Technischen Universität Wien

unter der Anleitung von
A.O. Univ. Prof. DI. Dr. techn. Maximilian Kreuzer

durch

Erwin Riegler

Aßmayergasse 23/5/41
A-1120 Wien

Wien, Jänner 2001

To Sylvia

First of all I want to thank my supervisor Max Kreuzer for raising my interest in the subject of string theory and algebraic geometry. I benefitted much of his knowledge and he always took his time to discuss occuring problems with me.

I like to thank Sylvia for her love and patience during the last years. I am grateful to my parents for rendering my studies possible and their support during that time.

My gratitude goes to all members of the Institut für Theoretische Physik for inspiring discussions in a fruitful atmosphere. Special thanks are due to my colleagues Robert for finishing off hard days at the university in Johnny's pub, Daniel for support in physics and having fun at (after) swimming, Waldi for introduction into Linux, Herbert for his willingness in discussing mathematical stuff on his blackboard, encouragement and KitKat and Axel for proof-reading the manuscript.

I am grateful to Eva and Rudi for lending us their flat, Anita and Markus for giving us their furniture, coffee, cake and for a lot of nice playing-sessions.

Finally, I would like to thank my colleague and friend Michael. He always supported me through hard times and explained me a lot of mathematics.

Contents

1 Introduction **1**
1.1 Motivation . . . 1
1.2 Mathematical Methods . . . 2

2 Affine Varieties **6**
2.1 The Affine Space and Regular Functions . . . 6
2.2 Algebraic Subsets . . . 6
2.3 Systems of Algebraic Equations; Ideals . . . 7
2.4 Hilbert's Nullstellensatz . . . 8
2.5 Affine Varieties . . . 9
2.6 Abstract Affine Varieties . . . 10
2.7 Affine Schemes . . . 11
2.8 Localization . . . 11
2.9 Affine Toric Varieties . . . 12

3 Convex Geometry **16**
3.1 Convex Polyhedral Cones . . . 16
3.2 Cones from Polytopes . . . 22

4 Algebraic Varieties **25**
4.1 Atlases and Varieties . . . 25
4.2 Gluing . . . 26
4.3 Morphisms of Algebraic Varieties . . . 26
4.4 Presheaves . . . 27
4.5 Separatedness . . . 28
4.6 Algebraic Toric Varieties . . . 29

5 Local Study of Toric Varieties **33**
5.1 The Tangent Space . . . 33
5.2 Singularities of Toric Varieties . . . 34
5.3 Quotient Singularities . . . 35

6 Orbits, Divisors and Line Bundles **38**
6.1 Support Functions . . . 41
6.2 Equivariant Line Bundles . . . 41
6.3 Divisors . . . 42

7 Complete Intersections **47**
7.1 Nef Partitions 47
7.2 Gorenstein Cones 49
7.3 Combinatorial Polynomials of Eulerian Posets 51

8 Results **55**
8.1 Comparison with Weighted Projective Space 55
8.2 New Hodge Numbers 56

A Some New Hodge Data **59**

1 Introduction

1.1 Motivation

In string theory, physical processes are described by the propagation of a one-dimensional object, the string, in spacetime. A propagating string traces out a two-dimensional surface, called a world sheet. The classical fields can be described as functions or sections of line bundles on the world sheet, and quantizing leads to a two-dimensional quantum field theory. String theories have the advantage that they eliminate some of the problems which occur when a particle splits into two particles. While representing a particle as a point leads to a singularity, the string representation is a smooth 2-manifold with boundary.

However, string theories still have some undesirable features, including many infinities which require renormalization. A remarkable discovery in recent times is that supersymmetry, a symmetry between particles of integer and half-integer spin, can eliminate many of these difficulties. Although as of now supersymmetry has not been experimentally verified, supersymmetric theories have become very important in theoretical physics because of their special properties.

In recent years, it has become clear that superstring theories are good candidates for mathematically consistent theories of quantum gravity. The discovery of anomaly cancellation in a modified version of $d = 10$ supergravity and superstring theory with gauge group $O(32)$ or $E_8 \times E_8$ has opened up the possibility that they might be phenomenologically realistic as well as mathematically consistent. For these theories to be realistic, it is necessary that the vacuum state is of Kaluza Klein type $K \times M_4$, where M_4 is a four-dimensional Minkowski spacetime and K is some six-dimensional compact manifold. Quantum numbers of quarks or leptons are then determined by topological invariants of K and an $O(32)$ or $E_8 \times E_8$ gauge field defined on K.

Such considerations, however, are far from determining K uniquely. To ensure unbroken $N = 1$ supersymmetry in four dimensions, K must have $SU(3)$ holonomy and a vanishing cosmological constant. A manifold has $SU(3)$ holonomy if and only if it is Ricci-flat and Kähler [Nak90]. While there are many examples of Kähler manifolds, there are few explicit examples of

Ricci-flat Kähler metrics. For a Kähler manifold, the Ricci tensor has only nonzero mixed components and gives rise to a closed two-form. Obviously, the first Chern class [Nak90] is trivial for a Ricci-flat metric. The hard theorem, conjectured by Calabi [Cal57] and proved by Yau [Yau77], is that for any Kähler manifold with vanishing first Chern class there exists a unique Ricci-flat metric with a given complex structure and Kähler class. A Kähler manifold with vanishing first Chern class is therefore known as Calabi-Yau manifold.

Another important insight of recent years was the discovery of duality. For example, it was realized that if one considers string theory on a circle of radius R, the resulting physics can be equally well described in terms of string theory on a circle of radius $1/R$. Mirror symmetry is a generalization of this so called T-duality to curved space. In this case, two topologically distinct Calabi-Yau compactifications of string theory give rise to identical physical models. The transformation relating these two distinct geometrical formulations of the same physical model is that strong sigma model coupling problems in one can be mapped to weak sigma coupling problems in the other. By a judicious choice of the model which one uses, seemingly difficult physical questions can easily be analysed with perturbation theory.

1.2 Mathematical Methods

Instead of specifying a Calabi-Yau manifold as a patchwork of coordinate patches together with gluing instructions, it may be regarded as a subspace of another, presumably more easily discribable space. For example, the circle may also be represented as a subspace of a plane defined by the requirement that all of its points are at equal distance from a specified center. Thus, the desired circle may be considered as the (complete) intersection of a horizontal plane and the sphere in 3-dimensional affine space [1]. This approach is particularly useful if an otherwise complicated space may be represented as a complete intersection of simpler spaces.

For the construction of Calabi-Yau complete intersections we need methods of algebraic geometry. The study of this mathematical region is based on

[1] In algebraic geometry, neglecting the fact that $\mathbb{R}$ is not an algebraically closed field, we would say that a circle of radius r is the set of common zeros of the polynomials $x^2 + y^2 + z^2 - R^2$ and $z - h$, where $r^2 + h^2 = R^2$.

algebraic functions, i.e. functions that are polynomials in the coordinate functions of an affine space K^n which form a ring denoted by $K[T_1, \dots, T_n]$ [Sha94a, Sha94b, Mil, Sha94c]. An important assumption is that the field K is algebraically closed, so that there exists a one to one correspondence between radical ideals I in this ring and their sets of vanishing $V(I)$ in affine space, which follows from Hilbert's Nullstellensatz. These sets together with each ring of regular functions, which is the quotient ring over the ideal I, are called affine varieties. An algebraic variety is obtained by gluing affine varieties. The points of affine varieties may be identified with the maximal ideals in $K[T_1, \dots, T_n]/I(V)$. This gives an abstract definition of an affine variety as the set of maximal ideals of a reduced K-algebra of finite type over an algebraically closed field, called the maximal spectrum.

In the case of toric varieties the field K is the field of complex numbers, and the ring of regular functions is generated by monomials with exponents contained in a submonoid, or cone, of a lattice. An algebraic toric variety is obtained by gluing affine toric varieties, where the gluing-data is given by a fan Σ of cones. The remarkable feature of toric varieties is that all of the information, like singularity-considerations, compactness and even the cohomology, is already contained in the combinatorial data of their fan.

Our main interest is therefore directed to fans corresponding to reflexive polyhedra Δ^* dual to Δ, where the vertices of Δ^* lie on the rays of the cones of such a fan Σ. We will show that hypersurfaces, which are invariant under the action of the torus, are determined by a torus-equivariant Cartier divisor. By intersecting such hypersurfaces Batyrev and Borisov showed in their paper [BB95] that the result is a Calabi-Yau manifold V with at most quotient-singularities if the divisors satisfy the so called Nef condition, which can also be expressed as a purely combinatorial condition on the polyhedron Δ. Moreover, one obtains the dual Calabi-Yau manifold W corresponding to a new reflexive polyhedron ∇. This pair of Calabi-Yau manifolds is called a Mirror pair, and their Hodge numbers satisfy the relation

$$h_{st}^{p,q}(V) = h_{st}^{n-p,q}(W), \quad n = \dim V.$$

The Hodge numbers are (up to a sign) the coefficients of the so called E-polynomial. We will show that there exists an equivalent combinatorial definition of the Nef condition, and one can write the E- polynomial in a way which makes it possible to calculate the Hodge numbers of Calabi-Yau complete intersections starting with a reflexive polyhedron.

The first sizeable sets of Calabi-Yau manifolds were constructed as complete intersections (CICY) in products of projective spaces [GH87, CDLS88]. These manifolds have many complex structure deformations but only few Kähler moduli, which are inherited from the ambient space. With the discovery of Mirror symmetry [LVW89] the main interest therefore turned to weighted projective ($W\mathbb{P}$) spaces, where the resolution of singularities contributes additional Kähler moduli and thus provides a much more symmetric picture [CLS90]. A list of transversal configurations for codimension two Calabi-Yau manifolds in $W\mathbb{P}$ spaces was produced by Klemm [Kle]. As in the case of hypersurfaces there is, however, in general no Mirror construction available in that context [KT94].

We will work out a number of examples of toric complete intersection Calabi-Yau manifolds and discuss the relation of this construction to $W\mathbb{P}$ spaces. Identifying CICYs in $W\mathbb{P}$ spaces as a special case of the toric construction will provide us with, among other benefits, the Mirrors for these manifolds. In the case of hypersurfaces in $W\mathbb{P}^4$, the Newton polytope of a transversal quasihomogeneous polynomial [Fle89, KS92] can be identified with the polyhedron Δ, whose dual provides the toric resolution of the ambient space. It is thus clear that, for codimensions $r > 1$, we should look for the identification by trying to relate the Newton polytopes of the defining polynomial equations of degrees d_i to a Nef partition Δ_i of some reflexive polyhedron Δ. In some cases, however, this procedure is not so straightforward and we will give a detailed discussion on some examples in the last chapter.

The thesis is organized as follows: Chapter 2 contains fundamental definitions like algebraic subsets and algebraic functions, where Hilbert's Nullstellensatz plays a central role, showing the unique correspondence between affine varieties and ideals. At the end of the chapter we give a definition of an affine toric variety, which motivates the purely combinatorial chapter 3, where we work out some basic properties of cones and polytopes.

In chapter 4 we define algebraic varieties. The method is similar to that when dealing with manifolds using atlases of open coverings. The difference is that the open sets in a covering are open in a very special topology, the

Zariski topology, and must themselves have the structure of affine varieties.

Chapter 5 is devoted to the discussion of local properties of affine toric varieties. We define the tangent space at a point in a purely algebraic way and give a characterization of singularities in terms of the corresponding cone.

The aim of chapter 6 is to work out the orbits of the algebraic torus and to characterize closed subvarieties, which are invariant under the action of the torus. Furthermore, we show that for a compact toric variety the group of toric invariant Cartier divisors modulo principal divisors is isomorphic to the Picard group, and relate the set of global sections with a polyhedron by considering divisors as line bundles.

In chapter 7 we define Nef partitions on reflexive polytopes and give a proof that there is an equivalent definition using certain decompositions into Minkowski sums. With these Nef partitions we construct a special higher-dimensional cone and give an explicit formula for the Hodge numbers as coefficients of a polynomial depending only on this cone.

Finally chapter 8 together with the appendix A finishes with some tables of new Hodge numbers and compares them with results on toric hypersurfaces and complete intersections in weighted projective spaces [Kle, KS].

2 Affine Varieties

2.1 The Affine Space and Regular Functions

Let K be an algebraically closed field and K^n the n-th Cartesian power of K. Although K^n is a vector space over K, algebraic geometry supplies K^n with a weaker structure; in fact, among all mappings $K^n \to K$ (functions), one selects the so-called algebraic, or regular, functions.

The question arises which functions on K^n may be called algebraic in a natural way? The most natural definition seems to be the following: First of all we include the constants which are identified with the numbers in K. Then the coordinate functions, that is, the projection maps $T_i : K^n \to K$, where $T_i(x_1, \ldots, x_n) = x_i$. And finally the functions that are built from them through the elementary algebraic operations of addition and multiplication. These functions are called *regular*. Thus, regular functions are expressed polynomially in terms of the coordinate functions T_i. Moreover, as K is infinite, we can identify the ring of regular functions on K^n with the polynomial ring $K[T_1, \ldots, T_n]$ in the variables $T_1, \ldots, T_n$ with coefficients in K. One could also say that all functions of the form $1/f$, where f is regular and always different from zero, are regular. But such a function is necessarily constant, so this does not lead to anything new. Here we use the fact that K is algebraically closed, since over $\mathbb{R}$ the function $1 + t^2$ does not vanish for any $t \in \mathbb{R}$.

2.2 Algebraic Subsets

The algebraic subsets of K^n are defined by the systems of algebraic equations. An algebraic equation is an expression $f = 0$, where f is a polynomial in $T_1, \ldots, T_n$. Given a family $F = (f_r, r \in R)$ of polynomials, the family of equations $(f_r = 0, r \in R)$ – or $F = 0$ for short – is called a *system of algebraic equations*. A solution (also called zero, or root) of this system is any point $x \in K^n$ such that $f_r(x) = 0 \ \forall \ r \in R$. The set of all solutions is denoted by $V(F)$ and called *vanishing set* of F.

Definition 2.2.1 A subset of K^n is said to be *algebraic* if it is of the form $V(F)$ for some family F of polynomials in $T_1, \ldots, T_n$.

For example, the empty set and also K^n are algebraic (take $F = \{1\}$ and $F = \{0\}$ respectively). The intersection of any number of algebraic subsets is again algebraic, since $\bigcap V(F_j) = V(\bigcup F_j)$. The union of any *finite* number of algebraic subsets is also algebraic. Indeed, $V(F_1) \cup V(F_1) = V(F_1 \cdot F_2)$, where $F_1 \cdot F_2$ consists of all products of the form $f_1 f_2$ with $f_1 \in F_1$ and $f_2 \in F_2$. On the other hand, the complement of an algebraic subset $V \subset K^n$ is not algebraic (except for $V = \emptyset$ and $V = K^n$).

Remark 2.2.2 By definition the algebraic subsets of K^n are the closed subsets of a topology, the *Zariski topology.*

2.3 Systems of Algebraic Equations; Ideals

Different systems of algebraic equations can have the same set of solutions. Indeed, if we adjoin to a system F the polynomial $\sum f_j g_j$, where $f_j \in F$ and $g_j \in K[T_1, \ldots, T_n]$, the set of solutions will remain unchanged. We shall say that $\sum f_j g_j$ can be expressed algebraically in terms of the family F. Two families F and F' are said to be *equivalent* if every member of F can be expressed algebraically in terms of F' and conversely. Clearly, F and F' are equivalent if and only if they generate the same ideal in the ring $K[T_1, \ldots, T_n]$. Going over to ideals is useful because of *Hilbert's Basis Theorem* [Spi95, Sha94a, Cig95]:

Theorem 2.3.1
Every ideal in the polynomial ring $K[T_1, \ldots, T_n]$ is generated by a finite set of elements.

In other words, the ring of polynomials over any field is *noetherian.* As a corollary we see that every system of algebraic equations is equivalent to a *finite* system of equations, or that every algebraic subset is the intersection of a finite number of hypersurfaces.

Remark 2.3.2 The following conditions on a ring A are equivalent ([Spi95, Mil, Cig95]):

- Every ideal is finitely generated.
- Every ascending chain of ideals $I_1 \subsetneq I_2, \ldots$ becomes stationary, i.e. $\exists m \in \mathbb{N} : I_m = I_n \; \forall \; n \geq m, \; n \in \mathbb{N}$.

As we have already said, equivalent systems of equations have identical sets of solutions; however, two nonequivalent systems may also define the same subset. The reason for this is quite simple: The polynomials $f, f^2, f^3, \ldots$ have the same zeros. In view of this we shall say that two families, F and F', are *weakly equivalent* if every $f \in F$ has a power f^r that can be expressed algebraically in terms of F' and conversely. Again, weakly equivalent systems of equations have identical sets of solutions. As we shall see now, the converse is also true. In any case, for each algebraic subset $V \subset K^n$ there is a largest ideal defining V, namely, the ideal $I(V)$ of all regular functions vanishing at all points of V.

2.4 Hilbert's Nullstellensatz

Let us start from the simplest situation. It is clear that the unit ideal $I = K[T_1, \ldots, T_n]$ defines the empty subset $V(I)$. Though this is much less obvious, the converse is also true; this assertion is called *Hilbert's Weak Nullstellensatz*:

Theorem 2.4.1
If the ideal $J \subset K[T_1, \ldots, T_n]$ is not the unit ideal then $V(J)$ is not empty.

Proof: It is essential here that the field K is algebraically closed. For example, the ideal generated by $1 + t^2$ has no zeros in $\mathbb{R}$. By 2.3.1, the ring of polynomials $K[T_1, \ldots, T_n]$ is noetherian, so it follows from 2.3.2 that every ideal J is contained in a maximal ideal I. Therefore it is sufficient to show that $V(I)$ is not empty for all maximal ideals $I \subset K[T_1, \ldots, T_n]$. Let $I \subset K[T_1, \ldots, T_n]$ be a maximal ideal. Then $K[T_1, \ldots, T_n]/I$ is a field which contains K. We shall prove that these two fields coincide. Then we have a canonical K-algebra homomorphism

$$\varphi : K[T_1, \ldots, T_n] \to K = K[T_1, \ldots, T_n]/I$$

with $\operatorname{kern} \varphi = I$. Now let $a = (a_1, \ldots, a_n) \in K^n$ with $a_i := \varphi(T_i)$. Then $a \in V(I)$ because for every $f \in I,\ f = \sum c_{i_1} \ldots c_{i_n} T_1^{i_1} \ldots T_n^{i_n}$ we have

$$0 = \varphi(f) = \sum c_{i_1} \ldots c_{i_n} \varphi(T_1)^{i_1} \ldots \varphi(T_n)^{i_n} = \sum c_{i_1} \ldots c_{i_n} a_1^{i_1} \ldots a_n^{i_n} = f(a).$$

Hence it remains to show that the field $K[T_1, \ldots, T_n]/I$ is isomorphic to K. Since K is algebraically closed, this assertion is a consequence of a purely algebraic proposition [Eig, Mil, Sha94c]:

Proposition 2.4.2 (Zariski's lemma). Let K be an arbitrary field and L a K-algebra of finite type. Then if L is a field it is algebraic over K (Hence L=K if K is algebraically closed).

Corollary 2.4.3 [Sha94c, Spi95] (Hilbert's Nullstellensatz). Let I be an ideal in $K[T_1, \ldots, T_n]$, and suppose the polynomial h vanishes at all points of the set $V(I) \subset K^n$. Then $h^r \in I$ for some integer $r \geq 0$.

Proof: By 2.3.1, there exists a finite set $\{f_1, \ldots, f_m\}$ generating I. Construct the ideal $J \subset K[T_1, \ldots, T_{n+1}]$ generated by the polynomials $f_1, \ldots, f_m, 1 - T_{n+1}h$. Since $V(J)$ is the empty set, 2.4.1 implies that $J = K[T_1, \ldots, T_{n+1}]$ and

$$1 = \sum_{i=1}^{m} f_i g_i + g(1 - T_{n+1}h)$$

must hold for some $g_1, \ldots, g_n, g \in K[T_1, \ldots, T_{n+1}]$. Substituting $1/h$ for T_{n+1}, we obtain the identity

$$1 = \sum_{i=1}^{m} f_i(T_1, \ldots, T_n) g_i(T_1, \ldots, T_n, \frac{1}{h})$$

in $K[T_1, \ldots, T_n]$ and there exists a $N \in \mathbb{N}$ such that

$$h^N = \sum_{i=1}^{m} f_i \tilde{g}_i,$$

with $\tilde{g}_i = h^N g_i$ $(i = 1, \ldots m)$.

2.5 Affine Varieties

Let $V \subset K^n$ and $W \subset K^m$ be two algebraic subsets. A mapping $f : V \to W$ is said to be *regular* (or a morphism) if it is given by m regular functions $f_1, \ldots, f_m \in K[T_1, \ldots, T_n]$, that is, if it extends to a regular mapping of the ambient space $K^n \to K^m$. The composite of two regular maps is still regular. Hence algebraic sets, together with regular mappings, form a category. The objects of this category are called *affine algebraic varieties* (or simply affine varieties).

A *regular function* on an algebraic set V is a regular mapping of V into K. Regular functions can be added and multiplied, so that they form a ring (and even a K-algebra) $K[V]$. Given an algebraic subset $V \subset K^n$, the algebra $K[V]$ identifies with the quotient algebra $K[T_1, \dots, T_n]/I(V)$. Also, the embedding $V \subset K^n$ can be recovered from the generators $t_i = T_i \mid_V$ of $K[V]$.

One can also think of a morphism in terms of regular functions. A mapping $f : V \to W$ is regular if and only if, for every regular function $g \in K[W]$, the function $f^*(g) = g \circ f$ is regular on V. In this case the map $f^* : K[W] \to K[V]$ is a K-algebra homomorphism. Conversely, such a homomorphism is always induced by a morphism $V \to W$.

2.6 Abstract Affine Varieties

The K-algebra $K[V]$ of regular functions on an algebraic set V has two specific properties. First of all it is of finite type, i.e. it is generated by finitely many elements. Second, as an algebra of functions with values in a field K, it is reduced, i.e. it has no nilpotent elements (other than 0). Finally, it follows from Hilbert's Nullstellensatz that by associating with a point $x \in V$ the maximal ideal $I(x) := \{f \in K[V] : f(x) = 0\}$ we get a bijection between V and the set $\operatorname{Specm} K[V]$ of all maximal ideals of the ring $K[V]$.

These properties enable us to give an abstract definition of an *affine variety* over K as a triple $(X, K[X], \varphi)$, where X is a set, $K[X]$ a reduced K-algebra of finite type and φ a bijection of X onto $\operatorname{Specm} K[X]$. The elements of X are the points of this variety, while those of $K[X]$ are called its regular functions. In fact, given $x \in X$ and $f \in K[X]$, it makes sense to talk about the value of f at the point x. By definition it is the image of f under the composite map

$$K[X] \xrightarrow{\alpha} K[X]/\varphi(x) \underset{\approx}{\xleftarrow{\beta}} K,$$

where α is the projection onto the quotient algebra and β the structure K-homomorphism which is one-to-one by virtue of Hilbert's Nullstellensatz. The fact that φ is bijective means that both the points and the functions are in good supply: There are enough functions to distinguish the points, and enough points to realize all K-algebra homomorphisms $K[X] \to K$. In what follows we shall no longer write φ , and the values at the points will be understood.

With this terminology a *morphism* of $(X, K[X])$ into $(Y, K[Y])$ is a pair (f, f^*) consisting of a mapping $f : X \to Y$ and a K-algebra homomorphism $f^* : K[Y] \to K[X]$ such that $f^*(g)(x) = g(f(x))\ \forall\ g \in K[Y],\ x \in X$. In fact, each of f and f^* is determined by the other.

2.7 Affine Schemes

Suppose that, in the definition of an abstract variety, we forget about the requirement that the ring $K[X]$ should be reduced. Then the object obtained will be called an *affine algebraic K-scheme* (more briefly: an affine scheme). An element of $K[X]$ defines a mapping $X \to K$; but in general these elements of $K[X]$ cannot be identified with functions. Indeed, some nonzero elements of $K[X]$ can give rise to functions which are identically zero on X. Besides, it follows from Hilbert's Nullstellensatz that this can only happen with nilpotent elements of $K[X]$.

2.8 Localization

The Zariski topology (see 2.2.2) makes it possible to define regular functions in a more local fashion: For every $f \in K[X]$ define the sets $\mathcal{D}(f) = X - V(f)$ which are a basis of the Zariski topology. Then an element $f \in K[X]$ defines functions

$$f : X \to K \quad \text{and} \quad 1/f : \mathcal{D}(f) \to K.$$

Let $U \subset X$ be a Zariski-open set. We call a function $h : U \to K$ *regular at a point* $x \in U$ if there exists $f, g \in K[X]$ with $f(x) \neq 0$ such that h coincides with g/f on an open neighbourhood U_x of x. The functions on U that are regular at every point of U form a ring which is denoted by $\mathcal{O}_X(U)$.

Proposition 2.8.1 Every open covering of an affine variety X has a finite subcovering.

Proof: Let $X = \bigcup_{i\in I} U_i$ be an open covering of X. Choose $i_0 \in I$; if $U_{i_0} \neq X$, there exists an $i_1 \in I$ such that $U_{i_0} \subsetneq U_{i_0} \cup U_{i_1}$. If $U_{i_0} \cup U_{i_1} \neq X$, there exists an $i_2 \in I$ etc. Let $\tilde{U}_k = U_{i_0} \cup \cdots \cup U_{i_k}$, $V_k = X \setminus \tilde{U}_k$ and $I_k = I(V_k)$. By 2.3.2 $\exists m \in \mathbb{N} : I_m = I_k\ \forall\ k \geq m,\ k \in \mathbb{N}$. Therefore $\tilde{U}_k = \tilde{U}_m\ \forall\ k \geq m$ and $\{U_{i_0}, \ldots, U_{i_m}\}$ is a finite subcovering of $\bigcup_{i\in I} U_i$.

Proposition 2.8.2 If X is an affine variety then $\mathcal{O}_X(X) = K[X]$.

Proof:
$\mathcal{O}_X(X) \supset K[X]$: is obvious.
$\mathcal{O}_X(X) \subset K[X]$: Let $h \in \mathcal{O}_X(X)$ be regular at every point $x \in X$. Then $h = g_x/f_x$ on an open neighbourhood U_x of x for all $x \in X = \bigcup_{x \in X} U_x$. By 2.8.1 there exists a finite sub-covering $\{U_{x_1}, \ldots, U_{x_n}\}$ of $\bigcup_{x \in X} U_x$. Since the set $\{f_{x_1}, \ldots, f_{x_n}\}$ has no common zeros, Hilbert's Nullstellensatz 2.4.3 implies that the functions f_{x_i} generate the unit ideal $K[X]$. So there exists a decomposition of the unit $1 = \sum a_{x_i} f_{x_i}$ ($a_i \in K[X]$) and it follows that $h = h \cdot 1 = \sum a_{x_i} h f_{x_i} = \sum a_{x_i} g_{x_i} \in K[X]$.

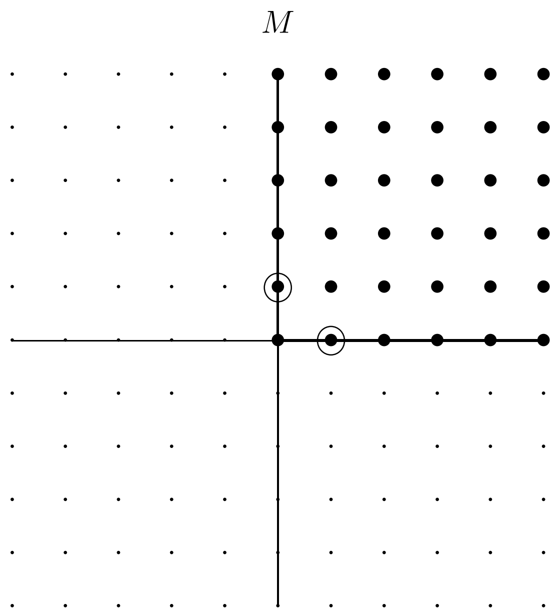
2.9 Affine Toric Varieties

Let M be a free abelian group of rank d and $M_{\mathbb{R}}$ its real scalar extension. Now let $C \subset M_{\mathbb{R}}$ be a cone. Then $S := C \cap M \subset M$ is a submonoid of M, i.e. a subset of M containing $\{0\} \in S$ which is closed under addition. We can form a semigroup K-algebra $K[S]$. This algebra is generated by all elements of the form x^m with $m \in S$, multiplication being defined by the rule $x^m \cdot x^{m'} = x^{m+m'}$. If S is finitely generated as a monoid then the K-algebra $K[S]$ is of finite type and defines an affine variety X, namely,

$$X = \operatorname{Specm} K[S].$$

Example 2.9.1 Let $K = \mathbb{C}$ be the field of complex numbers. For $S = M$ we get the d-dimensional complex algebraic torus $T = \operatorname{Specm}\mathbb{C}[M] = \operatorname{Specm}[T_1, \ldots, T_n, T_1^{-1}, \ldots, T_n^{-1}]$ which consists of all points $z \in \mathbb{C}^d$ with $T_i(z) \neq 0$ $(i = 1, \ldots, d)$ and coincides with the set $\mathbb{C}^{*d}$ with $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

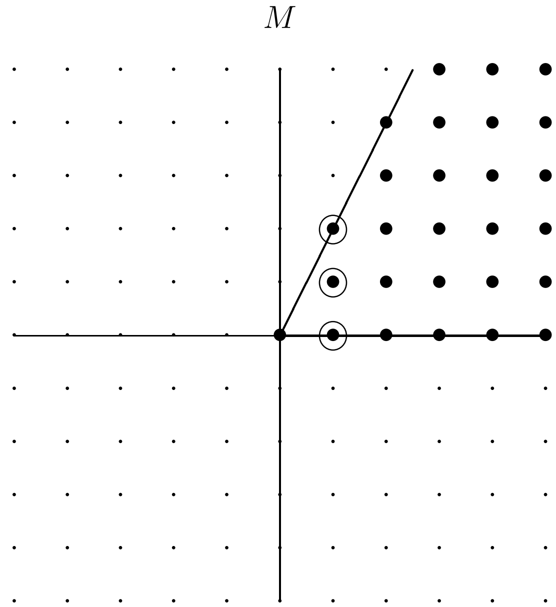
Example 2.9.2 Let $K = \mathbb{C}$ be the field of complex numbers. If $C = \{(x, y) \in \mathbb{R}^2 : x, y \geq 0\}$, we get a submonoid $S = \{(m_1, m_2) \in M : m_1, m_2 \geq 0\}$:



and

$$X = \operatorname{Specm}\mathbb{C}[S] = \operatorname{Specm}[T_1, T_2] = \mathbb{C}^2.$$

Example 2.9.3 Let $K = \mathbb{C}$ be the field of complex numbers. If $C \subset \mathbb{R}^2$ is the cone generated by e_1 and $e_1 + 2e_2$, where $\{e_1, e_2\}$ is the standard-basis of $\mathbb{R}^2$, S is generated as a semigroup by $e_1, e_1 + e_2$ and $e_1 + 2e_2$:



and we get

$$X = \operatorname{Specm} \mathbb{C}[S] = \operatorname{Specm}[T_1, T_1T_2, T_1T_2^2] = \mathbb{C}[U, V, W]/(V^2 - UW),$$

which is a quadric cone, i.e. a cone over a conic.

Remark 2.9.4 [Ful93, Oda88] For $\mathbb{C}[S]$ generated by a submonoid $S \subset M$, the points x of an affine toric variety are easy to describe: they correspond to homomorphisms of semigroups from S to $\mathbb{C}$:

$$\operatorname{Specm} \mathbb{C}[S] = \operatorname{Hom}_{sg}(S, \mathbb{C}).$$

The value of $m \in S$ is the value of the corresponding function $\chi(m)$ evaluated at the point x:

$$x : \; S \to \mathbb{C} \quad m \mapsto x(m) := \chi(m)(x).$$

Let $t \in T$ (see example 2.9.1). Then the *action of the torus* T is defined as

$$t : \mathrm{Hom}_{sg}(S, \mathbb{C}) \to \mathrm{Hom}_{sg}(S, \mathbb{C}) \quad x \mapsto tx,$$

with $tx(m) := t(m)x(m) \ \forall m \in S$.

3 Convex Geometry

3.1 Convex Polyhedral Cones

At the end of the last section we saw that a K-algebra $K[S]$ of finite type can be constructed with a finitely generated semigroup. We constructed such semigroups by intersecting a cone $C \subset M_{\mathbb{R}}$ with the lattice M. It will turn out that toric varieties can be constructed by gluing affine toric varieties, each corresponding to a cone C, where the gluing data is given in terms of a fan of cones. Thus we will work out some basic properties of convex geometry in this chapter.

Let N and M be two *free abelian groups* of *rank* d which are dual to each other, i.e. $M = \mathrm{Hom}(N, \mathbb{Z})$. We denote by

$$\langle *, * \rangle : M \times N \to \mathbb{Z}$$

the canonical $\mathbb{Z}$-bilinear pairing, and by $N_{\mathbb{R}}$ (resp. by $M_{\mathbb{R}}$) the real scalar extension of N (resp. of M) with a canonical $\mathbb{R}$-bilinear pairing $\langle *, * \rangle : M_{\mathbb{R}} \times N_{\mathbb{R}} \to \mathbb{R}$.

Definition 3.1.1 A subset $C \subset N_{\mathbb{R}}$ is called a *d-dimensional convex polyhedral cone*, if there exists a finite set $\{e_1, \ldots, e_k\} \in N_{\mathbb{R}}$ such that

$$C = \{\lambda_1 e_1 + \cdots + \lambda_k e_k \in N_{\mathbb{R}} : \lambda_i \in \mathbb{R}_{\geq},\ i = 1, \ldots, k\}.$$

The *dimension* d of a cone is defined as the dimension of the space $-C+C = \{-x + y : x, y \in C\}$.

Definition 3.1.2 If $C \subset N_{\mathbb{R}}$ is a d-dimensional convex polyhedral cone, the *dual* C^* is defined as the set of equations of bounding hyperplanes, i.e.

$$C^* = \{\hat{z} \in M_{\mathbf{R}} : \langle \hat{z}, z \rangle \geq 0 \quad \forall z \in C\}.$$

Definition 3.1.3 A *face* F of a cone C is the intersection of C with any supporting hyperplane:

$$F = C \cap \hat{z}^{\perp} = \{z \in C : \langle \hat{z}, z \rangle = 0\} \text{ for some } \hat{z} \in C^*.$$

A cone is regarded as a face of itself while the others are called *proper* faces. The faces with codimension 1 are called *facets*.

Proposition 3.1.4 [Ful93, Oda88] An important role plays the following fundamental fact from the theory of convex sets:

$$\forall\ C \subset N_{\mathbb{R}},\ z \in N_{\mathbb{R}} \setminus C \quad \exists \hat{z} \in C^* : \langle \hat{z}, z \rangle < 0.$$

Proposition 3.1.5 For the faces $F \subset C$ of a convex polyhedral cone the following holds:

1. $(C^*)^* = C$.
2. Any linear subspace V of a cone is contained in every face.
3. Any face is also a convex polyhedral cone.
4. Any intersection of faces is also a face.
5. Any face of a face is a face.
6. Any proper face is contained in some facet.
7. Any proper face is the intersection of all facets containing it.

Proof:

1. $C \subset\ (C^*)^*$: follows from definition 3.1.2.
   $C \supset (C^*)^*$: Suppose there exists a $z \in (C^*)^*$ with $z \notin C$. By 3.1.4 $\exists\ z^* \in C^*$ with $\langle \hat{z}, z \rangle < 0$. A contradiction to $z \in (C^*)^*$.
2. Follows from $\langle \hat{z}, z \rangle = 0\ \forall \hat{z} \in C^*, z \in V$.
3. A face $F = C \cap \hat{z}^{\perp}$ is generated by those vectors of $\{e_1, \ldots, e_k\}$ for which $\langle \hat{z}, e_i \rangle = 0$. In particular, we see that a cone has only a finite number of faces.
4. $\bigcap \left(C \cap \hat{z}_i^{\perp}\right) = C \cap (\sum \hat{z}_i)^{\perp} \quad \forall \hat{z}_i \in C^*, \quad i = 1, \ldots, n.$
5. Let $F = C \cap \hat{x}^{\perp}$, $\tilde{F} = F \cap \hat{y}^{\perp}$ for some $\hat{x} \in C^*$ and $\hat{y} \in F^*$. Then, for large positive $p \in \mathbb{R}_{\geq}$, $\hat{y} + p\hat{x} \in C^*$ and $\tilde{F} = C \cap (\hat{y} + p\hat{x})^{\perp}$.
6. It is sufficient to show that if $F = C \cap \hat{z}^{\perp}$ has codimension greater than one F is contained in a larger face. We may assume that $V := N_{\mathbb{R}} = -C + C$, otherwise replace V by the span of C. Define $W := -F + F$ and let $\bar{e}_i$ be the images of the generators of C under the

canonical homomorphism $\varphi : V \to V/W$, which are all contained in the halfspace determined by $\hat{z}$. By moving this halfspace in the sphere of halfspaces in V/W, we can find one that contains these vectors $\bar{e}_i$ but with at least one such nonzero vector in the boundary hyperplane. This hyperplane determines a $\hat{w} \in C^*$, such that $C \cap \hat{w}^\perp$ is a larger face. If the codimension of F is two, so that V/W is a plane, there are exactly two such supporting lines, which proves that any face of codimension two is the intersection of exactly two facets.

7. Indeed, if F is any face of codimension lager than two, from (6) we can find a facet $\tilde{F}$ containing it; by induction F is the intersection of facets in $\tilde{F}$ and each of these is the intersection of two facets in C.

Definition 3.1.6 Let C be a convex polyhedral cone. Then we denote by:

- $C^\circ \ldots$ the *relative interior (topological interior)* of C.
- $\partial C \ldots$ the *topological boundary* of C.

Proposition 3.1.7 The topological boundary ∂C of a cone C that spans V is the union of its proper faces (or facets) F.

Proof: We have to show: $z \in \bigcup_{F \subset C} F \Leftrightarrow z \in \partial C$.
$\Rightarrow$: is obvious, because a face is the intersection of C with a bounding hyperplane
$\Leftarrow$: Let $z \in \partial C$. $\Rightarrow \exists\ (x_i)_{i\in\mathbb{N}} \to z$ with $x_i \notin C\ \forall\ i \in \mathbb{N}$. (3.1.4)$\Rightarrow \exists\ \hat{x}_i \in C^* : \langle \hat{x}_i, x_i \rangle < 0\ \forall i \in \mathbb{N}$. By taking the $\hat{x}_i$ in a sphere, we find a converging subsequence $(\hat{x}_{i_k})_{k\in\mathbb{N}} \to \hat{x} \in C^*$. Then $z \in F = C \cap \hat{x}^\perp$.

When C spans the space $N_\mathbb{R}$ and $\mathcal{F}$ is a facet, there exists, up to multiplication by a positive scalar, a unique $\hat{z}$ with $\mathcal{F} = C \cap \hat{z}^\perp$. Such a vector, which we denote by $z_\mathcal{F}$, corresponds to an *equation for the hyperplane* spanned by $\mathcal{F}$.

Proposition 3.1.8 If C spans $N_\mathbb{R}$ and $C \neq N_\mathbb{R}$, then C is the intersection of half-spaces $\mathrm{H}_\mathcal{F} = \{z \in N_\mathbb{R} : \langle \hat{z}_\mathcal{F}, z \rangle \geq 0\}$ as $\mathcal{F}$ ranges over facets of C.

Proof:
$C \subset \bigcap_{\mathcal{F} \subset C} \mathrm{H}_\mathcal{F}$: follows from $C = (C^*)^*$ and $\hat{z}_\mathcal{F} \in C^*\ \forall \mathcal{F}$.

$C \supset \bigcap_{\mathcal{F} \subset C} \mathrm{H}_{\mathcal{F}}$: Assume there exists a $z \in \bigcap_{\mathcal{F} \subset C} \mathrm{H}_{\mathcal{F}}$ with $z \notin C$. Take any $w \in C^{\circ}$ and let v be the last point in C on the line segment from z to w, so that $v \in \partial C$. (3.1.7) $\Rightarrow \exists$ facet $\mathcal{F}$ of $C : v \in \mathcal{F} \Rightarrow \exists \hat{z}_{\mathcal{F}} \in C^*$: $\langle \hat{z}_{\mathcal{F}}, w \rangle > 0$ and $\langle \hat{z}_{\mathcal{F}}, v \rangle = 0$ so $\langle \hat{z}_{\mathcal{F}}, z \rangle < 0$, a contradiction to $z \in \bigcap_{\mathcal{F} \subset C} \mathrm{H}_{\mathcal{F}}$.

Corollary 3.1.9 If C spans $N_{\mathbb{R}}$ the proof of (3.1.8) gives a practical procedure of finding generators for the dual cone C^*: Let n be the dimension of C. Then for each set of $(n-1)$ linear independent vectors search for a $\hat{z} \in M_{\mathbb{R}}$ annihilating this set. If neither $\hat{z}$ nor $-\hat{z}$ is nonnegative on all generators of C it is discarded; otherwise $\hat{z}$ (or $-\hat{z}$) is taken as a generator of C^*. The fact that the cone created by these generators is C^* follows from:

Proposition 3.1.10 *(Farkas' Theorem).* The dual of a convex polyhedral cone is a convex polyhedral cone.

Proof: We have to prove two cases:

1. C spans $N_{\mathbb{R}}$: Let $\tilde{C}^*$ be the cone from proposition 3.1.8, generated by the set $\{\hat{z}_{\mathcal{F}}\}$ as $\mathcal{F}$ ranges over facets of C. We prove that $\tilde{C}^*$ and C^* coincide:
   $\tilde{C}^* \subset C^*$: is obvious.
   $\tilde{C}^* \supset C^*$: Assume there exists a $\hat{z} \in C^*$ with $\hat{z} \notin \tilde{C}^*$. Applying 3.1.4 to $\tilde{C}^* \ \Rightarrow \exists z \in (\tilde{C}^*)^* : \langle \hat{z}, z \rangle < 0$ with $\langle \hat{z}_{\mathcal{F}}, z \rangle \geq 0 \ \forall$ facets $\mathcal{F} \subset C$. A contradiction to (3.1.8).

2. $(-C+C) \neq N_{\mathbb{R}}$: C^* is generated by lifts of generators of the dual cone in $M_{\mathbb{R}}/(-C+C)^{\perp}$ together with vectors $\hat{z}$ and $-\hat{z}$, as $\hat{z}$ ranges over a basis for $(-C+C)^{\perp}$.

Definition 3.1.11 A cone is said to be *rational* if its generators can be taken from M.

Remark 3.1.12 (3.1.10) shows that a convex polyhedral cone can also be given a dual definition as intersection of halfspaces: Let $\{\hat{e}_1, \ldots, \hat{e}_n\}$ be generators of C^*, then

$$C = \{z \in N_{\mathbb{R}} : \langle \hat{e}_i, z \rangle \geq 0 \ \forall i = 1, \ldots, n\}.$$

Remark 3.1.13 Since the generators in (3.1.9) can be taken from M, it follows that:

$$C \text{ is rational} \Leftrightarrow C^* \text{ is rational.}$$

Proposition 3.1.14 *(Gordon's Lemma).* If C is a rational convex polyhedral cone, then $S_C = C^* \cap M$ is a finitely generated semigroup.

Proof: Take the generators $\{\hat{e}_1, \ldots, \hat{e}_n\}$ of C^* from $M \cap C^*$. Let

$$K = \{\sum_{i=1}^{n} t_i \hat{z}_i \;:\; t_i \in [0,1] \subset \mathbb{R}_{\geq}\}.$$

Since K is compact and M is discrete the intersection $K \cap M$ is finite. Then $K \cap M$ generates the subgroup. Indeed, if $\hat{z} \in M \cap C^*$ write $\hat{z} = \sum r_i \hat{e}_i,\ r_i \in \mathbb{R}_{\geq}$ and set the $r_i = m_i + t_i$ with $m_i \in \mathbb{N}_0$ and $t_i \in [0,1] \subset \mathbb{R}$. Then $\hat{z} \sum m_i \hat{e}_i + \hat{u}$ with $\hat{u} = \sum t_i \hat{e}_i \in K \cap M$, so $S_C = C^* \cap M$ is generated as a semigroup by the finite set $\{\hat{e}_1, \ldots, \hat{e}_n\} \cup (K \cap M)$.

Proposition 3.1.15 If F is a face of C, then $C^* \cap F^{\perp}$ is a face of C^* with $\dim(F) + \dim(C^* \cap F^{\perp}) = n = \dim(N_{\mathbb{R}})$. This sets up a one to one order-reversing correspondence between the faces of C and the faces of C^*.

Proof: Let $\hat{C} \cap z^{\perp}, z \in C$ be a face of C^*. If F is the cone containing z in its relative interior, $C^* \cap z^{\perp} = C^* \cap (F^* \cap z^{\perp}) = C^* \cap F^{\perp}$, so every face has the asserted form. The map $F \to F^{\star} := C^* \cap F^{\perp}$ is clearly order-reversing, and from the obvious inclusion $F \subset (F^{\star})^{\star}$ it follows that $F^{\star} = ((F^{\star})^{\star})^{\star}$ and hence that the map is one-to-one onto. It follows from this that the smallest face of C is $(C^*)^* \cap (C^*)^{\perp} = (C^*)^{\perp} = C \cap (-C)$. In particular, we see that $\dim(C \cap (-C)) + \dim(C^*) = n$. The corresponding equation for a general face F can be deduced by putting F in a maximal chain of faces of C and comparing with the dual chain of faces in C^*.

Proposition 3.1.16 If $\hat{z} \in C^*$ and $F = C \cap \hat{z}^{\perp}$, then $F^* = C^* + \mathbb{R}_{\geq} \cdot (-\hat{z})$.

Proof: Since both sides of this equation are convex polyhedral cones, it is sufficient to show that their duals are equal: $F = C \cap \hat{z}^{\perp} = C \cap (\mathbb{R}_{\geq} \cdot (-\hat{z}))^*$, as required.

Proposition 3.1.17 Let C be a rational convex polyhedral cone and let $m \in S_C = C^* \cap M$. Then $F = C \cap m^\perp$ is a rational convex polyhedral cone. All faces of C have this form and

$$S_F = S_C + \mathbb{Z}_{\geq}(-m).$$

Proof: $F = C \cap \hat{z}^\perp$ for any $\hat{z} \in (C^* \cap F^\perp)^\circ$, and $\hat{z}$ can be taken from M since $C^* \cap F^\perp$ is rational, so $F = C \cap m^\perp$ with $m \in (C^* \cap F^\perp)^\circ \cap M$.
$S_C + \mathbb{Z}_{\geq}(-m) \subset S_F$: is obvious.
$S_C + \mathbb{Z}_{\geq}(-m) \supset S_F$: Given $\tilde{m} \in S_F$, then $\tilde{m} + nm \in C^*$ for large $n \in \mathbb{N}$, which shows, that $\tilde{m} \in S_C + \mathbb{Z}_{\geq}(-m)$.

Proposition 3.1.18 *(Separation Lemma).* If C and $\tilde{C}$ are convex polyhedral cones whose intersection is a face of each, then there exists a $\hat{z} \in C^* \cap (-\tilde{C}^*)$ with

$$F = C \cap \hat{z}^\perp = \tilde{C} \cap \hat{z}^\perp.$$

Proof: Let $\bar{C} = C - \tilde{C} = C + (-\tilde{C})$ and $\hat{z} \in (\bar{C}^*)^\circ$. From (3.1.15) it follows that $\bar{C} \cap \hat{z}^\perp = \bar{C} \cap (-\bar{C}) = (C - \tilde{C}) \cap (\tilde{C} - C)$ is the smallest face of $\bar{C}$. The claim is that this $\hat{z}$ works:
$F \subset C \cap \hat{z}^\perp : F = C \cap \tilde{C} \Rightarrow F \subset C \;\wedge\; F \subset \tilde{C} \Rightarrow F \subset C + (-\tilde{C}) \;\wedge\; F \subset \tilde{C} + (-C) \Rightarrow F \subset \bar{C} \cap (-\bar{C}) = C \cap \hat{z}^\perp$.
$F \supset C \cap \hat{z}^\perp$: Let $z \in C \cap \hat{z}^\perp = \tilde{C} - C \Rightarrow' = \tilde{x} - x$, with $\tilde{x} \in \tilde{C}$ and $x \in C \;\Rightarrow z + x \in F = C \cap \tilde{C} \;\Rightarrow z \in F$.
Since the same arguments hold for $-\hat{z}$ and $\tilde{C} \cap \hat{z}^\perp$ the proof is complete.

Corollary 3.1.19 If C and $\tilde{C}$ are rational convex polyhedral cones whose intersection F is a face of each, then

$$S_F = S_C + S_{\tilde{C}}.$$

Proof:
$S_F \supset S_C + S_{\tilde{C}}$: is obvious.
$S_F \subset S_C + S_{\tilde{C}}$: (3.1.18) $\Rightarrow \exists \hat{z} \in C^* \cap (-\tilde{C}^*) \cap M$ such that $F = C \cap \hat{z}^\perp = \tilde{C} \cap \hat{z}^\perp$. From 3.1.17 and $-\hat{z} \in S_{\tilde{C}}$ it follows that $S_F \subset S_C + \mathbb{Z}_{\geq}(-\hat{z}) \subset S_C + S_{\tilde{C}}$.

Definition 3.1.20 A convex polyhedral cone C is called *strongly convex* if $C \cap (-C) = \{0\}$.

Remark 3.1.21 For a convex polyhedral cone C the following are equivalent:

- $C \cap (-C) = \{0\}$.
- C contains no nonzero linear subspace.
- $\exists \hat{z} \in C^* : C \cap \hat{z}^\perp = \{0\}$.
- C^* spans $M_\mathbb{R}$.

Any cone is generated by some minimal set of generators. If the cone is strongly convex, then the rays generated by a minimal set of generators are exactly the one-dimensional faces of the cone, which can be seen by applying 3.1.4 to any generator that is not in the cone generated by the others.

Definition 3.1.22 A convex polyhedral cone C is called *simplicial* if it is generated by a set of linear independent generators.

Remark 3.1.23 Let $\{e_1, \dots, e_k\}$ be the set of generators of a simplicial cone C. Then

$$z = \sum_{i=1,\dots,k} \lambda_i z_i \in C^\circ \Leftrightarrow \lambda_i > 0 \; \forall i = 1, \dots, k.$$

3.2 Cones from Polytopes

Definition 3.2.1 Let V be a finite dimensional vector space. Then the convex hull of a finite set of points is called a *convex polytope* K. The *dimension* d of K is the dimension of the vector space spanned by K.

Definition 3.2.2 A *(proper) face* $F \subset K$ is the intersection with a bounding affine hyperplane, i.e.

$$F = \{z \in K : \langle \hat{z}, z \rangle = r \in \mathbb{R}_\geq\} \text{ with } \hat{z} \in V^* \text{ and } \langle \hat{z}, z \rangle \geq r \; \forall \; z \in K.$$

The faces of codimension one are called $facets$.

We assume for simplicity that $\dim(K) = \dim(V)$ and that K contains the origin in its interior. The results of 3.1 can be used to deduce the corresponding facts about the faces of convex polytopes. To see this let C be a strong convex polyhedral cone over $K \times 1$ in the vector space $V \times \mathbb{R}$. Then the faces of C are easily seen to be exactly the cones over the faces of K (with the cone $\{0\}$ corresponding to the empty face of K).

Definition 3.2.3 The *polar set* $K^*$of a convex polytope K is defined as

$$K^* = \{\hat{z} \in V^* : \langle \hat{z}, z \rangle \geq -1 \; \forall \; z \in K\}.$$

Proposition 3.2.4 The polar set K^* is a convex polytope, and $(K^*)^* = K$. If F is a face of K, then

$$F^\star = \{\hat{z} \in K^* : \langle \hat{z}, z \rangle = -1 \; \forall \; z \in F\}$$

is a face of K^* and the correspondence $F \mapsto F^\star$ is a one-to-one, order-reversing correspondence between the faces of K and K^*, with $\dim(F) + \dim(F^\star) = \dim(V) - 1$.

Proof: With C the cone over $K \times 1$, the dual cone C^* is given by $\{\hat{z} \times r \in V^* \times \mathbb{R} : \langle \hat{z}, z \rangle + r \geq 0 \; \forall \; z \in K\}$. It follows, that C^* is the cone over K^*. The assertions of the proposition are now simple consequences of the results in section 3.1.

A particularly important construction of toric varieties starts with a rational polytope P in the dual space $M_\mathbb{R}$. We assume that $\dim(P) = \dim(M_\mathbb{R})$, but it is not necessary that it contains the origin. From P a fan, denoted by $\triangle_P \subset N_\mathbb{R}$, is constructed as follows: There is a cone C_F of $\triangle_P$ for each face F of P defined by

$$C_F = \{z \in N_\mathbb{R} : \langle \hat{x} - \hat{y}, z \rangle \geq 0 \; \forall \; \hat{x} \in P \text{ and } \hat{y} \in F\}.$$

In other words, the dual cone C_F^* is generated by vectors $\hat{x} - \hat{y}$ where $\hat{x}$ and $\hat{y}$ vary among vertices of P and F, respectively.

Proposition 3.2.5 The cones C_F, as F varies over faces of P, form a fan $\triangle_P$. If P contains the origin as an interior point, then $\triangle_P$ consists of cones over faces of the polar polytope P^*.

Proof: If the origin is an interior point it is immediate from the definition that C_F is the cone over the dual face $F^\star$ of P^* and the second assertion follows. To see the first assertion, notice that $\triangle_P$ is unchanged when P is translated by some $m \in M$ or multiplied by a positive integer, so any P spanning $M_\mathbb{R}$ can be changed to contain the origin as an interior point.

Remark 3.2.6 Conversely, by the duality of polytopes, it follows that for a convex rational polytope K in N_R (containing the origin in its interior) the fan of cones over the faces of K is the same as $\triangle_P$ where $P = K^*$.

4 Algebraic Varieties

4.1 Atlases and Varieties

Let X be a topological space. An *affine chart* on X is an open subset $U \subset X$ equipped with a structure of affine variety, with the requirement that the induced topology on U should coincide with the Zariski topology. We say that two charts, U and $\tilde{U}$, are *compatible* if for every subset $V \subset U \cap \tilde{U}$ one has $\mathcal{O}_U(V) = \mathcal{O}_{\tilde{U}}(V)$.

An *atlas* on X is a collection $\mathcal{A} = (U_i)_{i \in I}$ of mutually compatible affine charts covering X. Two atlases are *equivalent* if their union is also an atlas. By a structure of *algebraic variety* on X we mean an equivalence class of atlases. In what follows we shall restrict our attention to the algebraic varieties that have a *finite* atlas. By a chart we mean an affine chart that belongs to some atlas defining the variety structure of X.

There are some trivial varieties:

- Every affine variety V is an algebraic variety (an atlas is given by the affine chart V open in V).
- Every closed subset $Y \subset X$ is an algebraic variety (for any atlas $(U_i)_{i \in I}$ on X, $(U_i \cap Y)_{i \in I}$ is an atlas on Y). V is called a *subvariety* of X.
- An open subset $U \subset X$ (obviously) has the structure of algebraic variety.

Example 4.1.1 As an example of an algebraic variety consider the projective space $\mathbb{P}^n$, which is the set of lines of a $n+1$ dimensional vector space V over K. If $f : V \to K$ is a nonzero linear map, we may define H_f as the set of lines $L \subset \mathrm{kern}(f)$. Then $U_f := \mathbb{P}^n \setminus H_f$ consists of those lines L for which $f\,|_L = K$ and can be identified with the affine subspace $f^{-1}(1) \subset V$. If we choose a basis of V, $\{e_0, \ldots, e_n\}$, we can define $n+1$ linear functions $f_i : V \to K$ by

$$f_i(e_j) = \begin{cases} 0 & \text{if } i = j \\ 1 & \text{otherwise.} \end{cases}$$

The sets $U_i \xrightarrow{\sim} K^n$ consist of all lines L with $x_i \neq 0$ and $A = (U_i)_{i=0,\dots,n}$ is a (Zariski-open) covering of $\mathbb{P}^n$. If we choose homogeneous coordinates

$$x = (x_0, \dots, x_n) \quad x \sim \lambda x \ \forall\ \lambda \in K \setminus \{0\}$$

for $\mathbb{P}^n$, there is a natural isomorphism $U_i \xrightarrow{\sim} K^n$ defined as

$$(x_0, \dots, x_n) \mapsto (x_0/x_i, \dots, x_n/x_i) \quad (i = 0, \dots, n),$$

which induces a structure of affine variety for each U_i with the ring of regular functions $\mathcal{O}_{U_i}(U_i)$ generated by the set $\{T_0/T_i, \dots, T_n/T_i\}$. These structures agree on the intersections $U_i \cap U_j$, since

$$(\mathcal{O}_{U_i}(U_i))_{T_j/T_i} = \mathcal{O}_{U_i}(U_i \cap U_j) = \mathcal{O}_{U_j}(U_i \cap U_j) = (\mathcal{O}_{U_j}(U_j))_{T_i/T_j},$$

where

$$(\mathcal{O}_U(U))_f = \{\frac{g}{f^n} : g \in \mathcal{O}_U(U), n \in \mathbb{N}\}.$$

4.2 Gluing

This operation yields some new varieties out of old ones. Let $(X_i)_{i \in I}$ be a finite covering of some set X, where each X_i has a structure of algebraic variety. We make two assumptions:

- For every pair $i, j \in I$ the set $X_i \cap X_j$ is open in X_i and X_j.
- The algebraic variety structures induced on $X_i \cap X_j$ from X_i and X_j coincide.

Every algebraic variety, for example, arises by gluing affine algebraic varieties.

4.3 Morphisms of Algebraic Varieties

Let X be an algebraic variety described by an atlas $(X_i)_{i \in I}$, and Y an affine variety. We say that a map $f : X \to Y$ is *regular* if the restriction of f to every chart $X_i \subset X$ is a morphism of affine varieties (see 2.5). If $Y = K$ we have the notion of a regular function. For any open set $U \subset X$, we denote by $\mathcal{O}_X(U)$ the K-algebra of functions regular at U.

Suppose now Y is an arbitrary algebraic variety. A continuous mapping $f : X \to Y$ is called a *morphism of algebraic varieties* if for every chart $V \subset Y$, the induced mapping $f^{-1}(V) \to V$ is regular. In other words, for every regular function g on V the function $f^* = g \circ f$ must be regular on $f^{-1}(V)$:

$$\begin{array}{ccc} f^{-1}(V) & \xrightarrow{f} & V \\ & \searrow^{f^*} & \downarrow g \\ & & K \end{array}$$

4.4 Presheaves

Let X be an arbitrary topological space, and $\mathrm{OP}(X)$ the category of open subsets of X. A *presheaf* of sets (respectively, groups, modules or rings) is a contravariant functor F from the category $\mathrm{OP}(X)$ to the category of sets (respectively, groups, modules or rings). In other words, the following must hold:

- For each open subset $U \subset X$ there exists a set $F(U)$.
- For each inclusion of open subsets $V \subset U$ we are given a map $\rho_{U,V} : F(U) \to F(V)$ with $\rho_{U,U}$ being the identity on U.
- Furthermore, if $W \subset V \subset U$ for open subsets U, V and W, $\rho_{U,W} = \rho_{V,W} \circ \rho_{U,V}$.

The elements of $F(U)$ are also referred to as the *sections* of F over U, and the mappings ρ as the *restriction maps*. One also writes $\rho_{U,V}(s) = s \mid_V$.

A presheaf is said to be a *sheaf* if it satisfies the following axiom: Suppose we are given a family $(U_i)_{i \in I}$ of open subsets of X, together with sections $s_i \in F(U_i)$ $\forall i \in I$ which agree on the intersections: $s_i \mid_{U_i \cap U_j} = s_j \mid_{U_i \cap U_j}$ $\forall i, j \in I$. Then there exists a section $s \in F(\bigcup_{i \in I} U_i)$ such that $s_i = s \mid_{U_i}$ $\forall i \in I$ and it is unique with this property.

Example 4.4.1 Consider the sheaf $\mathcal{O}_X$ of regular functions on an algebraic variety (see 4.3), the sheaf of smooth functions on a differentiable manifold, the sheaf of continuous functions on a topological space, etc.

A *morphism of presheaves* $\varphi : F \to G$ is a collection of mappings $\varphi_U : F(U) \to G(U)$, where U runs through $\mathrm{OP}(X)$, which are compatible with the restriction maps. For example, consider the morphism of sheaves which sends the sheaf of smooth functions to the sheaf of continuous functions on a differentiable manifold.

Remark 4.4.2 Of course, not every presheaf is a sheaf. It is, however, possible to attach to every presheaf F what is, in some sense, its closest sheaf F^+: For any open subset $U \subset X$, we denote by $\mathrm{Cov}(U)$ the set of all open coverings of U. For any covering $\mathcal{U} = (U_i)_{i \in I} \in \mathrm{Cov}(U)$, we define $\mathcal{F}(\mathcal{U})$ to be the set of families of sections $(s_i)_{i \in I}$ with $s_i \in F(U_i)$ and $s_i \mid_{U_i \cap U_j} = s_j \mid_{U_i \cap U_j} \ \forall i, j \in I$. Further, if $\mathcal{U}'$ is a refinement of $\mathcal{U}$, there is a canonical map $\mathcal{F}(\mathcal{U}) \to \mathcal{F}(\mathcal{U}')$. Hence we get a direct system $\mathcal{F}(\mathcal{U})$, $\mathcal{U} \in \mathrm{Cov}(U)$. We now define

$$F^+(U) = \varinjlim \mathcal{F}(\mathcal{U})$$

as the direct limit of this system.

4.5 Separatedness

As a motivati$U$-on, $consideratopologicalspace$X$andthediagonal \Delta = \{(x,x) : x \in X\} \subset X \times X. Then \Delta$ is closed in $X \times X$ (for the product topology) if and only if X is Hausdorff: Δ is closed in $X \times X \Leftrightarrow X \setminus \Delta$ is open in $X \times X \Leftrightarrow \ \forall\ (x,y) \in X \times X,\ x \neq y\ \exists\ U \times U'$ open in $X \times X$ with $U \cap U' = \emptyset \Leftrightarrow X$ is Hausdorff.

Definition 4.5.1 [Sha94c, Mil] Let X be an algebraic variety. Then X is *separated* if the diagonal Δ is closed in $X \times X$.

That a variety is non-separated has to do with the fact that, when we obtain it by gluing some affine pieces, these are glued imperfectly. To be precise, one has the following separatedness criterion: A variety X described by an atlas (X_i) is separated if and only if the image of all $X_i \cap X_j$ under the canonical injection into $X_i \times X_j$ is closed. In fact, the image of $X_i \cap X_j$ in $X_i \times X_j$ is just the intersection of $X_i \times X_j$ with the diagonal Δ_X in $X \times X$.

Example 4.5.2 Let us apply this criterion to the standard atlas (U_i), $i = 0, \ldots, n$ of projective space $\mathbb{P}^n$ (see example 4.1.1). It is easy to check that the image of $U_i \cap U_j$ in $U_i \times U_j$ is given by the equations

$$T_k/T_i = T_j/T_i T_k/T_j, \quad T_k/T_j = T_k/T_i T_i/T_j, \quad k = 0, \ldots, n,$$

whence we see that $\mathbb{P}^n$ is separated.

4.6 Algebraic Toric Varieties

Definition 4.6.1 A finite collection of rational strongly convex polyhedral cones

$$\Sigma = \{C_1, \ldots, C_n\}, \ C_i \subset N_{\mathbb{R}}$$

is called a *fan* if:

- Each face of a cone in Σ is also a cone in Σ.
- Σ is closed under intersection, i.e. $C_i \in \Sigma \wedge C_j \in \Sigma \Rightarrow C_i \cap C_j \in \Sigma \ \ \forall\, i, j = 1, \ldots, n.$

Let Σ be a fan. For each cone $C \in \Sigma$, $S_C = C^* \cap M$ is by proposition 3.1.14 a finitely generated semigroup and leads to an affine toric variety (see 2.9):

$$X_C = \operatorname{Specm} K[S_C].$$

We define an *algebraic toric variety* by taking the disjoint union of all affine toric varieties X_C with $C \in \Sigma$ and glue them together by the identifications:

$$X_C \supset X_{C \cap \tilde{C}} \xrightarrow{\sim} X_{\tilde{C} \cap C} \subset X_{\tilde{C}} \ \ \forall\, C, \tilde{C} \in S.$$

The fact that $X_{C \cap \tilde{C}}$ is an open subset of X_C (and $X_{\tilde{C}}$) follows from proposition 3.1.17:

$$\mathcal{O}_{X_C}(X_{C \cap \tilde{C}}) = \mathcal{O}_{X_C}(X_C)_{\chi(m)} \text{ for some } m \in C.$$

The fact that the algebraic structures of X_C and $X_{\tilde{C}}$ coincide on the intersection $X_{C \cap \tilde{C}} = X_C \cap X_{\tilde{C}}$ follows from 3.1.19, which was a consequence of the Seperation Lemma 3.1.18:

$$\mathcal{O}_{X_C}(X_{C \cap \tilde{C}}) = \mathcal{O}_{X_C}(X_C) \cdot \mathcal{O}_{X_{\tilde{C}}}(X_{\tilde{C}}) = \mathcal{O}_{X_{\tilde{C}}}(X_{C \cap \tilde{C}}).$$

Moreover, an algebraic toric variety is separated:

Proposition 4.6.2 If C and $\tilde{C}$ are cones that intersect in a common face, then the diagonal map $U_{C\cap\tilde{C}} \to U_C \times U_{\tilde{C}}$ is a closed embedding.

Proof: This is equivalent to the assertion that the natural mapping

$$\mathcal{O}_{X_C}(X_C) \times \mathcal{O}_{X_{\tilde{C}}}(X_{\tilde{C}}) \to \mathcal{O}_{X_{C\cap\tilde{C}}}(X_{C\cap\tilde{C}})$$

is surjective, which is again a consequence of 3.1.19.

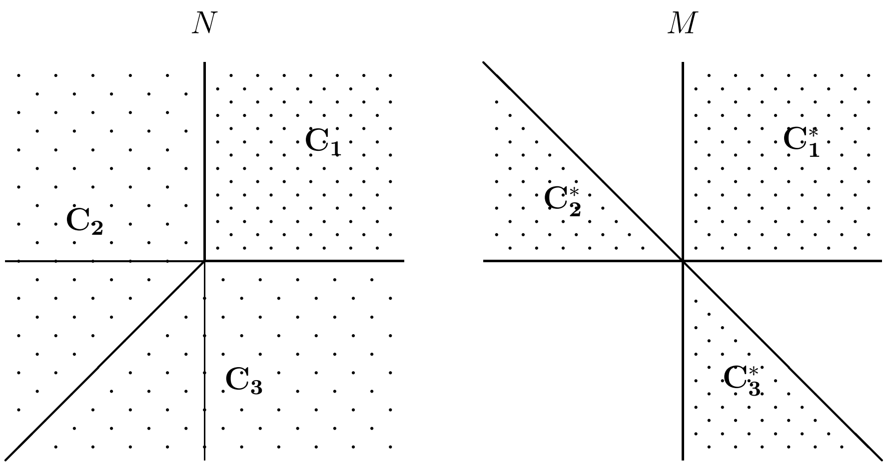
Whether an algebraic toric variety is compact (in the classical topology) or not also depends on the fan:

Proposition 4.6.3 [Ful93] Let X_Σ be an algebraic toric variety corresponding to a fan Σ. Then X_Σ is compact (in the classical topology) if and only if the support $\mid \Sigma \mid = \cup_{C\in\Sigma} C$ is the whole space $N_{\mathbb{R}}$.

Example 4.6.4 Let n_1, n_2 be a $\mathbb{Z}$-basis of N and m_1, m_2 be the dual basis of M. Define three cones

$$C_1 =< n_1, n_2 >, \quad C_2 =< n_2, -n_1 - n_2 > \quad \text{and} \quad C_3 =< n_1, -n_1, -n_2 > .$$

We obtain a fan Σ consisting of C_1, C_2, C_3 as well as their faces $\mathbb{R}_{\geq} n_1$, $\mathbb{R}_{\geq} n_2$, $\mathbb{R}_{\geq}(-n_1 - n_2)$ and $\{0\}$:



The resulting toric variety consists of three affine toric varieties

$$U_{C_1} = \mathrm{Specm}[X_1, X_2], \qquad U_{C_2} = \mathrm{Specm}\left[\frac{1}{X_1}, \frac{X_2}{X_1}\right] \quad \text{and}$$

$$U_{C_3} = \mathrm{Specm}\left[\frac{1}{X_2}, \frac{X_1}{X_2}\right]$$

which are glued together along

$$U_{C_1 \cap C_2} = \mathrm{Specm}\left[X_1, X_2, \frac{1}{X_1}\right], \quad U_{C_2 \cap C_3} = \mathrm{Specm}\left[\frac{1}{X_1}, \frac{X_2}{X_1}, \frac{X_1}{X_2}\right]$$

and

$$U_{C_3 \cap C_1} = \mathrm{Specm}\left[\frac{1}{X_2}, \frac{X_1}{X_2}, X_2\right].$$

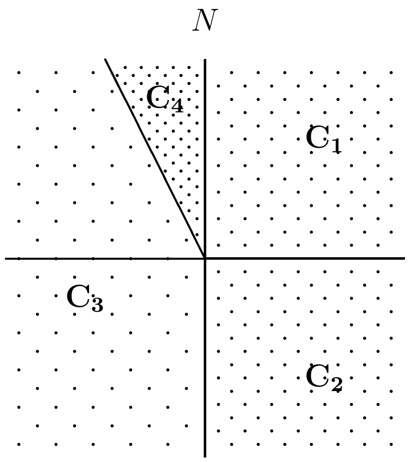
We get the projective space $\mathbb{P}^2(\mathbb{C})$. In terms of its homogeneous coordinate $[z_0, z_1, z_2]$ we have $z_1/z_0 = X_1$ and $z_2/z_0 = X_2$. Moreover, U_{C_1}, U_{C_2} and U_{C_3} are the affine planes $\{z_0 \neq 0\}, \{z_1 \neq 0\}$ and $\{z_2 \neq 0\}$, respectively.

Example 4.6.5 Let n_1, n_2 be a $\mathbb{Z}$-basis of N and m_1, m_2 be the dual basis of M. For a more interesting example, consider a fan generated by the cones

$$C_1 =< n_1, n_2 >, \quad C_2 =< n_1, -n_2 >, \quad C_3 =< -n_1 + an_2, -n_2 >$$

$$\text{and} \quad C_4 =< n_2, -n_1 + an_2 >,$$

for some positive integer a, and their faces:



The dual cones are given by

$$C_1^* =< m_1, m_2 >, \quad C_2^* =< m_1, -m_2 >, \quad C_3^* =< -m_1, -am_1 - m_2 >$$

$$\text{and} \quad C_4^* =< -m_1, am_1 + m_2 >,$$

and we have the patching

$$\begin{array}{ccc} U_{C_4} = \mathrm{Specm}(\frac{1}{X_1}, {X_1}^a X_2) & \longleftrightarrow & U_{C_1} = \mathrm{Specm}(X_1, X_2) \\ \updownarrow & & \updownarrow \\ U_{C_3} = \mathrm{Specm}(\frac{1}{X_1}, \frac{1}{{X_1}^a X_2}) & \longleftrightarrow & U_{C_2} = \mathrm{Specm}(X_1, \frac{1}{X_2}) \end{array}$$

The vertical arrows patch the varieties U_{C_1} and U_{C_2} (U_{C_3} and U_{C_4}) together to $\mathbb{C} \times \mathbb{P}^1$, and via the horizontal patching all together is a $\mathbb{P}^1$ bundle over $\mathbb{P}^1$. These rational ruled surfaces are called *Hirzebruch Surfaces*.

5 Local Study of Toric Varieties

5.1 The Tangent Space

Let $X \subset K^n$ be an affine variety defined by an ideal $I = I(X) \subset K[T_1, \ldots, T_n]$ (see 2.5). At every point $x \in X$ consider the surjective K-linear homomorphism

$$d_x : K[T_1, \ldots, T_n] \to T_x(K^n)^\wedge$$
$$f \mapsto df_x = \sum_{i=1,\ldots,n} \frac{\partial f}{\partial T_i} \mid_x dT_i \ ,$$

sending a regular function $f \in K[T_1, \ldots, T_n]$ to a linear form df_x in the cotangent-space $T_x(K^n)^\wedge$ of K^n at the point x. The *tangent space* $T_x(X)$ of X at the point x is defined as the subspace

$$T_x(X) = \{\xi \in T_x(K^n) : df_x(\xi) = 0 \ \forall f \in I\}.$$

Remark 5.1.1 Let $F = \{f_1, \ldots, f_n\} \subset I$ be a set of generators for I. Then

$$T_x(X) = \{\xi \in T_x(K^n) : df_x(\xi) = 0 \ \forall f \in F\},$$

since for all $g = \sum h_i f_i \in I$ we have

$$(dg)_x = d(\sum_{i=1}^{n} h_i f_i)_x = \sum_{i=1}^{n} (dh_i)_x f_i(x) + \sum_{i=1}^{n} (df_i)_x h_i(x) = \sum_{i=1}^{n} (df_i)_x h_i(x).$$

This definition of the tangent space uses the embedding of X into K^n. It is easy to see how we can get rid of that: Let $I_x \subset K[T_1, \ldots, T_n]/I$ be the maximal ideal corresponding to the point $x \in X$. Since $d_x k = 0 \ \forall \ k \in K = K[T_1, \ldots, T_n]/I_x$, we can replace the study of this map by that of

$$d_x : I_x \to T_x(X)^\wedge.$$

The kernel of this mapping is just ${I_x}^2 = \{f \cdot g : \ f \in I_x \ \wedge \ g \in I_x\}$, so we have the following invariant definition of the tangent space:

Definition 5.1.2 Let $x \in X$ be a point in an affine variety. Then the *tangent space* $T_x(X)$ of X at the point x is defined as the space dual to $I_x/{I_x}^2$.

5.2 Singularities of Toric Varieties

In what follows, let $K = \mathbb{C}$ be the field of complex numbers. For any cone $C \subset N_{\mathbb{R}}$, the corresponding affine toric variety $U_C = \mathrm{Specm}[C^* \cap M]$ has a distinguished point which we denote by $x_C \in U_C$. This point is defined by a homomorphism of semigroups:

$$x_C : C^* \cap M \to \{0, 1\} \subset \mathbb{C}$$

$$m \mapsto \begin{cases} 1 & \text{if } m \in C^\perp \\ 0 & \text{otherwise.} \end{cases}$$

Remark 5.2.1 [Ful93, Oda88] If $\dim C = \dim N_{\mathbb{R}}$, x_C is the unique fixed point under the action of the torus (see 2.9.4). If $\dim C \neq \dim N_{\mathbb{R}}$, there is no fixed point under the action of the torus.

Definition 5.2.2 If $\dim C = \dim N_{\mathbb{R}}$, we define the affine variety to be *nonsingular at the point* $x \in U_C$ if the dimension of the cotangent space ${I_x}/{I_x}^2$ is n-dimensional, since $\dim T = \dim U_C = n$.

Suppose $\dim C = \dim N_{\mathbb{R}}$. Then $C^\perp = \{0\} \subset M_{\mathbb{R}}$. The maximal ideal I_{x_C} corresponding to the distinguished point $x_C \in U_C$ is generated by the set

$$\{\chi(m) : m \in M \ \wedge m \neq 0\},$$

and ${I_{x_C}}^2$ by the set

$$\{\chi(m) : m \in M \ \wedge m \neq 0 \ \wedge \ m \neq m_1 + m_2 \text{ with } m_1 \neq 0 \ \wedge \ m_2 \neq 0\}.$$

By 5.1.2, the cotangent space of U_C at $x_C \in U_C$ is defined as $I_{x_C}/{I_{x_C}}^2$, so if we denote by $\{\varrho_1, \ldots, \varrho_k\}$ the first elements in M lying on the edges of C^*, the set

$$\{\chi(\varrho_1), \ldots, \chi(\varrho_k)\}$$

is a basis of the cotangent space $I_{x_C}/{I_{x_C}}^2$ at the distinguished point $x_C \in U_C$. The condition for non-singularity 5.2.2 at the distinguished point implies that $k \leq n$, and since $\dim C^* = \dim M_{\mathbb{R}}$ it follows that $k = n$. Since S_C generates M as a group, the set $\{\varrho_1, \ldots, \varrho_k\}$ must be a basis of M and the toric variety U_C is isomorphic to $\mathbb{C}^n$.

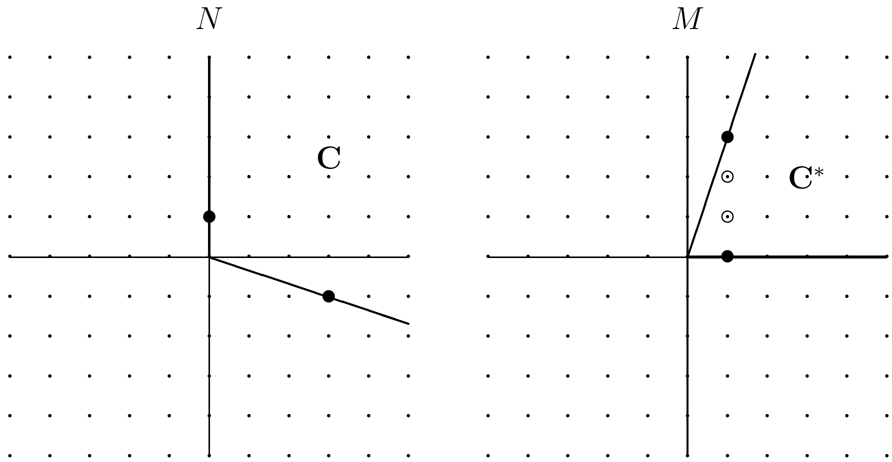
If $m = \dim C < \dim N_{\mathbb{R}} = n$, let $\tilde{N}_{\mathbb{R}} \subset N_{\mathbb{R}}$ be the subspace generated by C and denote by $\tilde{C}$ the same cone in the space $\tilde{N}_{\mathbb{R}}$. Then $U_{\tilde{C}}$ is nonsingular if the set of generators of $\tilde{C}$ are a basis of $\tilde{N}$, and since $U_{\mathbb{C}} \cong U_{\tilde{C}} \times \mathbb{C}^{n-m}$, we have proved:

Proposition 5.2.3 An affine toric variety U_C is nonsingular if and only if C is generated by a subset of a basis of N, in which case

$$U_{\mathbb{C}} \cong U_{\tilde{C}} \times \mathbb{C}^{n-m} \qquad m = \dim C, \ n = \dim N_{\mathbb{R}}.$$

5.3 Quotient Singularities

To motivate the following, consider the cone $C \subset N_{\mathbb{R}} = \mathbb{R}^2$ generated by $kn_1 - n_2$ ($k \in \mathbb{N}$) and n_2 and its dual $C^* \subset M_{\mathbb{R}}$ generated by $m_1 + km_2$ and m_1:



The resulting affine toric variety is $U_C = \text{Specm}(A_C)$ with

$$A_C = \mathbb{C}[X, XY, XY^2, \ldots, XY^k] = \mathbb{C}[U^k, U^{k-1}V, \ldots, UV^{k-1}, V^k] \subset \mathbb{C}[U, V],$$

where we have set $X = U^k$ and $Y = V/U$. The inclusion of $A_C \subset \mathbb{C}[U, V]$ corresponds to a mapping $\mathbb{C}^2 \to U_C$. The group

$$G = \{e^{\frac{2\pi i l}{k}} : l = 0, \ldots, k-1\} \cong \mathbb{Z}/k\mathbb{Z}$$

acts on $\mathbb{C}^2$ by $\xi(u,v) \mapsto (\xi u, \xi v)$ and $U_C \cong \mathbb{C}^2/G$, i.e. U_C is a *cyclic quotient singularity*. Algebraically, G acts on the coordinate ring $\mathbb{C}[U,V]$ by $f(U,V) \mapsto f(\xi U, \xi V)$ and via this action

$$A_C = \mathbb{C}[U,V]^G$$

is the ring of invariants.

Now consider the same cone denoted by $\tilde{C}$ but in the lattice $\tilde{N} \subset N$, generated by $kn_1 - n_2$ and n_2, with its dual lattice $\tilde{M} \supset M$, generated by m_1/k and $m_1/k + m_2$. Then $A_{\tilde{C}} = \mathbb{C}[U, UY] = \mathbb{C}[U,V]$ and we recover the previous description with

$$G \cong N/\tilde{N} \cong \tilde{M}/M.$$

A similar procedure applies to an arbitrary singular two-dimensional affine toric variety (see [Ful93, Oda88]). More generally consider a simplicial cone (see definition 3.1.22) $C \subset N_{\mathbb{R}}$ with arbitrary dimension of $N_{\mathbb{R}}$ and $\dim C = \dim N_{\mathbb{R}}$. Denote by $\tilde{N} \subset N$ the sublattice generated by the generators of C with its dual lattice $\tilde{M} \supset M$. There is a canonical duality pairing

$$\tilde{M}/M \times N/\tilde{N} \to \mathbb{Q}/\mathbb{Z} \hookrightarrow \mathbb{C}^*,$$

where the first map is the pairing $\langle *, * \rangle$, and the second is defined as $q \mapsto e^{2\pi i q}$. Now $G \cong N/\tilde{N}$ acts on $\mathbb{C}[\tilde{M}]$ by

$$\xi\chi(\tilde{m}) = e^{2\pi i \langle \tilde{m}, \xi \rangle} \chi(\tilde{m}),$$

and by this natural action,

$$\mathbb{C}[\tilde{M}]^G = \mathbb{C}[M].$$

Hence $G \cong N/\tilde{N}$ acts on the torus $T_{\tilde{N}}$ and $T_{\tilde{N}}/G = T_N$.

To prove this, take a basis $\{n_1, \ldots, n_m\}$ of N such that $\{k_1 n_1, \ldots, k_m n_m\}$ are generators for $\tilde{N}$ for some $k_i \in \mathbb{N}$. Then $\mathbb{C}[\tilde{M}]$ is the Laurant polynomial ring in generators X_i, and $\mathbb{C}[M]$ is the Laurant polynomial ring in generators U_i with $X_i = U_i^{k_i}$ $\forall\ i = 1, \ldots, m$. Now an element $(\xi_1, \ldots, \xi_m) \in N/\tilde{N}$ acts on monomials by

$$U_1^{l_1} \ldots U_m^{l_m} \mapsto e^{2\pi i \sum \frac{\xi_i l_i}{k_i}}\, U_1^{l_1} \ldots U_m^{l_m}.$$

To get the action of G on the toric variety U_C, we just have to intersect the ring $A_{\tilde{C}}$ with $\mathbb{C}[\tilde{M}]^G = \mathbb{C}[M]$. Thus we have proved:

Proposition 5.3.1 If $C \subset N_{\mathbb{R}}$ is an simplicial cone the toric variety U_C has only quotient singularities and

$$U_{\mathbb{C}} \cong U_{\tilde{C}}/G \times \mathbb{C}^{n-m}, \quad m = \dim C, \ n = \dim N_{\mathbb{R}}$$

with $G \cong N/\tilde{N}$.

6 Orbits, Divisors and Line Bundles

To motivate the following, consider the affine toric variety $\mathbb{C}^n$. The sets which are invariant under the action of the torus are

$$\{(z_1, \ldots, z_n) : \ z_i = 0 \ \forall \ i \in I \ \wedge z_i \neq 0 \ \forall \ i \notin I \quad i = 1, \ldots, n\},$$

as I ranges over subsets of $\{1, \ldots, n\}$. Note that each of these sets is equal to

$$O_F = \mathrm{Hom}(F^{\perp} \cap M, \mathbb{C}^*),$$

where $F \subset C$ is the face generated by the subset $\{e_i : i \in I\}$ of the standard basis of N. We will see that the sets O_F are open subvarieties of their closure, which is denoted by V_F.

In the general case, let $\Sigma \subset N_{\mathbb{R}}$ be a fan and denote by ${N^{\mathcal{C}}}_{\mathbb{R}} \subset N_{\mathbb{R}}$ the subspace generated by a cone $\mathcal{C} \in \Sigma$. In the quotient-space (with dual space $M(\mathcal{C})$),

$$N(\mathcal{C}) = N_{\mathbb{R}}/{N^{\mathcal{C}}}_{\mathbb{R}} \quad \text{and} \quad M(\mathcal{C}) = \mathcal{C}^{\perp} \cap M,$$

the cones $C \in \Sigma$ which contain $\mathcal{C}$ as a face are

$$\bar{C} = (C + {N^{\mathcal{C}}}_{\mathbb{R}})/{N^{\mathcal{C}}}_{\mathbb{R}} \ \subset N(\mathcal{C}).$$

They form a fan $\mathrm{Star}(\mathcal{C}) \subset N(\mathcal{C})$ in the quotient space:

$$\mathrm{Star}(\mathcal{C}) = \{\bar{C} : C \geq \mathcal{C}\}.$$

Now set

$$V(\mathcal{C}) = X[\mathrm{Star}(\mathcal{C})],$$

the corresponding $(n-k)$-dimensional ($k = \dim N^{\mathcal{C}}_{\mathbb{R}}$) toric variety with the torus embedding

$$T_{N^{\mathcal{C}}} = \mathrm{Hom}(M(\mathcal{C}), \mathbb{C}^*) = O_{\mathcal{C}}.$$

An affine open covering of $V(\mathcal{C})$ is given by

$$(U_C(\mathcal{C}))_{C \geq \mathcal{C}},$$

with

$$U_C(\mathcal{C}) = \mathrm{Specm}(\mathbb{C}[\bar{C}^* \cap M(\mathcal{C})]) = \mathrm{Specm}(\mathbb{C}[C^* \cap \mathcal{C}^{\perp} \cap M]) \ \forall \ C \geq \mathcal{C}.$$

To embed $V(\mathcal{C})$ as a closed subvariety, we first construct a closed embedding of $U_C(\mathcal{C})$ in U_C for all $C \geq \mathcal{C}$:

$$U_C(\mathcal{C}) = \mathrm{Hom}_{sg}(C^* \cap M(\mathcal{C}), \mathbb{C}) \hookrightarrow \mathrm{Hom}_{sg}(C^* \cap M, \mathbb{C})) = U_C \quad C \geq \mathcal{C},$$

regarding the points as semigroup-homomorphisms, defined as extension by zero. Since $\mathcal{C}$ is a face of C for all $C \geq \mathcal{C}$, this mapping is well defined. The corresponding surjection of rings

$$\mathbb{C}[C^* \cap M] \to \mathbb{C}[C^* \cap M(\mathcal{C})]$$

is the (obvious) projection: It takes $\chi(m) = \chi(m_{\mathcal{C}} + m_{\mathcal{C}}^{\perp})$ to $\chi(m_{\mathcal{C}})$ $\forall m = m_{\mathcal{C}} + m_{\mathcal{C}}^{\perp} \in M$ with $m_{\mathcal{C}} \in M(\mathcal{C})$ and $m_{\mathcal{C}}^{\perp} \in M(\mathcal{C})^{\perp}$.

If $\mathcal{C} \leq C \leq \tilde{C}$ we get the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Hom}_{sg}(C^* \cap M(\mathcal{C})) & \hookrightarrow & \mathrm{Hom}_{sg}(\tilde{C}^* \cap M(\mathcal{C})) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{sg}(C^* \cap M) & \hookrightarrow & \mathrm{Hom}_{sg}(\tilde{C}^* \cap M) \end{array}$$

where the horizontal maps are restrictions and the vertical maps are extension by zero. Therefore, these maps glue together to give a closed embedding

$$V(\mathcal{C}) \hookrightarrow X(\Sigma).$$

If $\mathcal{C} \leq \tilde{\mathcal{C}}$ is a face of˜
C, $wehaveclosedembeddings V(\tilde{\mathcal{C}}) \hookrightarrow V(\mathcal{C}) definedontheopensets U_C(\tilde{\mathcal{C}}) = \mathrm{Hom}_{sg}(C^* \cap M(\tilde{\mathcal{C}}), \mathbb{C}) \hookrightarrow U_C(\mathcal{C}) = \mathrm{Hom}_{sg}(C^* \cap M(\mathcal{C}), \mathbb{C}) forall$ C$\in$ Star($\mathcal{C}$) with $\tilde{\mathcal{C}} \leq C$.
In summary, we have an order-reversing correspondence between cones $\mathcal{C} \in \Sigma$ and orbit closures $V(\mathcal{C})$.

Remark 6.0.2 [Ful93] There are the following relations among orbits $O_{\mathcal{C}}$, orbit closures $V_{\mathcal{C}}$ and the affine open sets U_C:

1. $$U_C = \bigsqcup_{\mathcal{C} \leq C} O_{\mathcal{C}}.$$

2. $$V(\mathcal{C}) = \bigsqcup_{\tilde{\mathcal{C}} \geq \mathcal{C}} O_{\tilde{\mathcal{C}}}.$$

3. $$O_{\mathcal{C}} = V(\mathcal{C}) \setminus \bigcup_{\tilde{\mathcal{C}} \gneq \mathcal{C}} V(\tilde{\mathcal{C}}).$$

6.1 Support Functions

Definition 6.1.1 A real valued function $\varphi : |\Sigma| \to \mathbb{R}$ on the support $|\Sigma| := \cup_{C \in \Sigma}$ is called an *integral* Σ*-piecewise linear support function* if it is $\mathbb{Z}$-valued on $N \cap |\Sigma|$ and linear on each cone $C \in \Sigma$. Namely, there exists a $m_C \in M$ for each $C \in \Sigma$, such that $\varphi(z) = m_C(z) \; \forall \; z \in C$ and $m_C(z) = m_{\tilde{C}}(z) \; \forall \; z \in C \cap \tilde{C}$ and $C, \tilde{C} \in \Sigma$. We denote by $\mathrm{SF}(N, \Sigma)$ the additive group of integral convex Σ-piecewise linear support functions.

Note that the set $\{m_C : C \in \Sigma\} \subset M$ may not be uniquely determined, since $\{\tilde{m}_C : C \in \Sigma\}$ gives rise to the same $\varphi \in \mathrm{SF}(N, \Sigma)$ if $m_C - \tilde{m}_C \in M \cap C^{\perp} \; \forall \; C \in \Sigma$. Let us denote by

$$\Sigma(1) := \{\varrho \in \Sigma : \dim \varrho = 1\}$$

the set of one-dimensional cones of Σ. Then we get an injective homomorphism

$$\mathrm{SF}(N, \Sigma) \hookrightarrow \mathbb{Z}^{\Sigma(1)},$$

which sends φ to $(\varphi(n(\varrho)))$, where $n(\varrho) \in N \cap \varrho$ denotes the generator of ϱ. If the toric variety is nonsingular, we get an isomorphism:

$$\mathrm{SF}(N, \Sigma) \xrightarrow{\sim} \mathbb{Z}^{\Sigma(1)}.$$

6.2 Equivariant Line Bundles

Definition 6.2.1 An *equivariant line bundle* on a toric variety X is a fiber bundle $\pi : L \to X$ with fiber $\mathbb{C}$ and with an algebraic action of the torus T_N on L, such that $\pi(tz) = t\pi(z) \; \forall \; z \in L, \; t \in T_N$ and the action of each $t \in T_N$ on L induces a linear map from $\pi^{-1}(x)$ to $\pi^{-1}(tx)$ for each $x \in X$.

The set $\mathrm{ELB}(X)$ of isomorphism classes of equivariant line bundles is a commutative group with respect to the tensor product. The identity element $\mathbf{1_0}$ is the trivial bundle $X \times \mathbb{C}$ with the action $t(x, c) := (tx, c) \; \forall \; t \in T_N$. Each $m \in M$ gives rise to an equivariant line bundle $\mathbf{1_m}$, which is the trivial bundle $X \times \mathbb{C}$ with the action $t(x, c) := (tx, \chi(m)(t)c) \; \forall \; t \in T_N$, where $\chi(m)$ is the character corresponding to $m \in M$. The map which sends m to $\mathbf{1_m}$ is a homomorphism

$$M \to \mathrm{ELB}(X).$$

The *Picard group* $\mathrm{Pic}(X)$ is the group of isomorphism classes of line bundles on X. By disregarding the action of T_N, we get an homomorphism

$$\mathrm{ELB}(X) \to \mathrm{Pic}(X)$$

whose kernel obviously contains $\{\mathbf{1_m} : m \in M\}$.

6.3 Divisors

A *Weil divisor* is a formal finite $\mathbb{Z}$-linear sum of closed irreducible subspaces of X with codimension one. We denote by $\mathrm{Div}(X)$ the group of Weil divisors, $\mathrm{Div}_{\mathrm{T}}(X)$ being the subgroup of T_N-invariant Weil divisor:

$$\mathrm{Div}_{\mathrm{T}}(X) = \bigoplus_{\varrho \in \Sigma(1)} \mathbb{Z}\, V(\varrho),$$

where $V(\varrho)$ are the codimension one subvarieties corresponding to the rays $\varrho \in \Sigma(1)$. A divisor $D = \sum_{\varrho \in \Sigma(1)} a_i V(\varrho)$ is called *effective* and denoted by $D_{\geq}$ if all $a_i \in \mathbb{Z}$ are nonnegative. The subgroup of *principal divisors* $\mathrm{PDiv}(X)$ are those of the form

$$\mathrm{div}(f) = \sum_V \nu_V(f)\; V$$

for a nonzero rational function f on X, where $\nu_V(f)$ denotes the order of zero along each closed irreducible subspace $V \subset X$ of codimension one.

Of particular interest is the subgroup $\mathrm{CDiv}(X)$ of *Cartier divisors*, i.e. locally principal (Weil) divisors. We denote by $\mathrm{CDiv}_{\mathrm{T}}(X)$ the subgroup of T_N-invariant Cartier divisors. We can associate a line bundle to every Cartier divisors D: By definition there exists an open covering $X = \cup_i U_i$ and nonzero rational functions f_i, such that D coincides with the principal divisor $\mathrm{div}(f_i^{-1})$ on each open set U_i. Thus both, f_i/f_j and f_j/f_i, are regular functions on $U_i \cap U_j$. We obtain a line bundle $L = \cup_i (U_i \times \mathbb{C})$ by gluing $U_i \times \mathbb{C}$ and $U_j \times \mathbb{C}$ along $(U_i \cap U_j) \times \mathbb{C}$ via the map

$$U_i \times \mathbb{C} \supset (U_i \cap U_j) \times \mathbb{C} \xrightarrow{\sim} (U_i \cap U_j) \times \mathbb{C} \subset U_j \times \mathbb{C}$$

which sends (x, c) to $(x, f_i/f_j(c))$. The projections to the first factors are glued together to $\pi : L \to X$.

Now let $\varphi \in \mathrm{SF}(N,\Sigma)$. We can construct an equivariant line bundle $L_\varphi \in \mathrm{ELB}(X)$ as follows: Let $\{m_C : C \in \Sigma\} \subset M$ be as in the definition of $\varphi \in \mathrm{SF}(N,\Sigma)$. Then for two cones $C, \widetilde{C} \in \Sigma$, m_C and $m_{\tilde{C}}$ have the same values on $C \cap \tilde{C}$, so $m_C - m_{\tilde{C}} \in M \cap (C \cap \widetilde{C})^\perp$, and both $\chi(m_C - m_{\tilde{C}})$ and $\chi(m_{\tilde{C}} - m_C)$, are regular functions on the open set $U_C \cap U_{\tilde{C}} = U_{C \cap \tilde{C}}$. Define the line bundle $L_\varphi = \cup_{C \in \Sigma} U_C \times \mathbb{C}$ over X by gluing $U_C \times \mathbb{C}$ and $U_{\tilde{C}} \times \mathbb{C}$ along $U_{C \cap \tilde{C}} \times \mathbb{C}$ by the isomorphism

$$g_{C\tilde{C}} : U_C \times \mathbb{C} \supset U_{C \cap \tilde{C}} \times \mathbb{C} \xrightarrow{\sim} U_{C \cap \tilde{C}} \times \mathbb{C} \subset U_{\tilde{C}} \times \mathbb{C}$$

defined by $g_{C\tilde{C}}(x,c) = (x, \chi(m_C - m_{\tilde{C}})(x)c)$ for $(x,c) \in U_{C \cap \tilde{C}} \times \mathbb{C}$. We can define the action of the torus T_N on L_φ by

$$t(x,c) = (tx, \chi(-m_C)(t)c) \ \forall \ t \in T_N, \ (x,c) \in U_C \times \mathbb{C}$$

for all cones $C \in \Sigma$, which is obviously compatible with the gluing maps above. If $\{m_C : C \in \Sigma\}$ and $\{\tilde{m}_C : C \in \Sigma\}$ give rise to the same φ, we get isomorphisms $g_C : U_C \times \mathbb{C} \xrightarrow{\sim} U_C \times \mathbb{C}$ for all cones $C \in \Sigma$ defined by

$$g_C(x,c) = (x, \chi(\tilde{m}_C - m_C)(x)c) \ \forall \ (x,c) \in U_C \times \mathbb{C},$$

which are glued together to a bundle isomorphism $g : L_{\tilde{\varphi}} \xrightarrow{\sim} L_\varphi$ which commutes with the torus-action.

Since every $\varphi \in \mathrm{SF}(N,\Sigma)$ determines a T_N invariant Cartier divisor $D_\varphi = -\sum_{\varrho \in \Sigma(1)} \varphi(n(\varrho))V(\varrho)$, which can also be regarded as a line bundle, we have a commutative diagram of commutative groups:

$$\begin{array}{ccc} \mathrm{coker}[M \to \mathrm{SF}(N,\Sigma)] & & \\ \downarrow & \searrow & \\ \mathrm{CDiv_T}(X)/(\mathrm{Div_T}(X) \cap \mathrm{PDiv}(X)) & \longrightarrow & \mathrm{Pic}(X) \end{array}$$

Proposition 6.3.1 [Oda88] Let Σ be a complete fan, i.e. the corresponding toric variety is compact. Then there exists a canonical isomorphism

$$\begin{aligned} \mathrm{SF}(X,\Sigma)/M &\xrightarrow{\sim} \mathrm{CDiv}_T(X)/(\mathrm{CDiv}_T(X) \cap \mathrm{PDiv}(X)) \\ &\xrightarrow{\sim} \mathrm{ELB}(X)/\{\mathbf{1_m} : m \in M\} \xrightarrow{\sim} \mathrm{Pic}(X). \end{aligned}$$

Let $L_\varphi \in \mathrm{ELB}(X)$ be an equivariant line bundle coming from a Σ-linear support function $\varphi \in \mathrm{SF}(X, \Sigma)$. Suppose $m \in M$ satisfies $\langle m, z\rangle \geq \varphi(z)\ \forall\ z \in |\Sigma|$. Then $m - m_C \geq 0\ |_C\ \ \forall\ C \in \Sigma$, and $\chi(m - m_C)$ is a regular function on U_C. The maps $s_C : U_C \to U_C \times \mathbb{C}$ are naturally glued together to a section $s : X \to L_\varphi$ which satisfies $s(tx) = \chi(m)(t)(ts(x))\ \forall\ t \in T_N,\ x \in X$.

Proposition 6.3.2 Let Σ be a complete fan. Then for every $\varphi \in \mathrm{SF}(X, \Sigma)$

$$\Delta_\varphi = \{m \in M_\mathbb{R} : \langle m, z\rangle \geq \varphi(z)\ \forall\ z \in N_\mathbb{R}\}$$

is a convex polytope in $M_\mathbb{R}$. The set of global sections of L_φ is a finite dimensional $\mathbb{C}$-vector space with basis $\{\chi(m) : m \in M \cap \Delta_\varphi\}$.

Remark 6.3.3 Let $m \in M$. Then the following is equivalent:

$$\langle m, z\rangle \geq \varphi(z)\ \forall z \in |\Sigma| \Leftrightarrow -m + \varphi \text{ has nonpositive values on } |\Sigma|$$

$$\Leftrightarrow D_{-m+\varphi} \geq 0 \Leftrightarrow \mathrm{div}(\chi(m)) + D_\varphi \geq 0.$$

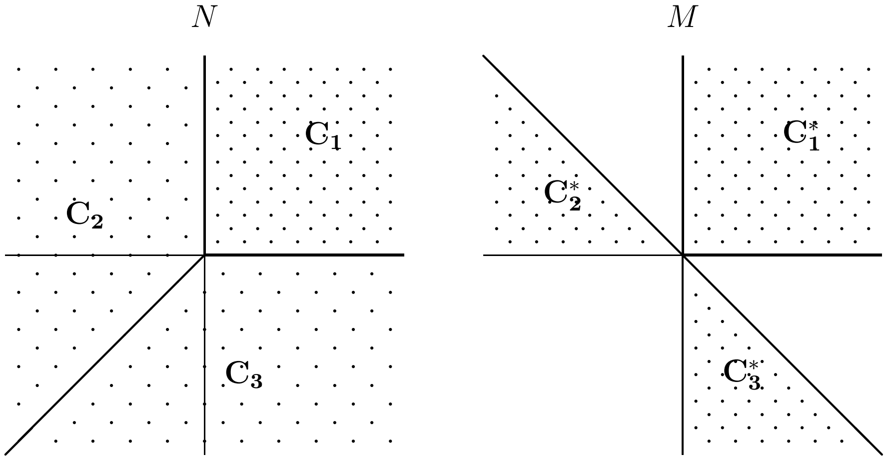
Definition 6.3.4 Let Σ be a complete fan, $\varphi \in \mathrm{SF}(N, \Sigma)$ and $\{m_C : C \in \Sigma\}$ as in the definition of φ. Then φ is said to be *(lower) convex* if

$$\varphi(z) \leq m_C(z)\ \forall\ C \in \Sigma,\ z \in N_\mathbb{R}.$$

φ is called *strictly (lower) convex* if the graph of φ lies strictly under the graph of m_C on the complement of C for all cones with $\dim(C) = \dim(N_\mathbb{R})$.

Remark 6.3.5 The convexity of φ is just the condition that $\{m_C : C \in \Sigma \wedge \dim C = \dim M_\mathbb{R}\} \subset \Delta_\varphi$, where Δ_φ is from proposition 6.3.2. Moreover, the set $\{m_C : C \in \Sigma\ \wedge\ \dim C = \dim M_\mathbb{R}\}$ are the vertices of Δ_φ.

Example 6.3.6 Let n_1, n_2 be a $\mathbb{Z}$-basis of N and m_1, m_2 the dual basis of M. Consider the fan for the projective space $\mathbb{P}^2(\mathbb{C})$ from example 4.6.4:



Let $\varphi \in \mathrm{SF}(N, \Sigma)$ and $\{m_C : C \in \Sigma\} \subset M$ be as in the definition of $\varphi \in \mathrm{SF}(N, \Sigma)$. For the m_C corresponding to cones of maximal dimension we need the following relations to coincide on the intersections:

$$m_{C_1} = a_{11}m_1 + a_{12}m_2,$$

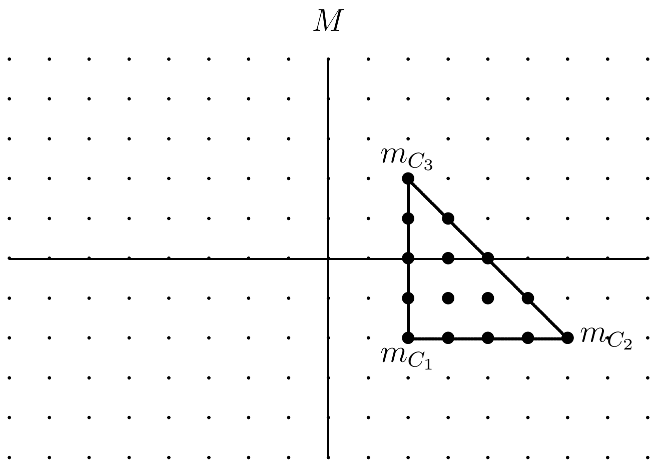
$$m_{C_2} = a_{21}m_1 + a_{12}m_2,$$

and

$$m_{C_3} = a_{11}m_1 + (a_{21} + a_{12} - a_{11})m_2.$$

The condition for φ to be convex is $a_{21} \geq a_{11}$. Choosing $a_{11} = 2,\ a_{12} = -2$

and $a_{21} = 6$, we get $\Delta_\varphi = \text{Conv}(\{m_{C_1}, m_{C_2}, m_{C_3}) :$



7 Complete Intersections

7.1 Nef Partitions

Let $\Delta \subset M_{\mathbb{R}}$ be a reflexive polytope (with $\dim(\Delta) = \dim(M_{\mathbb{R}})$ and interior point $\{0\} \in M_{\mathbb{R}}$) and $\Delta^* \subset N_{\mathbb{R}}$ its dual. From now on we denote by Δ^v the *set of vertices* of a polytope Δ. Let $E = \Delta^{*v}$ be the set of vertices of Δ^*. We define a $d-$dimensional complete fan $\Sigma[\Delta^*]$ as the union of the zero-dimensional cone $\{0\}$ together with the set of all cones

$$C[F] = \{0\} \cup \{z \in N_{\mathbb{R}} : \lambda z \in F \text{ for some } \lambda \in \mathbb{R}_{>}\}$$

supporting faces F of Δ^*.

Assume there exists a representation of $E = E_1 \cup \cdots \cup E_r$ as the union of disjoint subsets $E_1, \ldots, E_r$ and integral convex $\Sigma[\Delta^*]$-piecewise linear support functions $\varphi_i : N_{\mathbb{R}} \to \mathbb{R}$ $(i = 1, \ldots, r)$, such that

$$\varphi_i(e) = \begin{cases} 1 & \text{if } e \in E_i, \\ 0 & \text{otherwise.} \end{cases}$$

Each φ_i, which corresponds to a line bundle L_i defines a supporting polyhedron Δ_i for the global sections:

$$\Delta_i = \{\hat{z} \in M_{\mathbb{R}} : \langle \hat{z}, z \rangle \geq -\varphi_i(z) \ \forall \ z \in N_{\mathbb{R}}\}.$$

Conversely, each function φ_i is uniquely defined by the polyhedron Δ_i. A *Calabi-Yau complete intersection* is then determined by the intersection of the closure of r hypersurfaces, each corresponding to a global section of a line bundle L_i (see [BB95, BB]).

Definition 7.1.1 If there exists a reflexive polytope Δ and r such functions $\varphi_1, \ldots, \varphi_r$, we call the data

$$\Pi(\Delta) = \{\Delta_1, \ldots, \Delta_r\}$$

a *Nef partition.*

Equivalent to $\Pi(\Delta) = \{\Delta_1, \ldots, \Delta_r\}$ being a Nef partition is that the Δ_i have only $\{0\}$ as common point and that Δ can be written as the Minkowski sum $\Delta_i + \cdots + \Delta_r = \Delta$, which is shown by the following proposition:

Proposition 7.1.2 $\Pi(\Delta) = \{\Delta_1, \ldots, \Delta_r\}$ is a Nef partition if and only if Δ is the Minkowski sum of r rational polyhedra $\Delta = \Delta_1 + \cdots + \Delta_r$ and $\Delta_i \cap \Delta_j = \{0\}\ \forall\ i \neq j$.

Proof:
$\Rightarrow$: Assume Δ can be written as the Minkowski sum of r rational polyhedra $\Delta = \Delta_1 + \cdots + \Delta_r$ with $\Delta_i \cap \Delta_j = \{0\}\ \forall\ i \neq j$. Define r functions $\varphi_i : N_{\mathbb{R}} \to \mathbb{R}$ as

$$\varphi_i(z) = -\min_{\hat{z} \in \Delta_i} \langle \hat{z}, z \rangle\ \forall\ z \in N_{\mathbb{R}}.$$

- The φ_i are linear on the cones of $\Sigma[\Delta^*]$: It is sufficient to consider restrictions of the φ_i to cones of maximal dimension $C[F]$, where

  $$F = \Delta^* \cap \{z \in N_{\mathbb{R}}\ : \langle z, \hat{e} \rangle = -1\}$$

  is a facet of Δ^* corresponding to a vertex $\hat{e} \in \Delta^v$. Now let $\hat{e} = \hat{e}_1 + \cdots + \hat{e}_i + \cdots + \hat{e}_r$, where $\hat{e}_i \in \Delta_i^v$ denotes a vertex of Δ_i $(i = 1, \ldots, r)$. If we take another vertex $\hat{e}'_i \neq \hat{e}_i \in \Delta_i^v$ then the sum $\hat{e}' = \hat{e}_1 + \cdots + \hat{e}'_i + \cdots + \hat{e}_r$ denotes another vertex of Δ. Clearly $\langle \hat{e}, z \rangle \leq \langle \hat{e}', z \rangle\ \forall z \in C[F]$, i.e. $\langle \hat{e}_i, z \rangle \leq \langle \hat{e}'_i, z \rangle\ \forall z \in C[F]$. Hence $\varphi_i(z) = -\langle \hat{e}_i, z \rangle\ \forall\ z \in C[F]$.

- Convexity for the φ_i follows immediately from their definition.

- $\varphi_i(e) \in \{0, 1\}\ \forall\ e \in E,\ i = 1, \ldots, r$: For every function φ_i we have : $0 \in \Delta_i \Rightarrow \varphi_i \geq 0$ and $\Delta_i \subset \Delta \Rightarrow \varphi_i(e) \leq 1\ \forall\ e \in E$.

- $\varphi_i(e) = 1 \Rightarrow \varphi_j(e) = 0\ \forall\ j \neq i$: Assume $\varphi_i(e) = \varphi_j(e) = 1$ for $i \neq j$ $\Rightarrow \exists \hat{z}_i \in \Delta_i,\ \hat{z}_j \in \Delta_j : \langle \hat{z}_i, e \rangle = \langle \hat{z}_j, e \rangle = -1 \Rightarrow \exists \hat{z} = \hat{z}_i + \hat{z}_j \in \Delta$ with $\langle \hat{z}, e \rangle = -2$. This contradicts Δ^* being dual to Δ.

- $\forall e \in E\ \exists\ i \in \{1, \ldots, r\}$ with $\varphi_i(e) = 1$: Assume $\exists e \in E : \varphi_i(e) = 0\ \forall\ i = 1, \ldots, r$. By duality of Δ and Δ^* $\exists \hat{z} \in \Delta : \langle \hat{z}, e \rangle = -1$, where $\hat{z}$ is contained in the facet dual to e. Now $\hat{z} = \hat{z}_1 + \cdots + \hat{z}_r$ with $\hat{z}_i \in \Delta_i\ \forall\ i = 1, \ldots, r$. $\Rightarrow$ $\exists \hat{z}_k \in \Delta_k$ with $\langle \hat{z}_k, e \rangle < 0$. A contradiction to $\varphi_i(e) = 0\ \forall\ i = 1, \ldots, r$.

$\Leftarrow$: Follows from

$$\Delta = \{\hat{z} \in M_{\mathbb{R}} : \langle \hat{z}, z \rangle \geq -\varphi(z)\ \forall\ z \in N_{\mathbb{R}}\},$$

where $\varphi = \sum_i \varphi_i$ $(i = 1, \ldots, r)$ (see [Bor93]).

It can be shown that every Nef partition of a reflexive polytope Δ gives a dual Nef partition of a reflexive polytope ∇, which turns out to be an involution on the set of reflexive polytopes with Nef partitions:

Remark 7.1.3 [Bor93] Let $\Pi(\Delta) = \{\Delta_1, \ldots, \Delta_r\}$ be a Nef partition and denote by $E = E_1 \cup \cdots \cup E_r$ the set of vertices Δ^{*v}. Define r rational polyhedra $\nabla_i \subset N_{\mathbb{R}}$ $(i = 1, \ldots, r)$ as

$$\nabla_i = \mathrm{Conv}(E_i \cup \{0\}).$$

Then there is the following relation between Δ_i and ∇_j $(i, j = 1, \ldots, r)$:

$$\langle \Delta_i, \nabla_j \rangle = \begin{cases} \geq -1 & \text{if } \; i = j \\ \geq 0 & \text{otherwise.} \end{cases}$$

In particular $\nabla = \nabla_1 + \cdots + \nabla_r$ is a reflexive polyhedron with a Nef partition $\Pi(\nabla) = \{\nabla_1, \ldots, \nabla_r\}$, and there is a natural involution on the set of reflexive polyhedra with Nef partitions:

$$\iota \; : \; \Pi(\Delta) = \{\Delta_1, \ldots, \Delta_r\} \mapsto \Pi(\nabla) = \{\nabla_1, \ldots, \nabla_r\}.$$

Remark 7.1.4 The following procedure can be used to find all Nef partitions of a reflexive polyhedron $\Delta \subset M_{\mathbb{R}}$:

- First calculate $\Delta^* \subset N_{\mathbb{R}}$.
- Take all disjoint unions $E = E_1 \cup \ldots, \cup E_r$ of vertices of Δ^*.
- Check if $\nabla = \nabla_1 + \cdots + \nabla_r$ $(\nabla_1 = \mathrm{Conv}(E_i \cup \{0\}))$ is reflexive and $\nabla_i \cap \nabla_j = \{0\}$ $\forall \; i \neq j$.

7.2 Gorenstein Cones

A rational cone $C \subset M_{\mathbb{R}}$ is called *Gorenstein* if there exists a $n \in N$ in the dual lattice such that $\langle v, n \rangle = 1$ for all generators of C (see [BB95, BB]). With a Nef partition $\Pi(\Delta) = \{\Delta_1, \ldots, \Delta_r\}$ we can construct such a cone. First we go to a larger space and extend the canonical pairing: Let $\mathbb{Z}^r$ be the standard r-dimensional lattice and $\mathbb{R}^r$ its real scalar extension. We put

$\bar{N} = \mathbb{Z}^r \oplus N$, $\bar{d} = d + r$ and $\bar{M} = \mathrm{Hom}(\bar{N}, \mathbb{Z})$. We extend the canonical $\mathbb{Z}$-bilinear pairing $\langle *, * \rangle : M \times N \to \mathbb{Z}$ to a pairing between $\bar{M}$ and $\bar{N} = \mathbb{Z}^r \oplus N$ by the formula

$$\langle (a_1, \ldots, a_r, m), (b_1, \ldots, b_r, n) \rangle = \sum_{i=1}^{r} a_i b_i + \langle m, n \rangle.$$

The real scalar extension between $\bar{N}$ (resp. $\bar{M}$) is denoted by $\bar{N}_{\mathbb{R}}$ (resp. $\bar{M}_{\mathbb{R}}$), with the corresponding $\mathbb{R}$-bilinear pairing $\langle *, * \rangle : \bar{M}_{\mathbb{R}} \times \bar{N}_{\mathbb{R}} \to \mathbb{R}$.

Definition 7.2.1 From a Nef partition $\Pi(\Delta) = \{\Delta_1, \ldots, \Delta_r\}$ we construct a $\bar{d}$-dimensional Gorenstein cone $C_\Delta \subset \bar{M}_{\mathbb{R}}$ as

$$C_\Delta = \{(\lambda_1, \ldots, \lambda_r, \lambda_1 \hat{z}_1 + \cdots + \lambda_r \hat{z}_r) \in \bar{M}_{\mathbb{R}} : \lambda_i \in \mathbb{R}_{\geq}, \hat{z}_i \in \Delta_i,\ i = 1, \ldots r\},$$

with $n_\Delta \in \bar{N}$ uniquely defined by the conditions

$$\begin{aligned} \langle \hat{z}, n_\Delta \rangle &= 0 \ \forall\ \hat{z} \in M_{\mathbb{R}} \subset \bar{M}_{\mathbb{R}} \\ \langle \hat{e}_i, n_\Delta \rangle &= 1 \text{ for } i = 1, \ldots, r, \end{aligned}$$

where $\{\hat{e}_1, \ldots, \hat{e}_r\}$ is the standard basis of $\mathbb{Z}^r \subset \bar{M}$.

Note that all generators of C_Δ lie on the hyperplane $\langle \hat{z}, n_\Delta \rangle = 1$. They are the vertices of a $(\bar{d} - 1)$-dimensional rational polytope:

$$K_\Delta = \{\hat{z} \in C_\Delta : \langle \hat{z}, n_\Delta \rangle = 1\}.$$

Since $K_\Delta \cap \bar{M}$ has no interior point, we get

$$K_\Delta \cap \bar{M} = \bigcup_{i=1,\ldots,r} (\hat{e}_i \times \Delta_i) \cap \bar{M}.$$

Remark 7.2.2 [BB] Let $\Pi(\nabla) = \{\nabla_1, \ldots, \nabla_r\}$ be the dual Nef partition. Then the Gorenstein cone

$$C_\nabla = \{(\mu_1, \ldots, \mu_r, \mu_1 z_1 + \cdots + \mu_r z_r) \in \bar{N}_{\mathbb{R}} : \mu_i \in \mathbb{R}_{\geq}, z_i \in \nabla_i,\ i = 1, \ldots r\}$$

is the dual cone of C_Δ defined in 7.2.1. Note that C_Δ and C_∇ are dual to each other, but K_Δ is *not* dual to K_∇!

7.3 Combinatorial Polynomials of Eulerian Posets

Let P be an *Eulerian Poset* (i.e. a finite partially ordered set, see [BB95]) with unique minimal element $\hat{0}$, maximal element $\hat{1}$ and same length d of every maximal chain of P. For any $x \leq y \in P$ define the *interval* $I = [x, y]$ as

$$[x, y] = \{z \in P : x \leq z \leq y\}.$$

In particular, we have $P = [\hat{0}, \hat{1}]$. Define the *rank function* $\rho : P \to \{0, \ldots, d\}$ on P by setting $\rho(x)$ equal to the length of the interval $[\hat{0}, x]$. Note that for any Eulerian Poset P, every interval $I = [x, y]$ is again an Eulerian Poset with rank function $\rho(z) - \rho(x)\ \forall z \in I$. If an Eulerian Poset has rank d, then the *dual Poset* P^* is also an Eulerian Poset with rank function $\rho^* = d - \rho$.

Example 7.3.1 Let $C \in N_{\mathbb{R}}$ be a d-dimensional cone with its dual $C^* \in M_{\mathbb{R}}$. By proposition 3.1.15, there is a canonical bijective correspondence $F \leftrightarrow F^\star = C^* \cap F^\perp$ between faces $F \subset C$ and $F^\star \subset C^*$ ($\dim F + \dim F^\star = d$), which reverses the inclusion relation between faces. We denote the faces of C by indices x and define the poset $P = [\hat{0}, \hat{1}]$ as the poset of all faces $C_x \subset C$ with maximal element C, minimal element $\{0\}$ and rank function $\rho(x) = \deg(C_x)\ \forall x \in P$. The dual poset P^* can be identified with the poset of faces $C^*_x = F^\star_x \subset C^*$ of the dual cone C^* with rank function $\rho^*(x^*) = \dim(C^*_x)\ \forall\, x^* \in P^*$.

Definition 7.3.2 Let P be an Eulerian Poset of rank d as above. Define the *polynomial* $B(P; u, v) \in \mathbb{Z}[u, v]$ by the following rules [BB95]:

- $B(P; u, v) = 1$ if $d = 0$.
- The degree of $B(P; u, v)$ with respect to v is less than $d/2$.
- $\sum\limits_{\hat{0} \leq x \leq \hat{1}} B([\hat{0}, x]; u^{-1}, v^{-1})(uv)^{\rho(x)}(v-u)^{d-\rho(x)} = \sum\limits_{\hat{0} \leq x \leq \hat{1}} B([x, \hat{1}]; u, v)(uv-1)^{\rho(x)}.$

Let us consider how we can construct the B-polynomial for an interval $I = [x, y] \subset P$ with $d = \rho(y) - \rho(x)$: Suppose we know the B-polynomials $B(\tilde{I}; u, v)$ for all sub-intervals $\tilde{I} = [\tilde{x}, \tilde{y}] \subsetneq I$. Then we know all terms of the relation-formula for the B-polynomials in 7.3.2 except $B(I; u, v)$ on the right hand side and $B(I; u^{-1}, v^{-1})(uv)^d$ on the left hand side. Because the v-degree of $B(I; u, v)$ is less than $d/2$, the possible degrees of monomials with respect to v in $B(I; u, v)$ and $B(I; u^{-1}, v^{-1})(uv)^d$ do not coincide and we can calculate $B(I;u,v)$. So if we have to compute $B(P;u,v)$, we first have to calculate the B-polynomials for all intervals with rank 0 (which per definition are 1), then those intervals with rank 1, and so on.

Remark 7.3.3 Let P be an Eulerian Poset of rank d and P^* its dual. Then the polynomial defined in (7.3.2) satisfies

$$B(P;u,v) = (-u)^d B(P^*;u^{-1},v).$$

Definition 7.3.4 Let P be the Eulerian Poset corresponding to the Gorenstein cone $C = C_\Delta \subset \bar{M}_{\mathbb{R}}$ from definition 7.2.1. Define two functions on the set of faces of C by

$$\begin{aligned} S(C_x,t) &= (1-t)^{\rho(x)} \sum_{m \in C_x \cap \bar{M}} t^{\deg(m)} \\ T(C_x,t) &= (1-t)^{\rho(x)} \sum_{m \in C_x{}^\circ \cap \bar{M}} t^{\deg(m)}, \end{aligned}$$

where $C_x{}^\circ$ denotes the relative interior of $C_x \in C$ (see definition 3.1.6) and $\deg(m) = \langle m, n_\Delta \rangle$.

The following statement is a consequence of the Serre duality [BB95]:

Proposition 7.3.5 For the Gorenstein cone $C = C_\Delta \subset \bar{M}_{\mathbb{R}}$ the functions S and T are polynomials: $S(C_x,t)$, $T(C_x,t) \in \mathbb{Z}[t]$, and they satisfy the relation

$$S(C_x,t) = t^{\rho(x)} T(C_x,t^{-1}).$$

Remark 7.3.6 For $S = \sum_i a_i t^i$ and $T = \sum_i b_i t^i$ as defined above, 7.3.5 implies that

$$a_0 + a_1 t + \cdots + a_n t^n = b_0 t^n + b_1 t^{n-1} + \cdots + b_{n-1} t + b_n,$$

where $n = \dim C_x$, and we get the relations

$$a_i = b_{n-i} \quad (i = 1, \ldots, n)$$

for the coefficients of S and T. Since $a_0 = 1$ and $b_0 = 0$, the leading coefficients are determined to be $a_n = 0$ and $b_n = 1$. Thus it is sufficient to calculate $m \cdot (C_x \cap K_\Delta)$ and $m \cdot (C_x{}^\circ \cap K_\Delta)$ for $m = 0, \ldots, [\dim(C_x)/2]$ and to use the fact that $a_i = b_{n-i}$ for $i > \dim(C_x)/2$.

Definition 7.3.7 [BB95] Denote by $h_{st}^{p,q}$ the *string-theoretical Hodge numbers* of a Calabi-Yau complete intersection V arising from a Nef partition $\Pi(\Delta) = \{\Delta_1, \ldots, \Delta_r\}$. Then the *string-theoretical E-polynomial* is defined as

$$E_{st}(V; u, v) = \sum (-1)^{p+q} h_{st}^{p,q} u^p v^q.$$

Batyrev and Borisov showed in their paper [BB95] that the string-theoretical E-polynomial of a Nef partition can be calculated from the data of the corresponding Gorenstein cone:

Proposition 7.3.8 [BB95] Let $\Pi(\Delta) = \{\Delta_1, \ldots, \Delta_r\}$ be a Nef partition and $C = C_\Delta \subset \bar{M}_{\mathbb{R}}$ the $\bar{d}$-dimensional reflexive Gorenstein cone defined in 7.2.1 (with dual cone $C^* = C_\nabla \subset \bar{N}_{\mathbb{R}}$). Denote by P the poset of faces $C_x \subset C$ (see example 7.3.1). Then the string-theoretical E-polynomial is given by

$$E_{st}(V; u, v) = \sum_{I=[x,y]\subset P} \frac{(-1)^{\rho(y)}}{(uv)^r} (v-u)^{\rho(x)} B(I^*; u, v)(uv-1)^{\bar{d}-\rho(y)} A_{(x,y)}(u, v),$$

with

$$A_{(x,y)}(u, v) = \sum_{(m,n)\in C_x{}^\circ \cap \bar{M} \times C_y^*{}^\circ \cap \bar{N}} \left(\frac{u}{v}\right)^{\deg(m)} \left(\frac{1}{uv}\right)^{\deg(n)}.$$

The dual partition $\Pi(\nabla) = \{\nabla_1, \ldots, \nabla_r\}$ corresponds to the Calabi-Yau complete intersection W and (V, W) is called a *Mirror pair* of (singular) Calabi-Yau varieties. The following relation holds for the string-theoretical E-polynomial of V and W:

$$E_{st}(V; u, v) = (-u)^{d-r} E_{st}(W; \frac{1}{u}, v),$$

inducing the following duality of Hodge numbers between V and W:

$$h_{st}^{p,q}(V) = h_{st}^{n-p,q}(W) \quad 0 \le p, q \le n = \dim(V) = \dim(W),$$

which is called *Mirror duality*.

Remark 7.3.9 Using duality 7.3.3 for the B-polynomials and definition 7.3.4 with relation 7.3.5 between the S- and T-polynomials, we can write the E-polynomial as

$$E_{st}(V;u,v) = \sum_{I=[x,y]\subset P} \frac{(-1)^{\rho(x)}u^{\rho(y)}}{(uv)^r} S\left(C_x, \frac{v}{u}\right) S\left(C_y^*, uv\right) B(I;u^{-1},v),$$

which can be used for explicit calculation.

8 Results

8.1 Comparison with Weighted Projective Space

Using the formula for the E-polynomial 7.3.9 we are able to construct Calabi-Yau complete intersections starting with a reflexive polytope $\Delta \subset M_{\mathbb{R}}$ (or $\Delta^* \subset N_{\mathbb{R}}$). Regarding complete intersections in $W\mathbb{P}^5$ spaces as a special case of the toric construction we tried to reproduce the Hodge data of codimension two Calabi-Yau manifolds in $W\mathbb{P}^5$ spaces listed in [Kle]. In what follows we will analyze some examples from this list and discuss the different situations that can occur.

In the simplest case the Newton polyhedron $\Delta(d)$ corresponding to degree (d_1, d_2) equations is reflexive and the Hodge numbers of a Nef partition $\Pi(\Delta(d)) = \{\tilde{\Delta}_1, \tilde{\Delta}_2\}$ agree with those in [Kle]. This works for example in case of the first weight system 6 1 1 1 1 1 1 d=4 2 in this list.

In general, $\Delta(d_1 + d_2)$ may differ from the Minkowski sum $\Delta(d_1) + \Delta(d_2)$ and none of the two polytopes has to be reflexive. In many cases we find a simple modification of these polytopes that makes the Hodge data agree:

- Already in the second example of this list the Newton polyhedron $\Delta(7)$ of the weight system 7 1 1 1 1 1 2 d=3 4 is not reflexive. It is, however, possible to reduce $\Delta(7)$ to a reflexive polyhedron Δ by omitting 5 points, so that its dual provides a toric resolution of singularities of the weighted projective space. Indeed, the Hodge data matches for one Nef partition of the resulting polytope.
- Another possibility is that the Newton polyhedron is reflexive, but the Hodge numbers do not agree. In such a case we can compute the Minkowski sum $\tilde{\Delta} = \Delta(d_1) + \Delta(d_2)$ and check if it is reflexive and gives the right Hodge numbers. This works for example with the weight system 8 1 1 1 1 2 2 d=5 3.

There are still some examples where we are not able to reproduce the Hodge data. For example, in case of the weight system 9 1 1 1 1 2 3 d=5 4, neither $\Delta(9)$ nor the Minkowski sum $\Delta(5) + \Delta(4)$ is reflexive. Omitting up to 20 points yields a reflexive polyhedron, but there exists no Nef partition which agrees with the corresponding Hodge data in [Kle].

8.2 New Hodge Numbers

We are also interested in computing new Hodge numbers. In [KS] the full set of 30108 pairs of Hodge numbers (15122 if they are restricted to $h_{11} \leq h_{12}$) corresponding to hypersurfaces in toric varieties spaces is listed. Picking out only the new Hodge numbers from the list of 2387 ordered pairs in [Kle], i.e. those not contained in [KS], there remain only 15 pairs[2]:

New Hodge Numbers in [Kle]:

h^{11}	h^{21}	R	h^{11}	h^{21}	R	h^{11}	h^{21}	R
1	61	x	2	62	x	7	26	
1	73	x	2	68	x	8	20	
1	79		3	47		12	12	x
1	89	x	3	55	x	13	13	x
1	129	x	3	61		17	11	

where R=x means that we could reproduce the Hodge numbers.

Using the toric construction we found 28 (56 with Mirror duality) pairs of new Hodge numbers. They are listed in appendix A, table 1-3 together with a detailed information about the starting polyhedron.

[2](0,36) obviously is an error

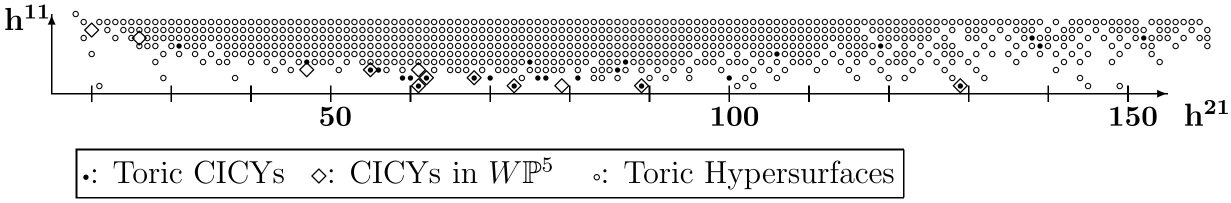
The resulting numbers are:

New Hodge Numbers with Toric CICYs:

h^{11}	h^{21}	h^{11}	h^{21}	h^{11}	h^{21}	h^{11}	h^{21}
1	61	2	68	3	56	6	31
1	73	2	70	3	86	6	119
1	89	2	76	4	47	6	139
1	129	2	77	4	75	6	251
2	59	2	81	4	87	7	138
2	60	2	100	5	106	7	152
2	62	3	55	5	166	9	164

Drawing all the new Hodge numbers together with the "background" of hypersurfaces in toric varieties in the range of $1 \leq h^{11} \leq 9$ and $10 \leq h^{21} \leq 160$ results in the following picture:

CICYs in Hypersurface Background:

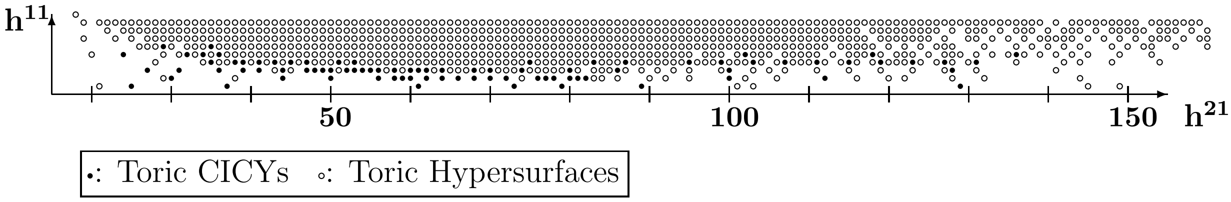


Most of the new Hodge numbers lie in the lower boundary region $h^{11} \leq 4$ which is less covered by the "background" of toric hypersufaces. It is remarkable that almost every pair of new Hodge numbers corresponds to a starting polyhedron $\Delta^* \cap N$ in the N-lattice with less than 20 points (see appendix table 1-3).

To calculate new spectra in a systematical way it seems to be suggestive to construct as many "small" reflexive polyhedra as possible, and to restrict our attention to these examples. Indeed, as a temporary result we found 86 pairs of Hodge numbers not contained in [KS]. They are listed in table 4 of appendix A and again compared with the 'background' of hypersurfaces in

$W\mathbb{P}^4$ spaces in a range of $0 \leq h^{11} \leq 9$ and $10 \leq h^{11} \leq 160$ in the following picture:

CICYs in Hypersurface Background:



The advantage of this strategy now lies in restricting our interest in a "small" class of reflexive polytopes which also dissipate less time in computing the corresponding Nef partitions because of their small number of vertices. Pursuing this strategy, a further step will be to increase the codimension by one and to construct complete intersections using six-dimensional starting polytopes with small number of points.

A Some New Hodge Data

Table 1

d_1	$w_{1_1}, \ldots, w_{n_1}$	d_2	$w_{1_2}, \ldots, w_{n_2}$	h^{11}	h^{21}	$-\chi$	$\#\Delta \cap M$	$\#\Delta^v$	$\#\Delta^* \cap N$	$\#\Delta^{*v}$
3	1 1 1 0 0 0 0	4	0 0 0 1 1 1 1	2	59	114	350	12	8	7
3	1 1 1 0 0 0 0	5	1 0 0 1 1 1 1	2	60	116	379	12	8	7
3	1 1 1 0 0 0 0	4	0 0 0 1 1 1 1	2	62	120	350	12	8	7
3	1 1 1 0 0 0 0	6	0 1 1 1 1 1 1	2	62	120	381	12	8	7
3	1 1 1 0 0 0 0	5	1 0 0 1 1 1 1	2	70	136	379	12	8	7
3	1 1 1 0 0 0 0	6	0 1 1 1 1 1 1	2	76	148	381	12	8	7
3	1 1 1 0 0 0 0	4	0 0 0 1 1 1 1	2	77	150	350	12	8	7
3	1 1 1 0 0 0 0	10	2 0 0 1 1 2 4	2	81	158	379	18	10	8
3	1 1 1 0 0 0 0	11	2 0 0 1 1 2 5	2	81	158	418	19	11	9
3	1 1 1 0 0 0 0	8	3 0 0 1 1 2 1	2	100	196	496	16	9	8
4	1 1 2 0 0 0 0	5	1 0 0 1 1 1 1	3	55	104	292	12	9	7
4	1 1 2 0 0 0 0	6	1 0 0 1 1 1 2	3	55	104	282	12	9	7
4	2 1 1 0 0 0 0	8	0 1 2 1 1 1 2	3	55	104	247	9	9	7
4	2 1 1 0 0 0 0	12	0 1 3 2 2 2 2	3	55	104	265	9	9	7
4	1 1 2 0 0 0 0	4	0 0 0 1 1 1 1	3	55	104	315	12	9	7
3	1 1 1 0 0 0 0	5	0 0 0 1 1 1 2	3	56	106	340	18	9	8
3	1 1 1 0 0 0 0	7	1 0 0 1 1 2 2	4	75	142	304	20	9	8
3	1 1 1 0 0 0 0	8	0 1 1 1 1 2 2	4	75	142	305	20	9	8
3	1 1 1 0 0 0 0	12	0 1 1 1 1 2 6	4	75	142	453	24	14	10
3	1 1 1 0 0 0 0	12	0 3 3 2 2 1 1	4	87	166	402	20	9	8
6	1 2 3 0 0 0 0	20	0 1 5 1 2 1 10	5	106	202	727	18	29	10
3	1 1 1 0 0 0 0	18	9 0 0 1 2 5 1	5	166	322	867	18	13	9
4	1 1 2 0 0 0 0	18	9 0 0 1 2 5 1	5	166	322	641	18	16	9
4	1 1 2 0 0 0 0	24	0 1 6 1 3 1 12	6	119	226	887	18	20	10
3	1 1 1 0 0 0 0	18	9 0 0 1 3 4 1	6	139	266	725	14	12	8
4	1 1 2 0 0 0 0	36	0 4 9 3 1 1 18	6	139	266	1392	16	20	9
4	1 1 2 0 0 0 0	60	0 15 30 3 10 1 1	6	251	490	3191	6	14	6
6	1 2 3 0 0 0 0	60	0 10 15 3 1 1 30	6	251	490	2722	12	31	7
3	1 1 1 0 0 0 0	16	7 0 0 1 2 5 1	7	138	262	689	18	13	9
4	1 1 2 0 0 0 0	16	7 0 0 1 2 5 1	7	138	262	509	18	16	9
6	1 2 3 0 0 0 0	36	0 4 9 3 1 1 18	7	138	262	1226	14	31	8
3	1 1 1 0 0 0 0	20	10 0 0 1 3 5 1	7	152	290	794	14	12	8
4	1 1 2 0 0 0 0	20	10 0 0 1 3 5 1	7	152	290	587	14	14	8
4	1 1 2 0 0 0 0	96	0 24 48 4 16 1 3	9	164	310	2051	6	17	6

Table 2

d	$w_1, \ldots, w_n$	h^{11}	h^{21}	$-\chi$	$\#\Delta \cap M$	$\#\Delta^v$	$\#\Delta^* \cap N$	$\#\Delta^{*v}$
12	1 1 2 2 3 3	1	61	120	407	6	7	6
6	1 1 1 1 1 1	1	73	144	462	6	7	6
8	1 1 1 1 2 2	1	73	144	483	6	7	6
6	1 1 1 1 1 1	1	89	176	462	6	7	6
12	1 2 2 2 2 3	2	62	120	321	6	8	6
9	1 1 1 2 2 2	2	68	132	434	12	8	7
10	1 1 2 2 2 2	2	68	132	378	6	8	6
15	1 1 2 2 3 6	3	86	166	607	16	9	8
33	1 1 3 3 10 15	6	251	490	1624	12	13	7
40	1 1 3 10 10 15	6	251	490	1297	14	13	8
60	1 1 3 10 15 30	6	251	490	3191	6	14	6
96	1 3 4 16 24 48	9	164	310	2051	6	17	6

Table 3

d	$w_1, \ldots, w_n$	d_1	d_2	h^{11}	h^{21}	$-\chi$	$\#\Delta \cap M$	$\#\Delta^v$	$\#\Delta^* \cap N$	$\#\Delta^{*v}$
12	1 1 2 2 3 3	6	6	1	61	120	407	6	7	6
6	1 1 1 1 1 1	2	4	1	73	144	462	6	7	6
8	1 1 1 1 2 2	4	4	1	73	144	483	6	7	6
6	1 1 1 1 1 1	2	4	1	89	176	462	6	7	6
8	1 1 1 1 1 3	4	4	1	129	256	636	10	8	7
12	1 2 2 2 2 3	6	6	2	62	120	321	6	8	6
9	1 1 1 2 2 2	4	5	2	68	132	434	12	8	7
10	1 1 2 2 2 2	4	6	2	68	132	378	6	8	6
14	1 2 2 3 3 3	6	8	4	47	86	294	12	9	7
13	1 1 2 2 3 4	5	8	4	75	142	448	16	9	8
16	1 1 3 3 3 5	6	10	4	75	142	454	13	10	8
12	1 1 2 2 3 3	4	8	4	87	166	402	20	9	8
22	2 2 3 5 5 5	10	12	6	31	50	207	13	12	8

Table 4

h^{11}	h^{21}	h^{11}	h^{21}	h^{11}	h^{21}	h^{11}	h^{21}	h^{11}	h^{21}	h^{11}	h^{21}
1	25	2	64	3	36	3	62	4	47	4	131
1	37	2	66	3	39	3	64	4	51	5	24
1	61	2	68	3	41	3	68	4	53	5	32
1	73	2	70	3	47	3	70	4	75	5	34
1	79	2	72	3	48	3	74	4	83	5	36
1	89	2	76	3	49	3	80	4	87	5	102
1	129	2	77	3	50	3	86	4	95	5	118
2	30	2	78	3	52	3	100	4	99	5	136
2	44	2	80	3	53	3	128	4	103	6	29
2	50	2	81	3	54	4	35	4	107	6	35
2	56	2	82	3	55	4	38	4	111		
2	58	2	100	3	56	4	39	4	115		
2	59	2	112	3	58	4	41	4	119		
2	60	3	27	3	60	4	43	4	123		
2	62	3	31	3	61	4	45	4	127		

References

[BB] V. V. Batyrev and L. A. Borisov. Dual Cones and Mirror Symmetry for Generalized Calabi-Yau Manifolds. In *Greene, B. (ed.): Yau, S.T. (ed.): Mirror symmetry II* 71-86.

[BB95] V. V. Batyrev and L. A. Borisov. Mirror Duality and String-Theoretic Hodge Numbers. 1995.

[Bor93] L. A. Borisov. Towards the Mirror Symmetry for Calabi-Yau Complete Intersections in Gorenstein Toric Fano Varieties. 1993.

[Cal57] E. Calabi. *Algebraic Geometry and Topology: A Symposium in Honor of S. Lefschetz.* Princeton University Press, 1957.

[CDLS88] P. Candelas, A.M. Dale, C.A. Lutken, and R. Schimmrigk. Complete intersection Calabi-Yau manifolds. *Nucl. Phys.*, B298:493, 1988.

[Cig95] J. Cigler. *Körper Ringe Gleichungen.* Spektrum Akademischer Verlag Heidelberg Berlin Oxford, 1995.

[CLS90] P. Candelas, M. Lynker, and R. Schimmrigk. Calabi-Yau manifolds in weighted p(4). *Nucl. Phys.*, B341:383–402, 1990.

[Eig] G. Eigenthaler. *Algebra.* Vorlesungsskriptum.

[Fle89] A.R. Fletcher. Working with complete intersections. *Bonn preprint MPI/89–35*, 1989.

[Ful93] W. Fulton. *Introduction to Toric Varieties.* Princeton University Press, 1993.

[GH87] P. Green and T. Hübsch. Calabi-Yau manifolds as complete intersections in products of complex projective spaces. *Commun. Math. Phys*, 109:99, 1987.

[Kle] A. Klemm. `http://ars-www.uchicago.edu/~aklemm/HTML/cy.htm`.

[KS] M. Kreuzer and H. Skarke. `http://tph16.tuwien.ac.at/~kreuzer/CY/`.

[KS92] M. Kreuzer and H. Skarke. On the classification of quasihomogeneous functions. *Commun Math. Phys.*, 150:137, 1992.

[KT94] A. Klemm and S. Theisen. Mirror maps and instanton sums for complete intersections in weighted projective space. *Mod. Phys. Lett.*, A9:1807, 1994.

[LVW89] W. Lerche, C. Vafa, and N. Warner. Chiral rings in n=2 superconformal theories. *Nucl. Phys.*, B324:427, 1989.

[Mil] J. S. Milne. Algebraic geometry. `http://www.jmilne.org/math/`.

[Nak90] M. Nakahara. *Geometry, Topology and Physics*. IOP Publishing, Bristol, 1990.

[Oda88] T. Oda. *Convex Bodies and Algebraic Geometry*. Springer-Verlag Berlin Heidelberg, 1988.

[Sha94a] I. R. Shafarevich. *Basic Algebraic Geometry 1*. Springer-Verlag Berlin Heidelberg, 1994.

[Sha94b] I. R. Shafarevich. *Basic Algebraic Geometry 2*. Springer-Verlag Berlin Heidelberg, 1994.

[Sha94c] I. R. Shafarevich. *Encyclopaedia of Mathematical Sciences: Algebraic Varieties 1*, volume 23. Springer-Verlag Berlin Heidelberg, 1994.

[Spi95] H. Spindler. *Algebraische Geometrie 1*. Selbstverlag der Universität Osnabrück, 1995. Vorlesungsskriptum.

[Yau77] S. T. Yau. volume 74. Proc. Natl. Acad. Sci., 1977.