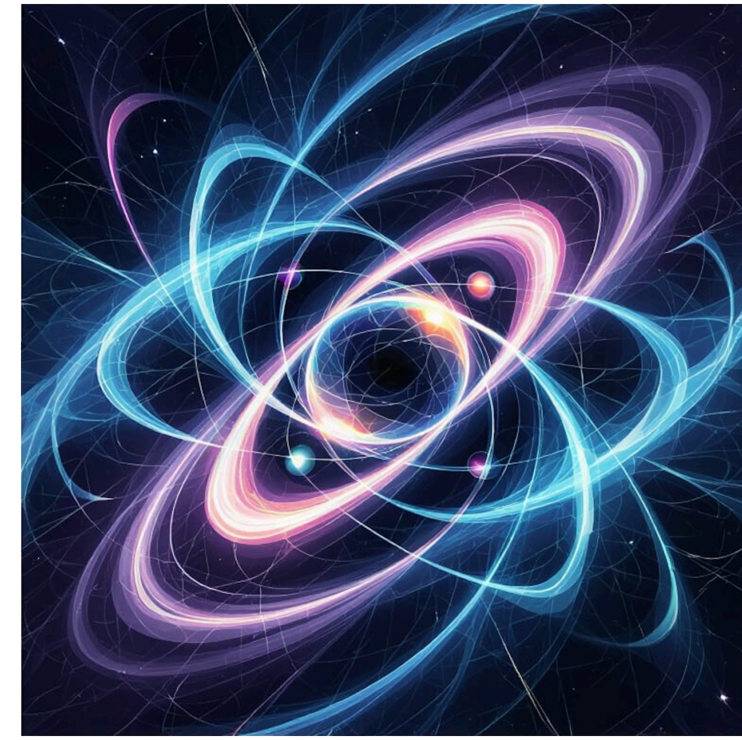
Higher spin theory:
Higher-spin gauge theories in three spacetime dimensions

132.071 Seminar on Fundamental Interactions 1

Lecture notes: https://arxiv.org/pdf/2403.16567, Andrea Campoleoni, Stefan Fredenhagen

Seminar organizers:
Iva Lovrekovic,
Daniel Grumiller

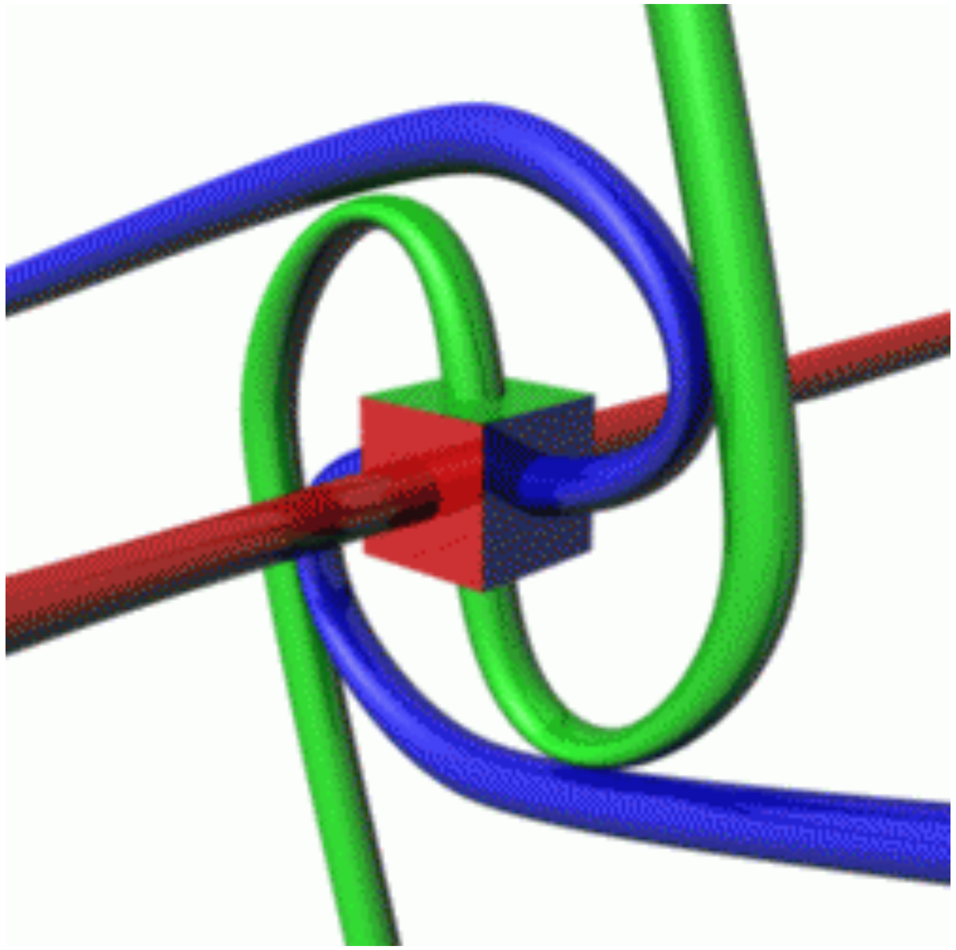
Outline



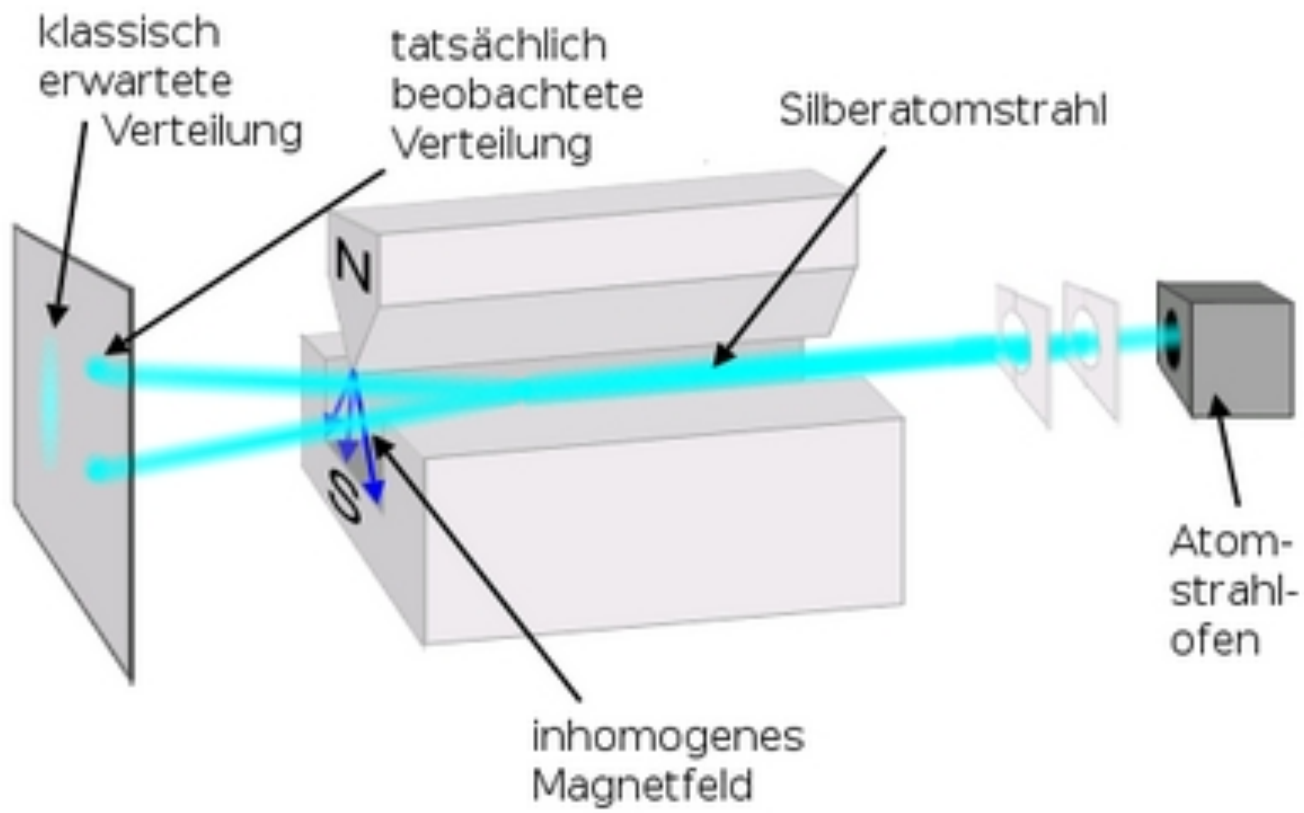
- Introduction and motivation
- Future prospects
- Chapter 2. Higher spins and Chern-Simons theory
- Chapter 3. Asymptotic symmetries
- Chapter 4. Quantum W-algebras and minimal model holography
- Chapter 5. Coupling to matter

Introduction and motivation

- Brief review:
- Spin - intrinsic angular momentum of a particle
- It is a property of a particle (like mass or charge for example)
- **L**=m**vr**
- Spin is quantum - if a particle of the size of an electron rotated with speed required to explain experiments, it would be faster than a speed of light



Introduction and motivation



- Evidence that the spin is quantum was given by the Stern-Gerlach experiment
- Silver atoms have one electron in their outer shell which grants the atom magnetic moment
- Because of that moment external magnetic field will induce a force on the atoms
- The direction of that force depends on where the little magnetic moments are pointing, relative to magnetic field

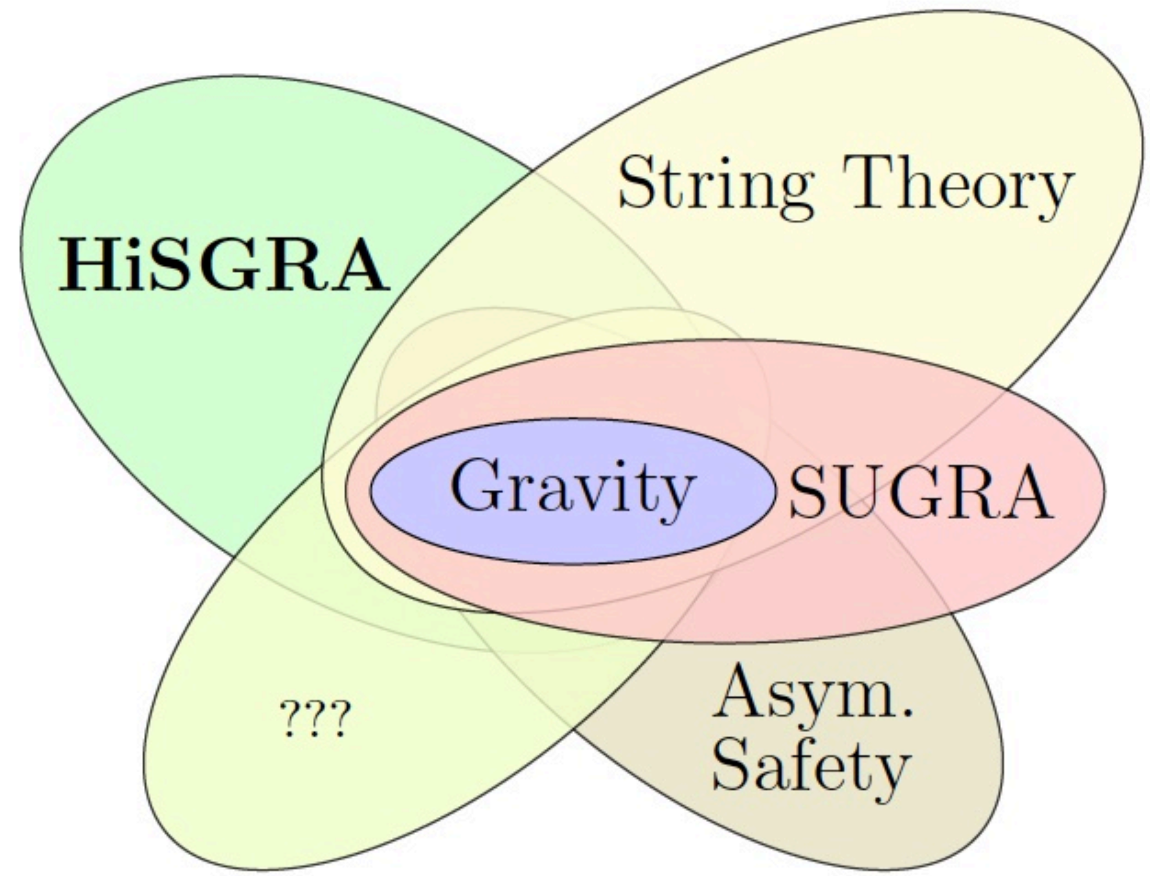
Introduction and motivation

- The motivation for their introduction is to improve the UV behaviour of gravity

- Theories in Standard Model can be quantised, renormalised, and unified, up to the point when we need to include gravity

- There are many candidates for quantum gravity

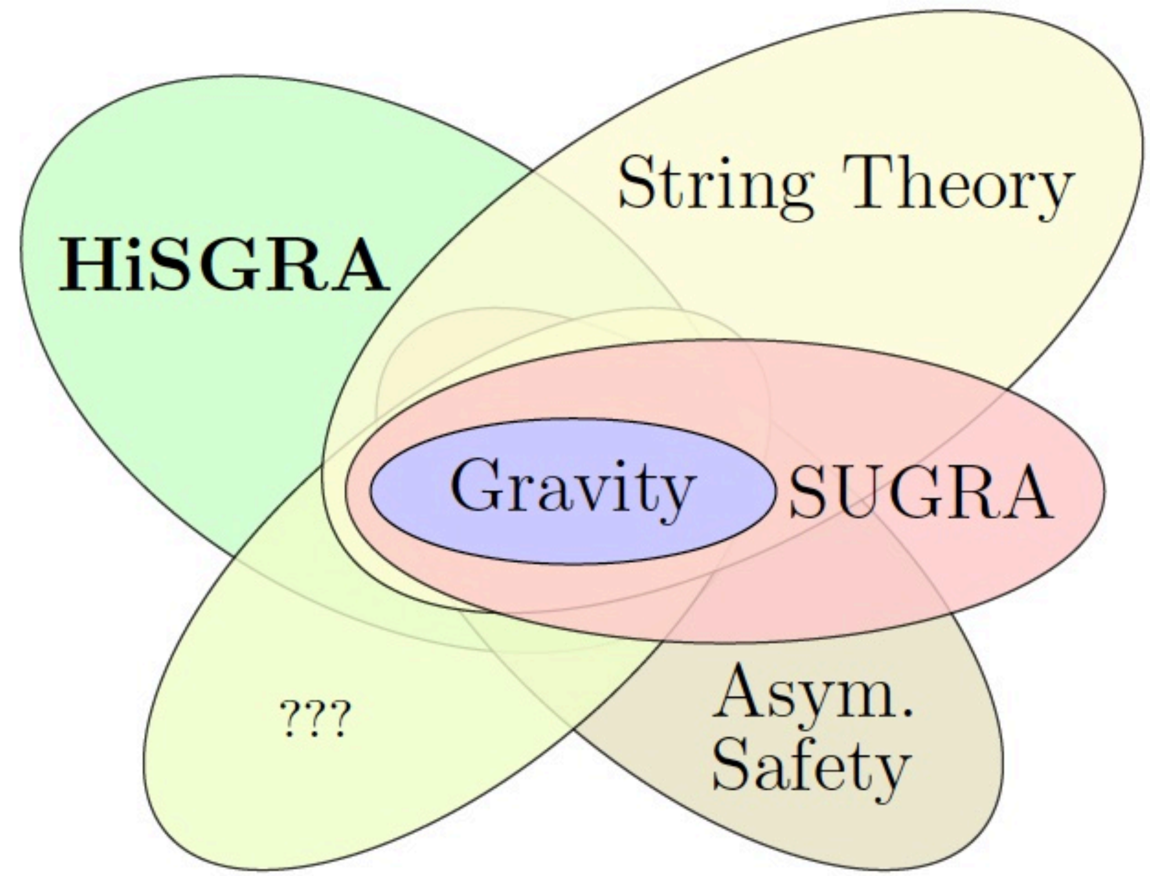
- HS fields naturally emerge in string theory, which includes infinite tower of massive HS fields



Introduction and motivation

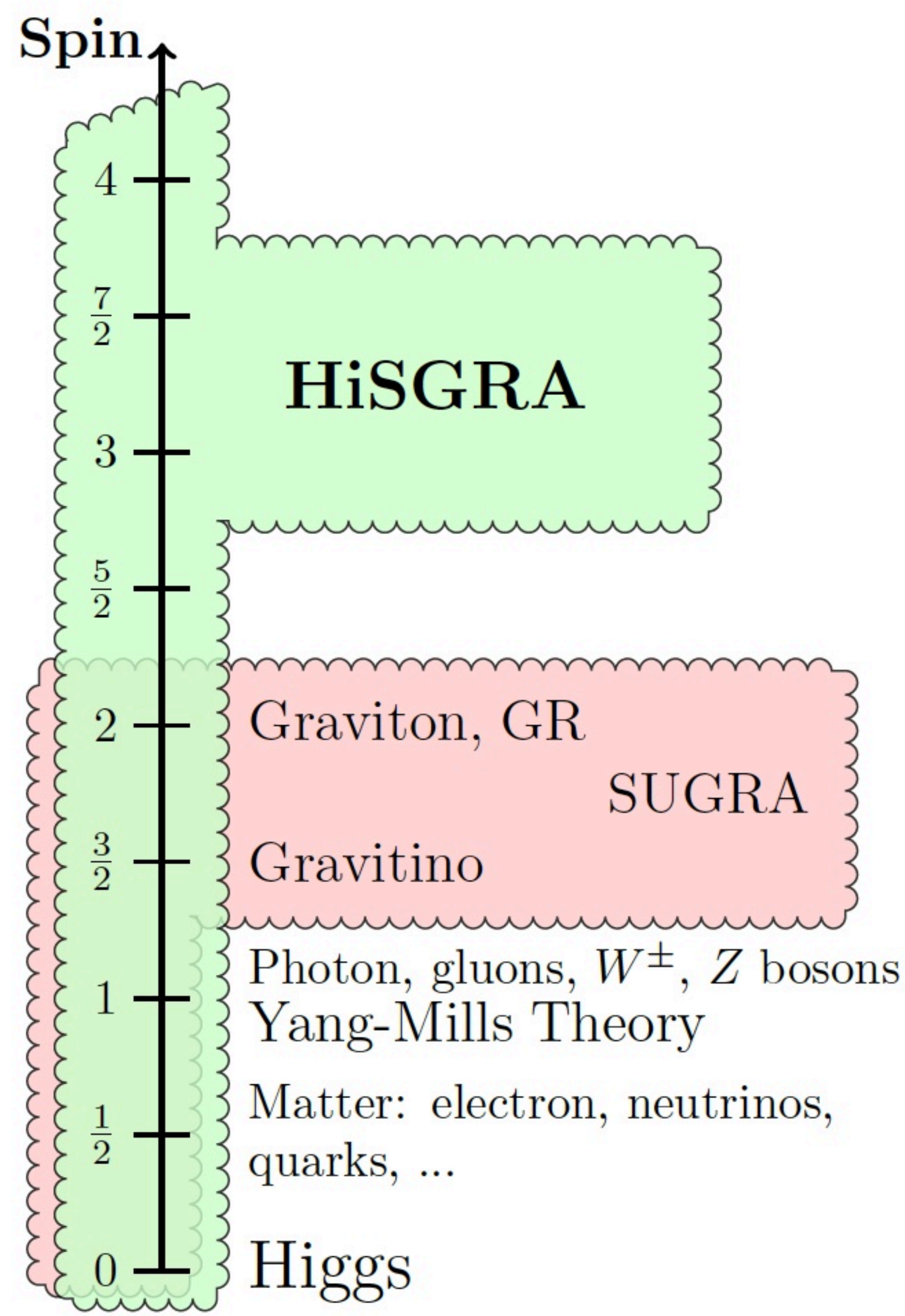
- Alternative speculation: String theory appears as a broken-phase of some underlying higher spin gauge theory

- Independently of string theory, the AdS/CFT correspondence indicates that higher spin states may be important to resolve the quantum gravity problem.



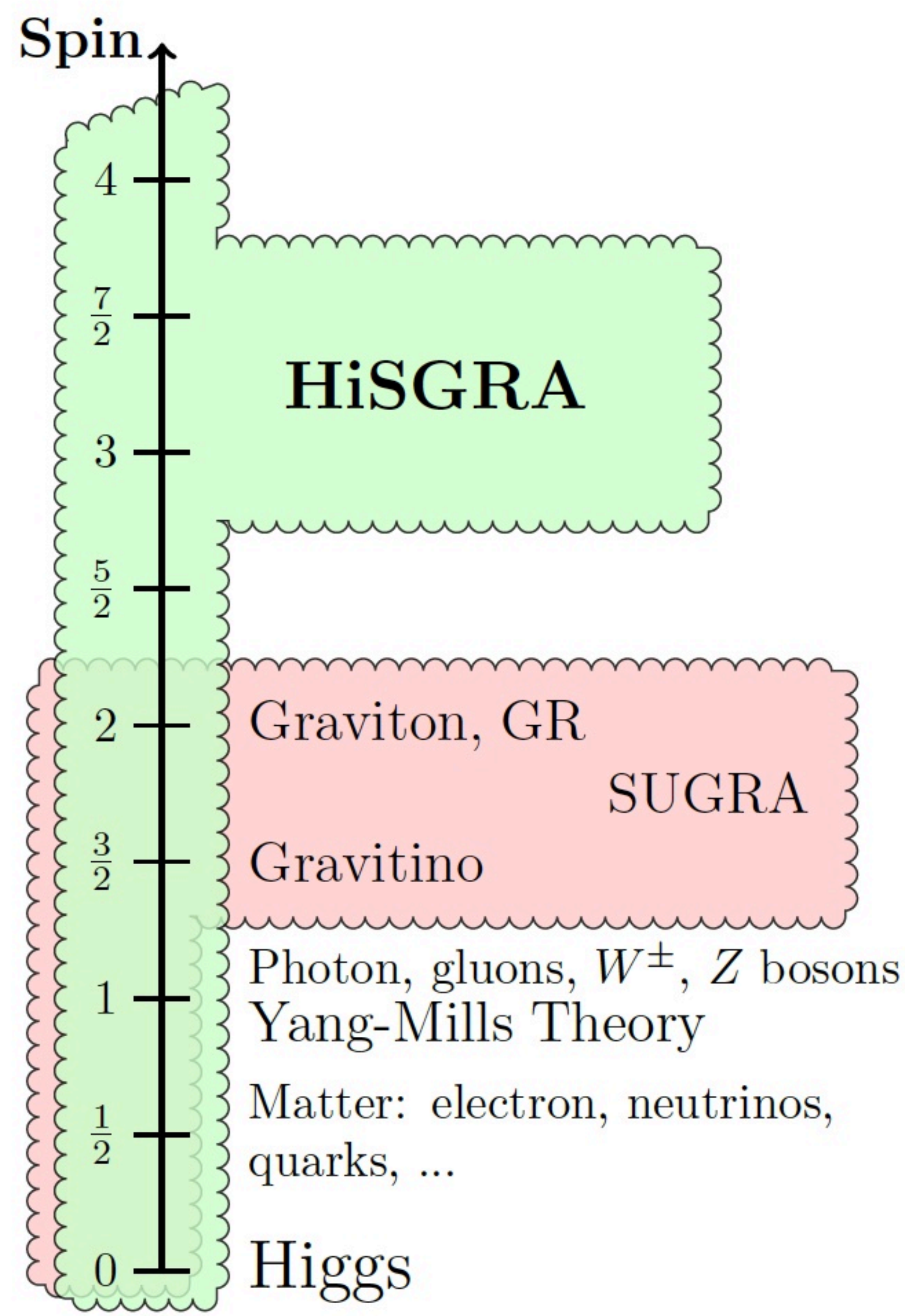
Introduction and motivation

- HS theory describes higher spin particles and their behaviour
- First introduced theories were for the free non-interacting massless higher spin particles
- Adding interaction - the action becomes more and more complicated by adding each higher spin particle.



Introduction and motivation

- Initially, HS theories encountered number of no-go theorems
- e.g. Coleman-Manudla theorem (restriction on symmetry of S-matrix); Weinberg low energy theorem (all HS fields must have constant coupling, assumes Lorentz invariance) - HS fields on AdS background avoid them
- in HS theory - there is a huge gauge symmetry
- There are few explicit examples of HS theory



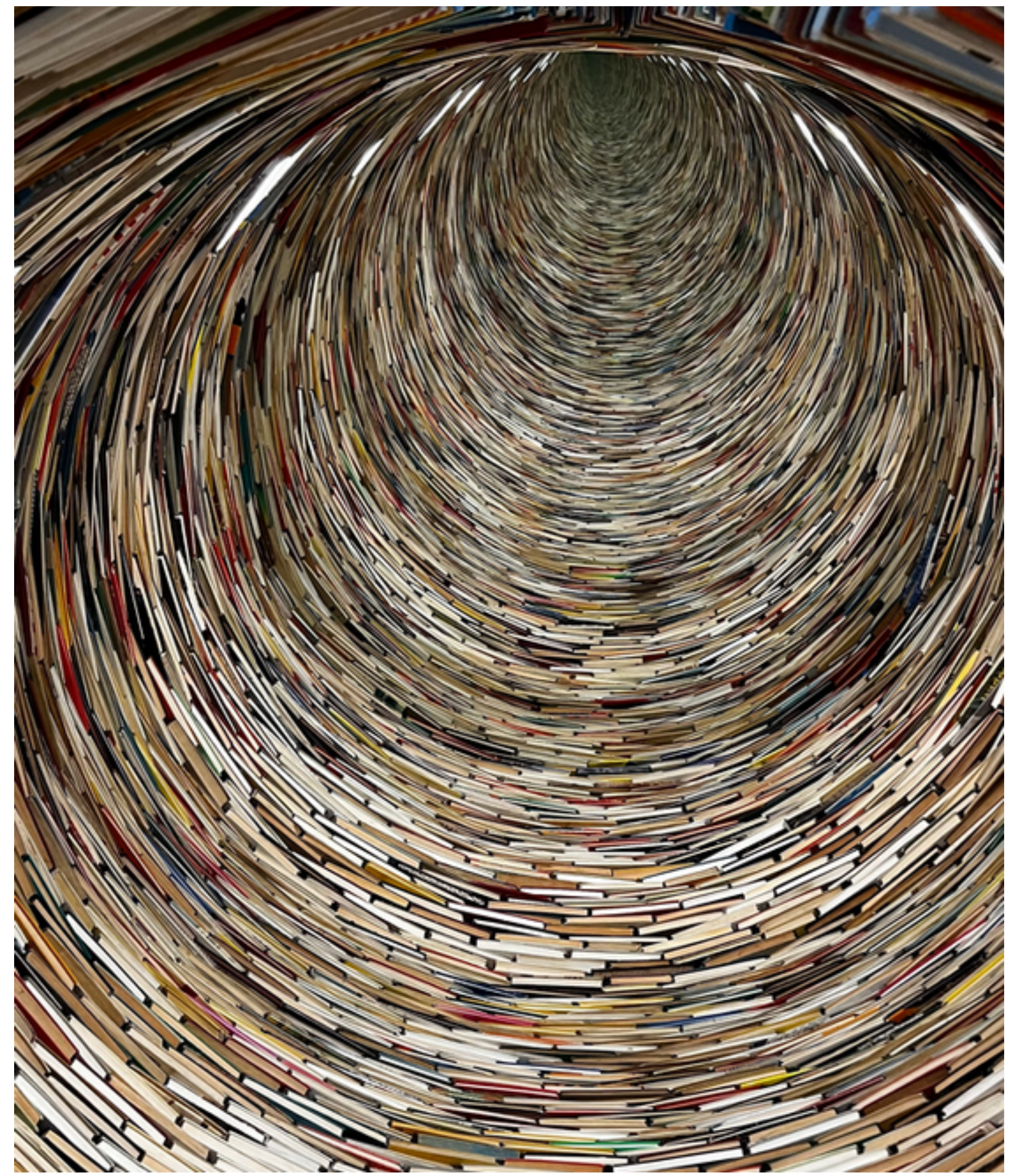
Introduction and motivation

- Vasiliev invented formalism that describes interacting HS field theory at the level of equations of motion

- Fronsdal - free massless HS theory

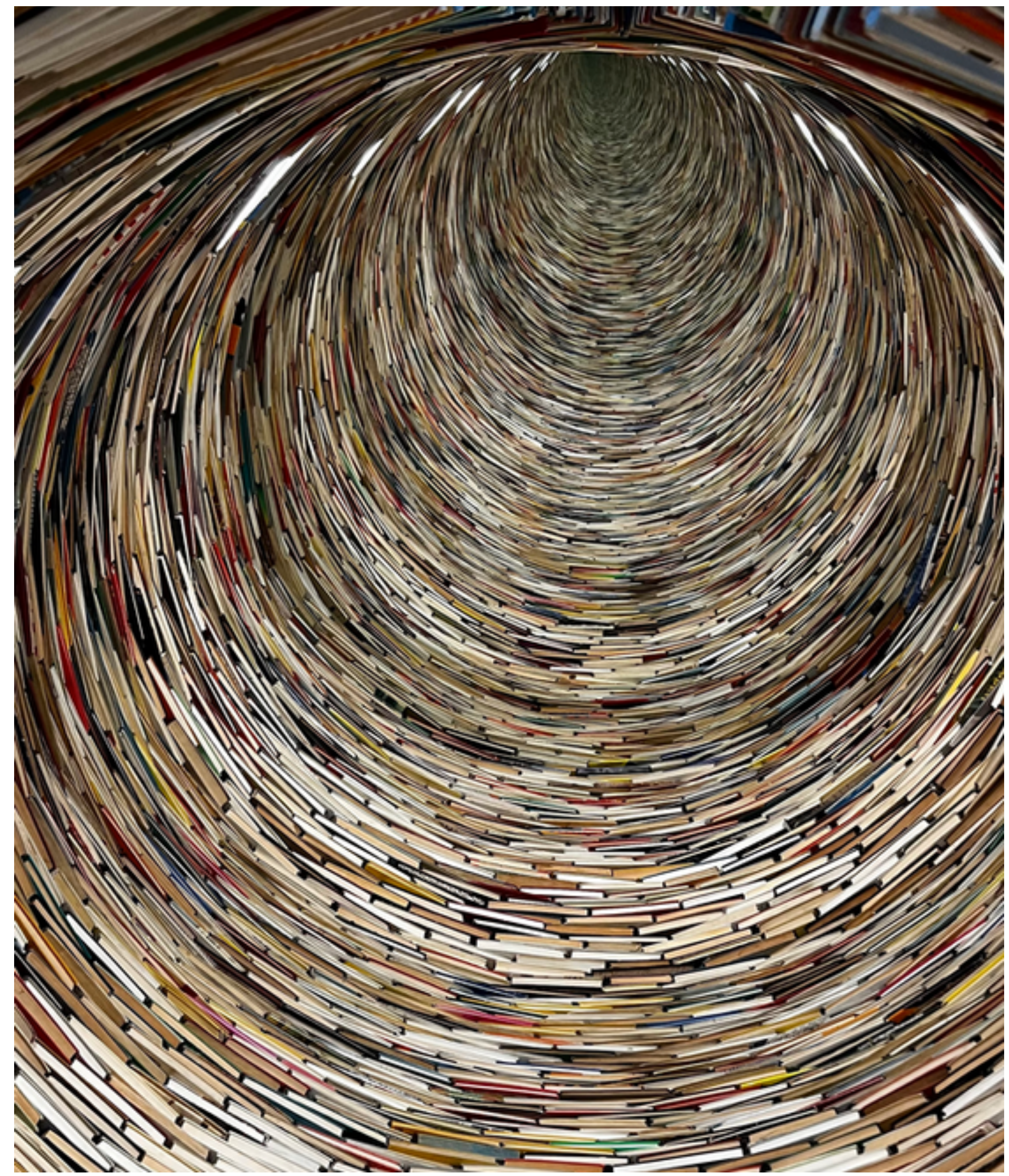
- Fradkin, Tseytlin, Segal - conformal HS theory

- Important: symmetries, 4D Chiral model



Introduction and motivation

- Theories in 3D - great playground for studying the HS properties - it is simpler
- It is possible to have interacting HS theory with finite number of particles
- There are Einstein equations and equations of motion, however there are no propagating d.o.f.-s
- There are no gravitational waves in 3d, however there are BHs



Future prospects

- Choice of the topics - such that they provide a basis to understand minimal model holography
- Higher-spin black holes
- Semiclassical methods to compute observables in CFT with W-symmetry
- Higher spins and strings
- 3D higher spin theories in flat space
- Metric-like formulation
- Higher spin theories in 3D
- Non-relativistic 3D higher-spin theories

Higher spins and Chern-Simons Theory

- In 3D the theories we can write Einstein-Hilbert action as:

$$S_{EH} = \frac{1}{16\pi G} \int \epsilon_{abc} \left(e^a \wedge R^{bc} + \frac{1}{3l^2} e^a \wedge e^b \wedge e^c \right)$$

Newton's constant

$$e^a = e_\mu{}^a dx^\mu$$

$$\Lambda = -\frac{1}{l^2}$$

$$\omega^{ab} = \omega_\mu{}^{ab} dx^\mu$$

$$R^{ab} = d\omega^{ab} + \omega^{ac} \wedge \omega_c{}^b$$

Higher spins and Chern-Simons Theory

- To write it in the Chern-Simons form we use

$$e^a = \frac{l}{2}(A^a - \tilde{A}^a) \qquad \omega^a = \frac{1}{2}\epsilon^{abc}\omega_{bc} \quad \omega^a = \frac{1}{2}(A^a + \tilde{A}^a)$$

- CS action (up to redefinition of constants and boundary terms)

$$S_{EH} = S_{CS}[A] - S_{CS}[\tilde{A}]$$

$$S_{CS}[A] = \frac{k}{4\pi}\int tr\left(A \wedge dA + \frac{2}{3}A \wedge A \wedge A\right)$$

Higher spins and Chern-Simons Theory

- Inclusion of higher spins

$$e^{a_1..a_{s-1}} = e_\mu{}^{a_1...a_{s-1}} dx^\mu, \quad \omega^{a_1...a_{s-1}} = \omega_\mu{}^{a_1...a_{s-1}} dx^\mu$$

- In Chern-Simons formulation

$$A^{a_1...a_{s-1}} = \left(\omega + \frac{1}{l} e\right)^{a_1...a_{s-1}}, \quad \tilde{A}^{a_1...a_{s-1}} = \left(\omega - \frac{1}{l} e\right)^{a_1...a_{s-1}},$$

- Why is this important?

Higher spins and Chern-Simons Theory

- We can write A as Lie algebra valued one form, gauge parameter is Lie algebra valued zero form

- We can choose the algebra in which we evaluate A

- Generalization from the EH action to HS theory is generalisation from the

$$so(2,2) \sim sl(2,\mathbb{R}) \oplus sl(2,\mathbb{R})$$

- to $sl(N,\mathbb{R}) \oplus sl(N,\mathbb{R})$

- CS theory with that gauge algebra describes gauge fields of spin s=2,3,…,N

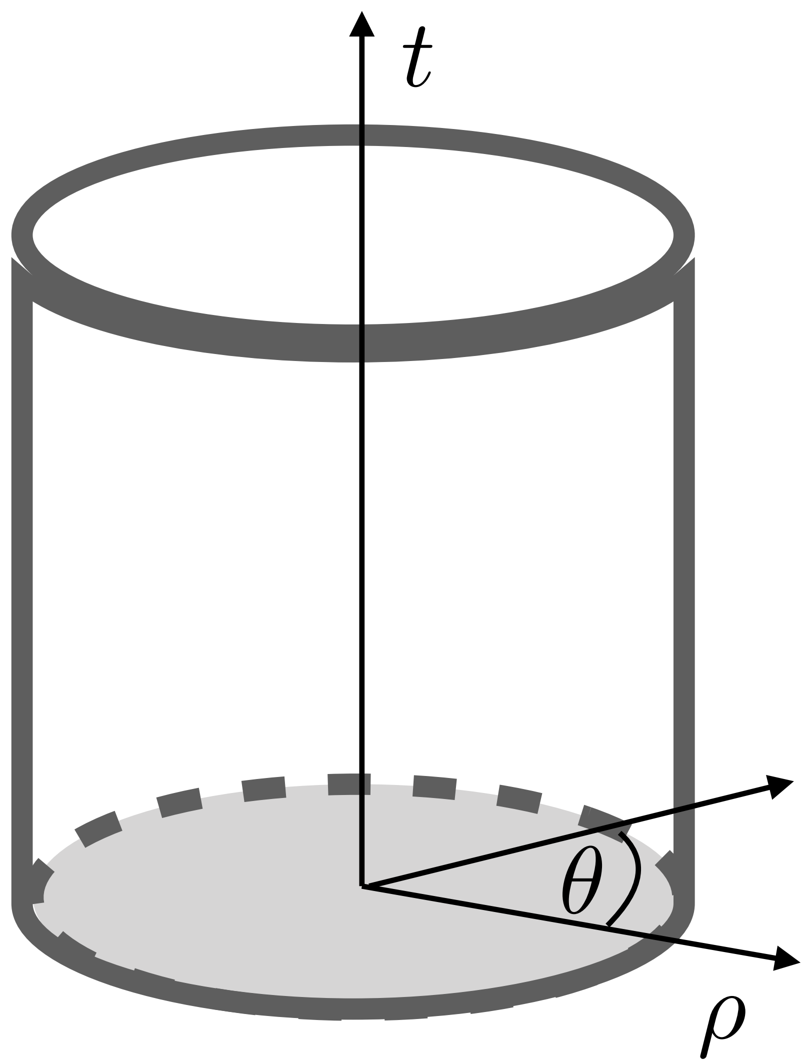
Asymptotic symmetries

- This section talks about asymptotic symmetries of 3d HS theory on a asymptotically AdS spacetime.
- These transformations correspond to global symmetries of the system
- They preserve boundary conditions on the fields while acting on boundary data
- Transformations of this kind contain a subset that is canonically generated by global charges defined on the boundary of spacetime

Asymptotic symmetries

- Here, we start with CS formulation, explain boundary conditions and how they lead to Lie algebra of asymptotic symmetries
- Then - we include additional constraints which characterise asymptotically AdS field configurations
- That leads to reduction of Lie algebra to a classical W algebra - which is called Drinfeld-Sokolov reduction
- For gravity - that leads to Virasoro algebra (Brown, Henneaux)

Asymptotic symmetries



$$\Sigma = \mathbb{R} \times D_2$$

disk

$$S_{CS}[A] = \frac{k}{4\pi} \int_\Sigma tr\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right)$$

$$S_{CS} = \frac{k}{4\pi}\left(\int_\Sigma dt d\rho d\theta tr(A_\theta \dot{A}_\rho - A_\rho \dot{A}_\theta + 2A_t F_{\rho\theta}) - \int_{\partial\Sigma} dt d\theta tr(A_t A_\theta)\right)$$

derivative with respect to *t*

Asymptotic symmetries

- A_t only enters algebraically and enforces the constraint

$$F_{\rho\theta} \equiv \partial_\rho A_\theta - \partial_\theta A_\rho + [A_\rho, A_\theta] = 0$$

$$\delta S_{CS} = \frac{k}{2\pi}\int_\Sigma dtd\rho d\theta \Big((\dot{A}_\rho - \partial_\rho A_t + [A_t, A_\rho])\delta A_\theta - (\dot{A}_\theta - \partial_\theta A_t + [A_t, A_\theta])\delta A_\rho$$

$$+F_{\rho\theta}\delta A_t) - \frac{k}{4\pi}\int_{\partial\Sigma} dtd\theta tr(A_\theta \delta A_t - A_t \delta A_\theta)$$

- boundary conditions on $\delta A_t|_{\partial\Sigma} = 0$ but not on δA_θ

$$S_{bdy} = -\frac{k}{4\pi}\int_{\partial\Sigma} dtd\theta tr(A_t A_\theta)$$

$$A = \mathcal{A}^A t_A$$

$$\{\mathcal{F}, \mathcal{H}\} = \frac{2\pi}{k}\int_{D_2} d\rho d\theta tr\left(\frac{\delta\mathcal{F}}{\delta A_\rho(\rho,\theta)}\frac{\delta\mathcal{H}}{\delta A_\theta(\rho,\theta)} - \frac{\delta\mathcal{F}}{\delta A_\theta(\rho,\theta)}\frac{\delta\mathcal{H}}{\delta A_\rho(\rho,\theta)}\right)$$

Quantum W-algebras and minimal model holography

- Classical W-algebras appear as asymptotic symmetries of HS gauge theories on AdS_3 backgrounds

- They are classical, would they appear as well in a quantised theory?

- example: Virasoro algebra

$$i\{\hat{\mathcal{L}}_m \hat{\mathcal{L}}_n\} = (m-n)\hat{\mathcal{L}}_{m+n} + \delta_{m,-n}\frac{c}{12}m(m^2-1)$$

Quantum W-algebras and minimal model holography

- Quantum Virasoro algebra:

$$[L_m, L_n] = (m-n)L_{m+n} + \delta_{m,-n}\frac{c}{12}m(m^2-1)$$

- it is expected appear in the spirit of holographic correspondence between a gravitational theory on asymptotically AdS_{d+1} spacetimes and a conformal field theory on d-dimensional boundary, for d=2

- Virasoro algebra is linear - replacing classical modes with quantum operators was simple

- HS algebras are non-linear, and we need to be careful of the ordering quantum operators

Quantum W-algebras and minimal model holography

- Conformal field theory (CFT) is a quantum field theory which is invariant under conformal transformations

- In 2D, in Euclidean formulation, it is considered on the compactified complex plane $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$

- Group of global conformal transformations that is mapping $\overline{\mathbb{C}}$ to itself is $PSL(2, \mathbb{C})$

- Special set of fields, that transform covariantly under these transformations $z \to w = f(z)$ are quasi-primary fields

Quantum W-algebras and minimal model holography

$$\phi_{h,\overline{h}}(z,\overline{z}) \to \tilde{\phi}_{h,\overline{h}}(w,\overline{w}) = (f'(z))^{-h}\,\overline{(f'(z))}^{-\overline{h}}\phi_{h,\overline{h}}(z,\overline{z})$$

Conformal weights, real and non-negative in unitary theory

scaling dimension $\Delta = h + \overline{h}$

spin $s = h - \overline{h}$

- Conservation of currents implies that z and $\overline{z}$ are separately conserved

- Currents can be split into holomorphic $W_{s_i}(z)$ and antiholomorphic $\overline{W}_{s_i}(\overline{z})$
- The set of (anti)holomorphic currents is closed under OPE. i.e. OPE of two such fields is again (anti)holomorphic field. These fields define **(quantum) W-algebra**

Coupling to matter

- Formulate equations of motion for the free scalar field on AdS_3 in the so-called **unfolded form**
- This allows to couple it to a HS background
- Unfolded equations of motion are studied in oscillator formulation of HS algebra $\mathfrak{hs}[\lambda]$
- First we want to describe a free scalar field coupled to gravity and to HS fields
- We need a scalar compatible with the first-order frame-like formulation of HS gravity.

Coupling to matter

- To do this, we consider unfolded reformulation of Klein-Gordon equation

$$d\Phi = \overline{h}_a C^a \qquad \overline{h}^a = \delta_\mu{}^a dx^\mu$$

constant vielbein

$$C^a = \overline{h}^{\mu a}\partial_\mu \phi$$

coefficient functions that encode derivatives of the scalar field

$$0 = d^2\Phi = -\overline{h}_a \wedge dC^a$$

- Solution of that eq. can be parametrised by symmetric field $\tilde{C}^{ab}$

$$\tilde{C}^{ab} = \overline{h}^{\mu a}\overline{h}^{\nu b}\partial_\mu\partial_\nu\Phi$$

Coupling to matter

- Impose the Klein-Gordon equation $\Box\Phi = M^2\Phi$, which constrains the trace part

$$dC^a = \overline{h}_b C^{ab} + \frac{1}{3} M^2 \overline{h}^a \Phi$$

- we got undetermined traceless symmetric part of C^{ab}
- applying derivative on it we get a condition on dC^{ab} which has as a solution traceless symmetric C^{abc}

$$dC^{a_1 \ldots a_n} = \overline{h}_b C^{a_1 \ldots a_n b} + \frac{n}{3} M^2 \overline{h}^{(a_1} C^{a_2 \ldots a_n)}$$