The BMS Bootstrap

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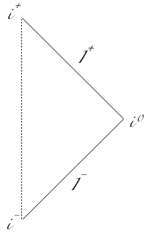
arXiv:1612.01730 [hep-th], Arjun Bagchi, MG, Zodinmawia

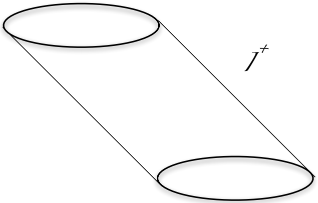
Der Wissenschaftsfonds.

Outline

Flat Space Holography

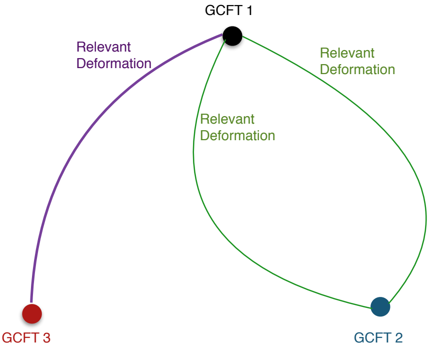
- Dual field theory to gravity in 3d asymptotically flat spacetimes is BMS_3 invariant
- A BMS bootstrap would be a powerful tool for solving the dual field theory
- Provide insights into flat holography, much as the CFT bootstrap has done for AdS holography





Understanding Non-relativistic Field Theories

- BMS_3 algebra is equivalent to GCA_2
- GCFTs are fixed points of non-relativistic RG flows



The BMS$_3$ Algebra

- Poincaré algebra in 3D is $\mathfrak{iso}(2,1)$

$$
\begin{aligned}
[J_n, J_m] &= (n-m)J_{n+m} \\
[J_n, P_m] &= (n-m)P_{n+m} \\
[P_n, P_m] &= 0 \\
n, m &\in \{-1, 0, 1\}
\end{aligned}
$$

- BMS$_3$ is an infinite dimensional generalization with central extensions

$$
\begin{aligned}
[L_n, L_m] &= (n-m)L_{n+m} + c_L(n^3 - n)\delta_{n,-m} \\
[L_n, M_m] &= (n-m)M_{n+m} + c_M(n^3 - n)\delta_{n,-m} \\
[M_n, M_m] &= 0
\end{aligned}
$$

- BMS$_3$ is the asymptotic symmetry algebra of 3d asymptotically flat spacetime [BC07]

BMS from CFT

Gallilean Conformal Algebra is an Inonu-Wigner contraction of 2 copies of the Virasoro algebra

$$[\mathcal{L}_n, \mathcal{L}_m] = (n-m)\mathcal{L}_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n,-m}$$
$$[\overline{\mathcal{L}}_n, \overline{\mathcal{L}}_m] = (n-m)\overline{\mathcal{L}}_{n+m} + \frac{\bar{c}}{12}(n^3 - n)\delta_{n,-m}$$

Let

$$L_n = \mathcal{L}_n + \overline{\mathcal{L}}_n$$
$$M_n = -\epsilon(\mathcal{L}_n - \overline{\mathcal{L}}_n)$$
$$c_L = \frac{c + \bar{c}}{12}$$
$$c_M = \epsilon\frac{c - \bar{c}}{12}$$

GCA arises in the $\epsilon \to 0$ limit [BGMM10]

The CFT Bootstrap: A Lighting Review [ZZ90a, ZZ90b]

- BIG IDEA: Use symmetries to determine correlation functions
- Conformal symmetry comletely fixes the behavior of descendant fields in terms of the primary operators, so restrict to primaries
- Conformal symmetry also completely fixes 2- and 3-point functions of primary operators

$$\langle \mathcal{O}_1(z_1, \bar{z}_1) \mathcal{O}_1(z_2, \bar{z}_2) \rangle = \frac{\delta_{1,2}}{z_{12}^{2h_a} \bar{z}_{12}^{2\bar{h}_a}}$$

$$\langle \mathcal{O}_1(z_1, \bar{z}_1) \mathcal{O}_2(z_2, \bar{z}_2) \mathcal{O}_3(z_3, \bar{z}_3) \rangle = \frac{C_{123}}{z_{12}^{h_{123}} z_{23}^{h_{231}} z_{31}^{h_{312}} \bar{z}_{12}^{\bar{h}_{123}} \bar{z}_{23}^{\bar{h}_{231}} \bar{z}_{31}^{\bar{h}_{312}}}$$

$$z_{ij} = z_i - z_j \qquad h_{ijk} = -(h_i + h_j - h_k)$$

- Use the OPE to reduce the 4-point function to 3-point functions
- Crossing symmetry restricts which primaries $\mathcal{O}_i$ and coefficients C_{ijk} can arise in the OPE

Steps to the Bootstrap

- Representation theory: Primary operators are highest weight states
- Use the global conformal symmetries to determine the form of 2 and 3 point functions
- Determine the form of the OPE
- Fix the behavior of the desendants using local conformal invariance
- Apply the OPE within 4-point functions and define conformal blocks
- Apply crossing symmetry to the conformal blocks

Steps of the BMS Bootstrap

- Representation theory
- Global symmetries determine 2- and 3-point functions
- The OPE
- Local BMS invariance and descendants
- 4-point functions and Conformal Blocks
- Crossing and the Bootstrap Equation

Highest Weight BMS Representations

- Plane representation

$$L_n = -u^{n+1}\partial_u - (n+1)u^n v\partial_v$$
$$M_n = u^{n+1}\partial_v$$

- Highest weight BMS modules

$$L_0 |\Delta, \xi\rangle = \Delta |\Delta, \xi\rangle \quad , \quad M_0 |\Delta, \xi\rangle = \xi |\Delta, \xi\rangle$$
$$L_n |\Delta, \xi\rangle = M_n |\Delta, \xi\rangle = 0 \qquad n > 0$$

- Primary operators are those that, when acting on the vacuum, generate highest weight states

Steps of the BMS Bootstrap

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2- and 3-point Functions

- Completely fixed by the global Poincaré subgroup $L_{0,\pm1}$, $M_{0,\pm1}$.

$$\langle \mathcal{O}_1(u_1, v_1)\mathcal{O}_2(u_2, v_2)\rangle = u_{12}^{-\Delta_1} e^{2\xi_1 \frac{v_{12}}{u_{12}}} \delta_{12}$$

$$\langle \mathcal{O}_1(u_1, v_1)\mathcal{O}_2(u_2, v_2)\mathcal{O}_3(u_3, v_3)\rangle = C_{123}\, u_{12}^{\Delta_{123}}\, u_{23}^{\Delta_{231}}\, u_{31}^{\Delta_{312}}\, e^{\xi_{123}\frac{v_{12}}{u_{12}}}\, e^{\xi_{231}\frac{v_{23}}{u_{23}}}\, e^{\xi_{312}\frac{v_{31}}{u_{31}}}$$

$$\Delta_{ijk} = -\Delta_i - \Delta_j + \Delta_k$$

$$\xi_{ijk} = \xi_i + \xi_j - \xi_k$$

- The two-point is only non-zero for operators of identical L_0, M_0 weight, choose to diagonalize
- C_{ijk} are unfixed by BMS invariance

Steps of the BMS Bootstrap

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The OPE

- General form fixed by demanding both sides transform the same way under L_0

$$\mathcal{O}_1(u,v)\mathcal{O}_2(0) = \sum_{p,\{\vec{k},\vec{q}\}} u^{\Delta_{12p}} e^{\xi_{12p}\frac{v}{u}} \left(\sum_{\alpha=0}^{K+Q} C_{12}^{p,\{\vec{k},\vec{q}\},\alpha} u^{K+Q-\alpha} v^{\alpha} \right) \mathcal{O}_p^{\{\vec{k},\vec{q}\}}$$

$$\mathcal{O}_p^{\{\vec{k},\vec{q}\}} = (L_{-1})^{k_1} \cdots (L_{-l})^{k_l} (M_{-1})^{q_1} \cdots (M_{-j})^{q_j} \mathcal{O}_p$$

- $\mathcal{O}_p^{\{\vec{k},\vec{q}\}}$ is a descendant field at level $K+Q = \sum l k_l + \sum j q_j$
- Using the OPE in a 3-point function reveals $C_{12}^{p,\{0,0\},0} = C_{p12}$, so we write $C_{12}^{p,\{\vec{k},\vec{q}\},\alpha} = C_{p12}\beta_{12}^{p,\{\vec{k},\vec{q}\},\alpha}$

Steps of the BMS Bootstrap

- Representation theory
- Global symmetries determine 2- and 3-point functions
- The OPE
- **Local BMS invariance and descendants**
- 4-point functions and Conformal Blocks
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Descendants and Recursion Relations

- Assume $\mathcal{O}_1 = \mathcal{O}_2$ for simplicity, act w/both sides of the OPE on $|0,0\rangle$

$$\mathcal{O}_1(u,v)\,|\Delta_1,\xi_1\rangle = \sum_p u^{\Delta_{11p}} e^{\xi_{11p}\frac{v}{u}} \sum_{N\geq\alpha} C_{11}^p u^{N-\alpha} v^\alpha \,|N,\alpha\rangle_p$$

$$|N,\alpha\rangle_p = \sum_{\{\vec{k},\vec{q}\}|K+Q=N} \beta_{12}^{p,\{\vec{k},\vec{q}\},\alpha} L_{\vec{k}} M_{\vec{q}}\,|\Delta_p,\xi_p\rangle$$

- $|N,\alpha\rangle_p$ is a descendant at level N, $L_0\,|N,\alpha\rangle_p = (\Delta_p + N)\,|N,\alpha\rangle_p$
- Acting on both sides with lowering operators L_n and M_n and equating coefficients yields recursion relations

$$\begin{aligned} L_n\,|N+n,\alpha\rangle_p &= (N + n\alpha + (n-1)\Delta_1 + \Delta_p)\,|N,\alpha\rangle_p \\ &\quad - n\left((n-1)\xi_1 + \xi_p\right)|N,\alpha-1\rangle_p \\ M_0\,|N,\alpha\rangle_p &= \xi_p\,|N,\alpha\rangle_p - (\alpha+1)\,|N,\alpha+1\rangle_p \\ M_n\,|N+n,\alpha\rangle_p &= ((n-1)\xi_1 + \xi_p)\,|N,\alpha\rangle_p - (\alpha+1)\,|N,\alpha+1\rangle_p \end{aligned}$$

Steps of the BMS Bootstrap

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4-point Functions

- 4-point functions are not completely fixed by global Poincaré symmetry

$$\langle \prod_{i=1}^{4} \mathcal{O}_i(u_i, v_i) \rangle = \prod_{i<j} u_{ij}^{\sum_k \Delta_{ijk}/3} e^{-\frac{v_{ij}}{u_{ij}} \sum_k \xi_{ijk}/3} \mathcal{G}(u, v)$$

$$u = \frac{u_{12} u_{34}}{u_{13} u_{24}} \qquad \frac{v}{u} = \frac{v_{12}}{u_{12}} + \frac{v_{34}}{u_{34}} - \frac{v_{13}}{u_{13}} - \frac{v_{24}}{u_{24}}$$

- Use Poincaré invariance to fix $\{(u_i, v_i)\} \to \{(\infty, 0), (1, 0), (u, v), (0, 0)\}$

$$\langle \mathcal{O}_1 | \, \mathcal{O}_2(1, 0) \mathcal{O}_3(u, v) \, | \mathcal{O}_4 \rangle = G_{34}^{21}(u, v)$$

BMS Blocks

- Starting with the 4-point function $G_{34}^{21}(u,v)$, apply the OPE to $\mathcal{O}_3(u,v)\mathcal{O}_4(0,0)$
- Using the fact that the 2-point function is diagonal, together with the OPE on $\mathcal{O}_1$ and $\mathcal{O}_2$, we find

$$G_{34}^{21}(u,v) = \sum_p C_{12}^p C_{34}^p A_{34}^{21}(p|u,v)$$

- $A(p|u,v)$ depends on $\beta_{34}^{p,\{\vec{k},\vec{q}\},\alpha}$ and is defined in the (u,v) plane in an annulus of radius 1 around $(0,0)$

$$A_{34}^{21}(p|u,v) = (C_{12}^p)^{-1} u^{\Delta_{34p}} e^{\xi_{34p}\frac{v}{u}} \sum_{N\geq\alpha} u^{N-\alpha} v^{\alpha} \langle\mathcal{O}_1|\,\mathcal{O}_2(1,0)\,|N,\alpha\rangle_p$$

Steps of the BMS Bootstrap

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The Bootstrap Equation

- Consider instead of G_{34}^{21} fixing $\{(u_i, v_i)\} \to \{(\infty, 0), (0, 0), (u, v), (1, 0)\}$ and defining $G_{32}^{41}(u, v)$
- Using BMS invariance we find $G_{34}^{21}(u, v) = G_{32}^{41}(1 - u, -v)$
- Applying this to the BMS blocks $A(p|u, v)$, we find the bootstrap equation

$$\sum_p C_{34}^p C_{12}^p A_{34}^{21}(p|u, v) = \sum_q C_{32}^q C_{41}^q A_{32}^{41}(p|1 - u, -v)$$

- If we know the BMS blocks this is a powerful constraint on the allowed values of the OPE coefficients C_{ijk}

The Global Block

- It is hard to find the conformal blocks because it involves solving an infinite set of recursion relations
- In the limit of large c_L, c_M, the equations become tractable
- In this limit, all descendants are generated by $L_{-1}^K M_{-1}^Q$

$$
\begin{aligned}
&\mathcal{O}_3(u,v)\,|O_4\rangle = \\
&\quad \sum_{p,K,Q} u^{\Delta_{34p}} e^{\xi_{34p}\frac{v}{u}} C_{34}^{p} \left(\sum_{\alpha=0}^{K+Q} \beta_{34}^{p,K,Q,\alpha} u^{K+Q-\alpha} v^{\alpha} \right) L_{-1}^K M_{-1}^Q \,|\Delta_p, \xi_p\rangle \\
&\quad + \mathcal{O}\left(1/c_L, 1/c_M\right)
\end{aligned}
$$

- Descendants that would arise from L_{-k}, M_{-q} with $k, q > 1$ generate terms of order $1/c_L, 1/c_M$ in the recursion relations
- Demanding both sides of the OPE transform identically under the quadratic casimirs of the global Poincaré algebra determines the global block $g_{34}^{21}(p|u,v)$

- Quadratic Casimirs of the global Poincaré algebra

$$\mathcal{C}_1 = M_0^2 - M_{-1}M_1$$
$$\mathcal{C}_2 = 2L_0M_0 - \frac{1}{2}\left(L_{-1}M_1 + L_1M_{-1} + M_1L_{-1} + M_{-1}L_1\right)$$

- $L_{-1}^K M_{-1}^Q \left|\mathcal{O}_p\right\rangle$ are eigenstates of $\mathcal{C}_1, \mathcal{C}_2$

$$\mathcal{C}_i L_{-1}^K M_{-1}^Q \left|\mathcal{O}_p\right\rangle = \lambda_i^p L_{-1}^K M_{-1}^Q \left|\mathcal{O}_p\right\rangle$$
$$\lambda_1^p = \xi_p^2 \qquad \lambda_2^p = (2\Delta_p\xi_p - 2\xi_p)$$

- Quadratic Casimirs act as differential operators on the left side of the OPE

$$\mathcal{C}_1\mathcal{O}_3(u,v)\left|\mathcal{O}_4\right\rangle = \left((\xi_3+\xi_4)^2 - 2\xi_3 u\partial_v\right)\mathcal{O}_3(u,v)\left|\mathcal{O}_4\right\rangle$$
$$\mathcal{C}_2\mathcal{O}_3(u,v)\left|\mathcal{O}_4\right\rangle = (2(\Delta_3+\Delta_4-1)(\xi_3+\xi_4) + 2u\xi_4\partial_u$$
$$+(2v\xi_4 - 2u\Delta_4)\partial_v)\,\mathcal{O}_3(u,v)\left|\mathcal{O}_4\right\rangle$$

- Act from the left with $\langle\mathcal{O}_1|\,\mathcal{O}_2(1,0)$ on both sides of the OPE and using the definition of the global conformal block gives an infinte set of PDEs

$$\mathcal{C}_i \sum_p C_{12}^p C_{34}^p g_{34}^{12}(p|u,v) = \sum_p \lambda_i^p C_{12}^p C_{34}^p g_{34}^{12}(p|u,v)$$

- These can be decoupled to two PDEs satisfied by each conformal block

$$\mathcal{C}_i g_{34}^{21}(p|u,v) = \lambda_i^p g_{34}^{21}(p|u,v)$$

- In the special case of identical operators, we find

$$g_{\Delta,\xi}(p|u,v) = \frac{2^{2\Delta_p-2}}{\sqrt{1-u}} e^{\frac{-\xi_p v}{u\sqrt{1-u}}+2\xi\frac{v}{u}} u^{\Delta_p-2\Delta}\left(1+\sqrt{1-u}\right)^{2-2\Delta_p}$$

Open Issues and Ongoing Research

- Liouville Theory
- More general conformal blocks (chiral block, etc)
- Minimal Models?
- Applications to Flat Holography
- Induced representations and the Ultra-Relativistic limit

Thank You

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