Dual Giant Gravitons & Emergent Curvature

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Motivation

- Understand emergent geometry in AdS/CFT.
- Anomalies are protected, so CFT calculations can be trusted.
- Explore $1/N^2$ and α' corrections.
- New predictions/tests for AdS/CFT.

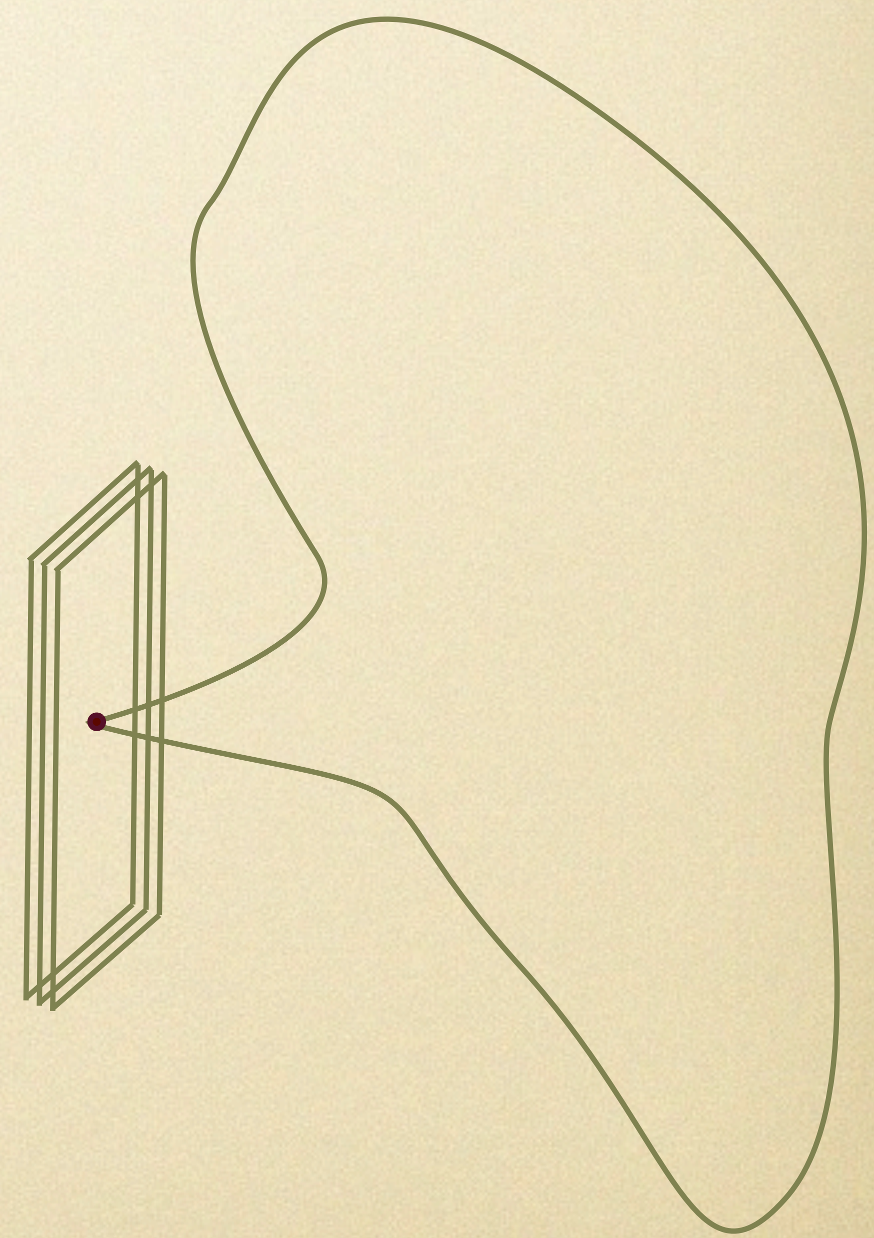
Outline

- Review of AdS/CFT & the Geometric Setup
- The Hilbert Series
- Dual Giant Gravitons
- The Hilbert Series Revisited
- Curvature from Counting
- Conclusions & Future Directions

Geometric Setup

- Compactify IIB String Theory on Calabi-Yau 3-fold X
- Place stack of N D3-branes at a singular point in X
- Near-Horizon Geometry is $AdS_5 \times L^5$ where L^5 is a Sasaki-Einstein manifold
- Metric cone over L^5 is X^6

$$ds_X^2 = dr^2 + r^2 ds_L^2$$



Dual Description

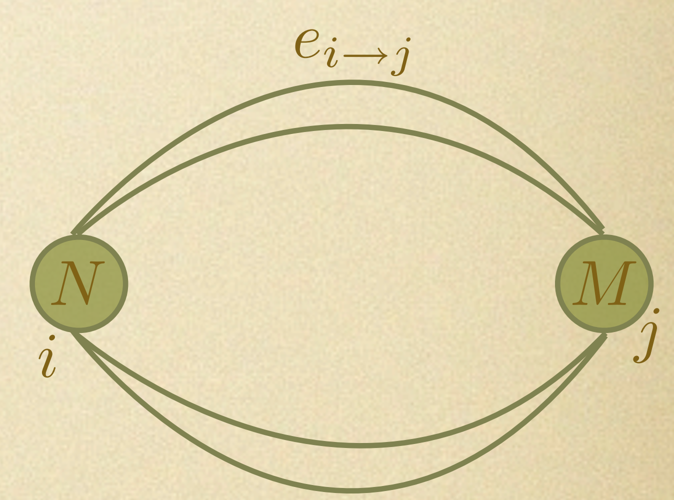
- The dual description is an $\mathcal{N}$=1 SCFT, the low energy theory on the stack of D3-branes.
- The $\mathcal{N}$=1 SCFT has a global U(1) R-Symmetry.
- The U(1) R-symmetry is dual to the U(1) generated by the Reeb vector $\xi = J(r\partial_r)$.

Specifics

- We consider theories arising from local CY Singularities X = Spec(R) where R is a Gorenstein ring.
- Consider only SCFTs from quiver gauge theories with superpotential algebra of form $A = \mathrm{End}_R(R + M_1 + \cdots + M_n)$.
- Often further restrict to X the total space of a (complex) line bundle $\mathcal{L} \to B^4$.

Hilbert Series

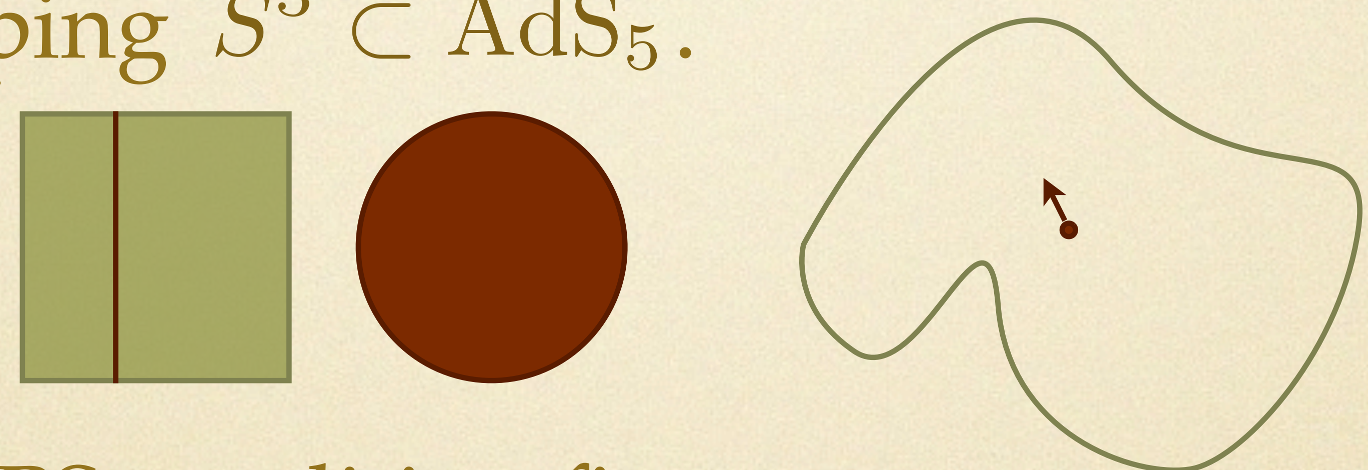
- The Hilbert Series of a gauge theory is defined to be $H(t) = \sum_{\mathcal{O}} t^{R(\mathcal{O})}$ where $\mathcal{O}$ are mesons and $R(\mathcal{O})$ is the R-charge.
- We define an adjacency matrix
$$M_{ij}(t) = \sum_{e \in \text{Arrows}(i \to j)} t^{R(e)}$$



- The Hilbert Series is given by the (0,0) component of
$$\mathbb{H} = \frac{1}{1 - \mathbb{M}(t) + t^2 \mathbb{M}^T(1/t) - t^2}$$

Dual Giant Gravitons

- Dual Giant Gravitons are BPS D3-branes wrapping $S^3 \subset \mathrm{AdS}_5$.



- The BPS condition fixes:
  - The radial position of the brane.
  - The position of the brane in B^4.

$$\begin{array}{ccc} \mathbb{C} & \rightarrow & X^6 \\ & & \downarrow \\ & & B^4 \end{array} \quad \xrightarrow{\text{restrict}} \quad \begin{array}{ccc} S^1 & \rightarrow & L^5 \\ & & \downarrow \\ & & B^4 \end{array}$$

Dual Giant Gravitons

- The Hamiltonian is equivalent to a BPS point particle moving in L^5.

$$H_{\rm BPS} = \frac{1}{\ell_{\rm AdS}} P_\psi$$

ψ is defined by $\xi = J(r\partial_r) = \partial_\psi$.

- Martelli & Sparks quantize this system by Geometric Quantization

Dual Giant Gravitons

- Martelli & Sparks compute the partition function for dual giant gravitons
$$Z(\beta) = \mathrm{Tr}_{\mathcal{H}} e^{-\beta H}$$
and show that the space of states is precisely the space of holomorphic functions on X^6.
    - Key Point: H generates flow along ξ.
- They show the partition function can be written as an equivariant character under the action of the Reeb vector and is the Hilbert Series.

Geodesic Motion

- Any Sasakian metric can locally be written in the form $ds_L^2 = ds_B^2 + (\frac{1}{3}d\psi + A)^2$.
- Consider the general case of motion on
$$ds^2 = -dt^2 + ds_B^2 + (\frac{1}{3}d\psi + A)^2$$
- Geodesic motion can be derived from the action
$$S = \int d\tau \left[-\dot{t}^2 + h_{ij}\dot{x}^i\dot{x}^j + (\frac{1}{3}\dot{\psi} + A_i\dot{x}^i)^2 \right]$$

BPS Geodesic Motion

$$S = \int d\tau \left[-\dot{t}^2 + h_{ij}\dot{x}^i\dot{x}^j + (\frac{1}{3}\dot{\psi} + A_i\dot{x}^i)^2 \right]$$

$$H = -p_t^2 + p_\psi^2 + h^{ij}(p_i - p_\psi A_i)(p_j - p_\psi A_j)$$

- $\partial_t, \partial_\psi$ are isometries, so $E = p_t, q = p_\psi$ are conserved quantities.
- On-shell solutions have H=0, so $E^2 = q^2 + h^{ij}(p_i - qA_i)(p_j - qA_j)$.
- There is a BPS bound $E^2 \geq q^2$ and the BPS states satisfy $p_i = qA_i, \dot{x}^i = 0, \dot{\psi} = q$.
- BPS geodesics are orbits of ξ with fixed momentum q.

Kaluza-Klein Reduction

$$ds_L^2 = h_{ij}dx^i dx^j + \phi^2(\frac{1}{3}d\psi + A)^2$$

- Dimensionally reduce the problem of geodesic motion on L^5 to motion of a charged particle on B^4 in a background magnetic field.

- The geodesic equation reduces to
$$\ddot{x}^i + \Gamma^i_{jk}\dot{x}^j\dot{x}^k = \frac{q}{m}F^i{}_j\dot{x}^j .$$

- The connection is proportional to the Kahler form on B^4.

Kaluza-Klein Reduction

- Fix re-parametrization invariance by choosing $t = \tau$.
- Fix to a holomorphic gauge for A
  $$S_{\mathrm{BPS,fixed}} = \int dt(-1 + h_{i\bar{\jmath}}\dot{x}^i\dot{x}^{\bar{\jmath}} + A_i\dot{x}^i + \bar{A}_{\bar{\jmath}}\dot{x}^{\bar{\jmath}})$$
- This action was studied by Douglas & Klevtsov, who counted the number of states in the Lowest Landau Level at large magnetic field.

Hilbert Series Revisited

- For a Calabi-Yau singularity X that is the total space of a line bundle $\mathcal{L} \to B^4$,
$$H(t,X) = \sum_{k=0}^{\infty} g(k) t^k \ .$$
- $g(k) = \mathrm{dim}\mathcal{H}^0(B, \mathcal{L}^{\otimes k})$
- At large k, $g(k) = ak^2 + bk + c + \mathcal{O}(1/k)$.

Hilbert Series Revisited

- Douglas & Klevtsov use the Tian-Yau-Zelditch asymptotic expansion to determine the number of states in the Lowest Landau Level in terms of the curvature of B^4 and the magnetic field.

$$g(k) = \frac{1}{2}\left(k^2 + k\frac{R}{2} + \left(\frac{1}{24}\text{Riem}^2 - \frac{1}{6}\text{Ric}^2 + \frac{1}{8}R^2\right)\right) + \mathcal{O}(1/k)$$

- Reversing the Kaluza-Klein reduction, we are able to re-write the TYZ expansion purely in terms of quantities defined on L^5.

Hilbert Series Revisited

- Since L^5 and B^4 are both Einstein spaces, R_{ij} and R can be fixed by convention.

$$ {}^{(4)}R_{ij} = 5h_{ij} \qquad {}^{(5)}R_{ij} = 4g_{ij} $$

- The only non-trivial calculation is the lift of Riem^2.

$$ {\mathrm{Riem}_L}^2 = {\mathrm{Riem}_B}^2 - 152 $$

Hilbert Series Revisited

- The general form of the Hilbert Series at large k is
$$H(t,X)=\sum_{k=0}^{\infty} g(k)t^k=\sum_{k=0}^{\infty}(ak^2+bk+c)t^k$$
- The Hilbert series has a pole at t=1. We expand about the singularity:
$$\lim_{s\to 0} H(e^{-s})=\frac{2a}{s^3}+\frac{b}{s^2}+\frac{c}{s}+\mathcal{O}(1)$$
- Using the lifted form of the TYZ expansion, we find
$$\lim_{s\to 0}\left(\frac{3\pi}{2}\right)^3 H(e^{-\frac{2}{3}s})=\frac{\mathrm{vol}}{s^3}+\frac{\mathrm{vol}}{s^2}+\left(\frac{91\mathrm{vol}}{216}+\frac{\mathrm{Riem}^2}{1728}\right)\frac{1}{s}+\mathcal{O}(1)$$

Curvature from Counting

- The result that the leading singularity in the Hilbert series is proportional to the volume of L^5 is well known from the work of Martelli, Sparks and Yau.

$$\lim_{s \to 0} s^3 H(e^{-s}) = \text{vol}$$

- The fact that the sub-leading term in the asymptotic expansion is also proportional to the volume follows from the Calabi-Yau condition on X^6.

Curvature from Counting

- That the order 1/s term in the asymptotic expansion of the Hilbert Series can be expressed in terms of the curvature of L^5 is non-trivial and a new result.

$$\left(\frac{91}{216}\mathrm{vol} + \frac{1}{1728}\int_{L^5} \mathrm{Riem}^2\right)\frac{1}{s}$$

Examples: S^5

- $X = \mathbb{C}^3, L = S^5, B = \mathbb{CP}^2$
- $\dim \mathcal{H}^0(\mathbb{CP}^2, \mathcal{O}(1)) = \dfrac{(k+1)(k+2)}{2}$
- $H(t) = \displaystyle\sum_{k=0}^{\infty} \frac{(k+1)(k+2)}{2} t^k = \frac{1}{(1-t)^3}$
- $\displaystyle\lim_{s \to 0} \pi^3 H(e^{-\frac{2}{3}s}) = \frac{27\pi^3}{8s^3} + \frac{27\pi^3}{8s^2} + \frac{3\pi^3}{2s}$
- $\mathrm{vol} = \pi^3$ $\qquad \displaystyle\int_{S^5} \mathrm{Riem}^2 = 40\pi^3$

Examples: $S^5/\mathbb{Z}_3$

- $X = \mathbb{C}^3/\mathbb{Z}_3, L = S^5/\mathbb{Z}_3$

- $H(t) = \dfrac{1 + 7t^2 + t^4}{(1 - t^2)^3}$

- $\lim_{s\to 0} \pi^3 H(t) = \dfrac{9\pi^3}{8s^3} + \dfrac{9\pi^3}{8s^2} + \dfrac{\pi^3}{2s}$

- $\text{vol} = \dfrac{\pi^3}{3} \qquad \int_{S^5/\mathbb{Z}_3} \text{Riem}^2 = \dfrac{40\pi^3}{3}$

Examples: $T^{1,1}$

- $X = \text{Conifold}, L = T^{1,1}$

- $H(t) = \dfrac{1+t}{(1-t)^3}$

- $\lim_{s\to 0} \pi^3 H(e^{-s}) = \dfrac{2\pi^3}{s^3} + \dfrac{2\pi^3}{s^2} + \dfrac{\pi^3}{s}$

- $\text{vol} = \dfrac{16\pi^3}{27} \qquad \int_{T^{1,1}} \text{Riem}^2 = \dfrac{2176\pi^3}{27}$

Examples: $Y^{2,1}$

- $$H(t) = \frac{8t^{2\sqrt{13}/3}}{27(t - t^{(1+2\sqrt{13})/3})^2(t^5 - t^{(1+2\sqrt{13})/3})^2}$$

  $$f(t) = 2t^{\frac{22}{3}} + t^{\frac{2(4+\sqrt{13})}{3}} + 3t^{\frac{2(7+\sqrt{13})}{3}} - 3t^{\frac{2(10+\sqrt{13})}{3}} - t^{\frac{2(13+\sqrt{13})}{3}} - 2t^{\frac{4(3+\sqrt{13})}{3}}$$

- $$\lim_{s\to 0} \pi^3 H(e^{-s}) = \frac{(46+13\sqrt{13})\pi^3}{324s^3} + \frac{(46+13\sqrt{13})\pi^3}{324s^2} + \frac{(22+7\sqrt{13})\pi^3}{324s}$$

- $$\mathrm{vol} = \frac{(46+13\sqrt{13})\pi^3}{324}$$

  $$\int_{Y^{2,1}} \mathrm{Riem}^2 = \frac{2\pi^3}{81}(566 + 329\sqrt{13})$$

Conclusions

- We are able to compute $\int_{L^5} \mathrm{Riem}^2$ purely from gauge theory data.
- While our proof is only valid in the regular case, we conjecture that the formula also holds in the quasi-regular case and provide evidence.
- On the gauge theory side, the sub-leading terms in the Hilbert Series are related to $1/N^2$ corrections to $a - c$, a topic explored in recent work by Minasian and Liu

Future Directions

- Generalize the proof to the quasi-regular case by properly treating singular points.
- The fact that the sub-leading terms can be written purely in terms of 5d quantities hints at a direct 5d proof.
- Extend to Generalized Complex Geometry, as recently considered by Gabella and Sparks.
- Understand relation to higher curvature corrections to bulk action.