Dual Giant Gravitons
&
Emergent Curvature

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Motivation

- Understand emergent geometry in AdS/CFT.
- Anomalies are protected, so CFT calculations can be trusted.
- Explore $1/N^2$ and $\alpha'$ corrections.
- New predictions/tests for AdS/CFT.
Outline

- Review of AdS/CFT & the Geometric Setup
- The Hilbert Series
- Dual Giant Gravitons
- The Hilbert Series Revisited
- Curvature from Counting
- Conclusions & Future Directions
Geometric Setup

- Compactify IIB String Theory on Calabi-Yau 3-fold $X$

- Place stack of $N$ D3-branes at a singular point in $X$

- Near-Horizon Geometry is $\text{AdS}_5 \times L^5$ where $L^5$ is a Sasaki-Einstein manifold

- Metric cone over $L^5$ is $X^6$

$$ds^2_X = dr^2 + r^2 ds^2_L$$
Dual Description

- The dual description is an $\mathcal{N}=1$ SCFT, the low energy theory on the stack of D3-branes.
- The $\mathcal{N}=1$ SCFT has a global U(1) R-Symmetry.
- The U(1) R-symmetry is dual to the U(1) generated by the Reeb vector $\xi = J(r \partial_r)$. 

Specifics

- We consider theories arising from local CY Singularities $X = \text{Spec}(R)$ where $R$ is a Gorenstein ring.

- Consider only SCFTs from quiver gauge theories with superpotential algebra of form $A = \text{End}_R(R + M_1 + \cdots + M_n)$.

- Often further restrict to $X$ the total space of a (complex) line bundle $\mathcal{L} \to B^4$. 

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The Hilbert Series of a gauge theory is defined to be
$$H(t) = \sum_{\mathcal{O}} t^{R(\mathcal{O})}$$
where $\mathcal{O}$ are mesons and $R(\mathcal{O})$ is the R-charge.

We define an adjacency matrix
$$M_{ij}(t) = \sum_{e \in \text{Arrows}(i \rightarrow j)} t^{R(e)}$$

The Hilbert Series is given by the $(0,0)$ component of
$$H = \frac{1}{1 - M(t) + t^2 M^T(1/t) - t^2}$$
Dual Giant Gravitons

- Dual Giant Gravitons are BPS D3-branes wrapping $S^3 \subset \text{AdS}_5$.

- The BPS condition fixes:
  - The radial position of the brane.
  - The position of the brane in $B^4$.

\[ \mathbb{C} \rightarrow X^6 \quad \text{restrict} \quad S^1 \rightarrow L^5 \]

\[ \downarrow \quad \downarrow \]

\[ B^4 \quad B^4 \]
The Hamiltonian is equivalent to a BPS point particle moving in $L^5$.

$$H_{\text{BPS}} = \frac{1}{\ell_{\text{AdS}}} P_\psi$$

$\psi$ is defined by $\xi = J(r \partial_r) = \partial_\psi$.

Martelli & Sparks quantize this system by Geometric Quantization.
Dual Giant Gravitons

- Martelli & Sparks compute the partition function for dual giant gravitons
  \[ Z(\beta) = \text{Tr}_\mathcal{H} e^{-\beta H} \]
  and show that the space of states is precisely the space of holomorphic functions on \( X^6 \).

  - Key Point: \( H \) generates flow along \( \xi \).

- They show the partition function can be written as an equivariant character under the action of the Reeb vector and is the Hilbert Series.
Geodesic Motion

- Any Sasakian metric can locally be written in the form \( ds^2_L = ds^2_B + \left( \frac{1}{3} d\psi + A \right)^2 \).

- Consider the general case of motion on \( ds^2 = -dt^2 + ds^2_B + \left( \frac{1}{3} d\psi + A \right)^2 \).

- Geodesic motion can be derived from the action

\[
S = \int d\tau \left[ -\dot{t}^2 + h_{ij} \dot{x}^i \dot{x}^j + \left( \frac{1}{3} \dot{\psi} + A_i \dot{x}^i \right)^2 \right]
\]
BPS Geodesic Motion

\[ S = \int d\tau \left[ -\dot{t}^2 + h_{ij} \dot{x}^i \dot{x}^j + \left( \frac{1}{3} \dot{\psi} + A_i \dot{x}^i \right)^2 \right] \]

\[ H = -p_t^2 + p_\psi^2 + h^{ij} (p_i - p_\psi A_i) (p_j - p_\psi A_j) \]

- \( \partial_t, \partial_\psi \) are isometries, so \( E = p_t, q = p_\psi \) are conserved quantities.

- On-shell solutions have \( H=0 \), so \( E^2 = q^2 + h^{ij} (p_i - q A_i) (p_j - q A_j) \).

- There is a BPS bound \( E^2 \geq q^2 \) and the BPS states satisfy \( p_i = q A_i, \dot{x}^i = 0, \dot{\psi} = q \).

- BPS geodesics are orbits of \( \xi \) with fixed momentum \( q \).
Kaluza-Klein Reduction

\[ ds^2_L = h_{ij} dx^i dx^j + \phi^2 \left( \frac{1}{3} d\psi + A \right)^2 \]

- Dimensionally reduce the problem of geodesic motion on \( L^5 \) to motion of a charged particle on \( B^4 \) in a background magnetic field.

- The geodesic equation reduces to
  \[ \ddot{x}^i + \Gamma^i_{jk} \dot{x}^j \dot{x}^k = \frac{q}{m} F^i_{jk} \dot{x}^j. \]

- The connection is proportional to the Kahler form on \( B^4 \).
Kaluza-Klein Reduction

- Fix re-parametrization invariance by choosing $t = \tau$.

- Fix to a holomorphic gauge for $A$

$$S_{\text{BPS, fixed}} = \int dt(-1 + h_{i\bar{j}} \dot{x}^i \dot{x}^\bar{j} + A_i \dot{x}^i + \bar{A}_j \dot{x}^\bar{j})$$

- This action was studied by Douglas & Klevtsov, who counted the number of states in the Lowest Landau Level at large magnetic field.
Hilbert Series Revisited

- For a Calabi-Yau singularity $X$ that is the total space of a line bundle $\mathcal{L} \to B^4$, 
  \[ H(t, X) = \sum_{k=0}^{\infty} g(k) t^k. \]

- $g(k) = \dim \mathcal{H}^0(B, \mathcal{L}^\otimes k)$

- At large $k$, $g(k) = ak^2 + bk + c + O(1/k)$. 
Douglas & Klevtsov use the Tian-Yau-Zelditch asymptotic expansion to determine the number of states in the Lowest Landau Level in terms of the curvature of $B^4$ and the magnetic field.

$$g(k) = \frac{1}{2} \left( k^2 + k \frac{R}{2} + \left( \frac{1}{24} \text{Riem}^2 - \frac{1}{6} \text{Ric}^2 + \frac{1}{8} R^2 \right) \right) + \mathcal{O}(1/k)$$

Reversing the Kaluza-Klein reduction, we are able to re-write the TYZ expansion purely in terms of quantities defined on $L^5$. 
Hilbert Series Revisited

- Since $L^5$ and $B^4$ are both Einstein spaces, $R_{ij}$ and $R$ can be fixed by convention.

  \[(4) R_{ij} = 5h_{ij} \quad (5) R_{ij} = 4g_{ij}\]

- The only non-trivial calculation is the lift of $\text{Riem}^2$. 

  $\text{Riem}_L^2 = \text{Riem}_B^2 - 152$
The general form of the Hilbert Series at large \( k \) is
\[
H(t, X) = \sum_{k=0}^{\infty} g(k) t^k = \sum_{k=0}^{\infty} (a k^2 + b k + c) t^k
\]

The Hilbert series has a pole at \( t=1 \). We expand about the singularity:
\[
\lim_{s \to 0} H(e^{-s}) = \frac{2a}{s^3} + \frac{b}{s^2} + \frac{c}{s} + \mathcal{O}(1)
\]

Using the lifted form of the TYZ expansion, we find
\[
\lim_{s \to 0} \left( \frac{3\pi}{2} \right)^3 H(e^{-\frac{2}{3}s}) = \frac{\text{vol}}{s^3} + \frac{\text{vol}}{s^2} + \left( \frac{91\text{vol}}{216} + \frac{\text{Riem}^2}{1728} \right) \frac{1}{s} + \mathcal{O}(1)
\]
The result that the leading singularity in the Hilbert series is proportional to the volume of $L^5$ is well known from the work of Martelli, Sparks and Yau.

$$\lim_{s \to 0} s^3 H(e^{-s}) = \text{vol}$$

The fact that the sub-leading term in the asymptotic expansion is also proportional to the volume follows from the Calabi-Yau condition on $X^6$. 
Curvature from Counting

That the order $1/s$ term in the asymptotic expansion of the Hilbert Series can be expressed in terms of the curvature of $L^5$ is non-trivial and a new result.

$$\left(\frac{91}{216}\text{vol} + \frac{1}{1728} \int_{L^5} \text{Riem}^2\right) \frac{1}{s}$$
Examples: $S^5$

- $X = \mathbb{C}^3$, $L = S^5$, $B = \mathbb{CP}^2$

- $\dim \mathcal{H}^0(\mathbb{CP}^2, \mathcal{O}(1)) = \frac{(k + 1)(k + 2)}{2}$

- $H(t) = \sum_{k=0}^{\infty} \frac{(k + 1)(k + 2)}{2} t^k = \frac{1}{(1 - t)^3}$

- $\lim_{s \to 0} \pi^3 H(e^{-\frac{2}{3} s}) = \frac{27\pi^3}{8s^3} + \frac{27\pi^3}{8s^2} + \frac{3\pi^3}{2s}$

- $\text{vol} = \pi^3 \int_{S^5} \text{Riem}^2 = 40\pi^3$
Examples: $S^5 / \mathbb{Z}_3$

- $X = \mathbb{C}^3 / \mathbb{Z}_3, L = S^5 / \mathbb{Z}_3$

- $H(t) = \frac{1 + 7t^2 + t^4}{(1 - t^2)^3}$

- $\lim_{s \to 0} \pi^3 H(t) = \frac{9\pi^3}{8s^3} + \frac{9\pi^3}{8s^2} + \frac{\pi^3}{2s}$

- $\text{vol} = \frac{\pi^3}{3} \int_{S^5 / \mathbb{Z}_3} \text{Riem}^2 = \frac{40\pi^3}{3}$
Examples: $T^{1,1}$

- $X = \text{Conifold, } L = T^{1,1}$

- $H(t) = \frac{1 + t}{(1 - t)^3}$

- $\lim_{s \to 0} \pi^3 H(e^{-s}) = \frac{2\pi^3}{s^3} + \frac{2\pi^3}{s^2} + \frac{\pi^3}{s}$

- $\text{vol} = \frac{16\pi^3}{27} \int_{T^{1,1}} \text{Riem}^2 = \frac{2176\pi^3}{27}$
Examples: $Y^{2,1}$

- $H(t) = \frac{8t^2\sqrt{13}/3}{27(t - t^{(1+2\sqrt{13})/3})^2(t^5 - t^{(1+2\sqrt{13})/3})^2}$

- $f(t) = 2t^{\frac{22}{3}} + t^{\frac{2(4+\sqrt{13})}{3}} + 3t^{\frac{2(7+\sqrt{13})}{3}} - 3t^{\frac{2(10+\sqrt{13})}{3}} - t^{\frac{2(13+\sqrt{13})}{3}} - 2t^{\frac{4(3+\sqrt{13})}{3}}$

- $\lim_{s \to 0} \pi^3 H(e^{-s}) = \frac{(46 + 13\sqrt{13})\pi^3}{324s^3} + \frac{(46 + 13\sqrt{13})\pi^3}{324s^2} + \frac{(22 + 7\sqrt{13})\pi^3}{324s}$

- $\text{vol} = \frac{(46 + 13\sqrt{13})\pi^3}{324}$

- $\int_{Y^{2,1}} \text{Riem}^2 = \frac{2\pi^3}{81} (566 + 329\sqrt{13})$
Conclusions

- We are able to compute $\int_{L^5} \text{Riem}^2$ purely from gauge theory data.

- While our proof is only valid in the regular case, we conjecture that the formula also holds in the quasi-regular case and provide evidence.

- On the gauge theory side, the sub-leading terms in the Hilbert Series are related to $1/N^2$ corrections to $a - c$, a topic explored in recent work by Minasian and Liu.
Future Directions

- Generalize the proof to the quasi-regular case by properly treating singular points.
- The fact that the sub-leading terms can be written purely in terms of 5d quantities hints at a direct 5d proof.
- Extend to Generalized Complex Geometry, as recently considered by Gabella and Sparks.
- Understand relation to higher curvature corrections to bulk action.