

Higher Spin Gravity in 3-Dimensions and Unitarity

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Outline

- 1 Motivation
- 2 Gravity in 3 Dimensions
- 3 Higher Spins

Motivation

- Black hole evaporation and unitarity
- Role of geometry in quantum gravity
- Experience of infalling observers?
- Microscopic description of black hole entropy?
- Want one theory in which semiclassical and quantum regime can both be understood

Gravity in 3D with $\Lambda \leq 0$

$$I = \frac{1}{16\pi G_N} \int \sqrt{-g} (R + 2)$$

- One dimensionless coupling G_N (in units of curvature radius)
- Vacuum solution: Anti de-Sitter space (AdS)
- Admits black holes with finite horizon size [BTZ92]
- (Boundary) Gravitons

First Order Formalism

- Dreibein e_μ^a and spin connection ω_μ^{ab} independent variables
- 3D trick: use invariant antisymmetric rank 3 symbol ϵ^{abc} to construct ω_μ^a
- Action becomes a difference of two \mathfrak{sl}_2 Chern-Simons theories [AT86, Wit88]

$$I = \frac{k}{4\pi} \left(\int \text{tr} \left[A \wedge dA + \frac{2}{3} A^3 \right] - \int \text{tr} \left[\bar{A} \wedge d\bar{A} + \frac{2}{3} \bar{A}^3 \right] \right)$$

where $A = \frac{1}{2}(\omega + e)$, $\bar{A} = \frac{1}{2}(\omega - e)$, $k = \frac{1}{4G_N}$

- $g_{\mu\nu} = \frac{1}{2} \text{tr} [(A - \bar{A})_\mu (A - \bar{A})_\nu]$

First Order Formalism: Gauge Symmetries

- Gauge symmetries

$$\delta_{\epsilon} A = d\epsilon + [A, \epsilon] \qquad \delta_{\bar{\epsilon}} \bar{A} = d\bar{\epsilon} + [\bar{A}, \bar{\epsilon}]$$

- $\epsilon - \bar{\epsilon}$ generate local translations (diffeomorphisms)
- $\epsilon + \bar{\epsilon}$ generate local Lorentz transformations

$$\begin{aligned} \delta_{\epsilon - \bar{\epsilon}} e &= d(\epsilon - \bar{\epsilon}) + [\omega, \epsilon - \bar{\epsilon}] & \delta_{\epsilon - \bar{\epsilon}} \omega &= [e, \epsilon - \bar{\epsilon}] \\ \delta_{\epsilon + \bar{\epsilon}} e &= [e, \epsilon + \bar{\epsilon}] & \delta_{\epsilon + \bar{\epsilon}} \omega &= d(\epsilon + \bar{\epsilon}) + [\omega, \epsilon + \bar{\epsilon}] \end{aligned}$$

Canonical Analysis

- Imposing appropriate boundary conditions \rightarrow gravity in asymptotically AdS_3

$$ds^2 = d\rho^2 + (e^{2\rho}\eta_{\mu\nu} + \mathcal{O}(1)) dx^\mu dx^\nu$$

- Canonical Analysis [BH86], originally in 2nd order formalism
- Later repeated in 1st order formalism, same result, but simpler to understand

Connection and Boundary Conditions

- Partially gauge fix $A = b^{-1}ab$, $\bar{A} = b\bar{a}b^{-1}$ with $b = e^{\rho L_0}$
- Split connection into background, state-dependent fluctuations, and state-independent (subleading in ρ) fluctuations $a = \hat{a}^{(0)}(t, \phi) + a^{(0)}(t, \phi) + a^{(1)}(\rho, t, \phi)$
- It is necessary [convenient] for $\hat{a}^{(0)}$ [$a^{(0)}$] to satisfy the asymptotic equations of motion $F = 0 = \bar{F}$

Boundary Conditions for Asymptotically AdS_3

$$\hat{a}^{(0)} = L_0 d\rho + L_1 dx^+$$

$$a^{(0)} = \mathcal{L}(x^+) L_{-1} dx^+$$

$$a^{(1)} = \mathcal{O}(e^{-\rho})$$

$$\hat{\bar{a}}^{(0)} = -L_0 d\rho - L_{-1} dx^-$$

$$\bar{a}^{(0)} = \bar{\mathcal{L}}(x^-) L_1 dx^-$$

$$\bar{a}^{(1)} = \mathcal{O}(e^{-\rho})$$

Canonical Analysis of CS Theories I: Hamiltonian

- Convenient to use a $2 + 1$ decomposition

$$I_{CS}[A] = \frac{k}{4\pi} \int_{\mathbb{R}} dt \int_{\mathcal{D}} d^2x \epsilon^{ij} \kappa_{ab} \left(\dot{A}_i^a A_j^b + A_0^a F_{ij}^b \right)$$

- Canonical momenta π_a^μ generate primary constraints φ_a^μ

$$\varphi_a^0 := \pi_a^0 \approx 0 \qquad \varphi_a^i := \pi_a^i - \frac{k}{4\pi} \epsilon^{ij} \kappa_{ab} A_j^b \approx 0$$

- Total Hamiltonian density $\mathcal{H}_T = -\frac{k}{4\pi} \epsilon^{ij} \kappa_{ab} A_0^a F_{ij}^b + u_\mu^a \varphi_a^\mu$
- Conservation of the primary constraints $\dot{\varphi}_a^\mu = \{\varphi_a^\mu, \mathcal{H}_T\} \approx 0$ leads to secondary constraints

$$\mathcal{K}_a := -\frac{k}{4\pi} \epsilon^{ij} \kappa_{ab} F_{ij}^b \approx 0 \qquad D_i A_0^a - u_i^a \approx 0$$

Canonical Analysis of CS Theories II: Charges

- Let $\bar{\mathcal{K}}_a = \mathcal{K}_a - D_i \varphi_a^i$. Then total Hamiltonian density expressed as sum of constraints

$$\mathcal{H}_T = A_0^a \bar{\mathcal{K}}_a + u_0^a \varphi_a^0$$

- $\varphi_a^0, \bar{\mathcal{K}}_a$ are first class, φ_a^i are second class
- Construct gauge generators via Castellani's algorithm.

$$\bar{\mathcal{G}}[\epsilon] = \int_{\mathcal{D}} d^2x (D_0 \epsilon^a \pi_a^0 + \epsilon^a \bar{\mathcal{K}}_a)$$

- Demanding functional differentiability determines the charges $\delta \mathcal{G}[\epsilon] = \delta \bar{\mathcal{G}}[\epsilon] + \delta \mathcal{Q}[\epsilon]$

$$\delta \mathcal{Q}[\epsilon] = \frac{k}{2\pi} \oint_{\partial \mathcal{D}} d\phi \text{tr}(\epsilon \delta A_\phi)$$

Asymptotic Symmetry Algebra

- 2 copies of Witt Algebra

$$\{\mathcal{L}(\theta), \mathcal{L}(\theta')\} = \delta(\theta - \theta') \mathcal{L}'(\theta') - 2\delta'(\theta - \theta') \mathcal{L}(\theta') - \frac{k}{4\pi} \delta^{(3)}(\theta - \theta')$$

- Quantizing \rightarrow 2 copies of Virasoro Algebra

$$[L_n, L_m] = (n - m)L_{n+m} + \frac{k}{2}(n^3 - n)\delta_{n+m,0}$$

- $c_L = c_R = 6k$
- Boundary gravitons are descendents of the vacuum created by $L_{-n_1} \cdots L_{-n_m} |0\rangle$ with $n_i > 1$

Dual CFT and Unitary Models

Requirements for Unitary CFT (partial list)

- Central charge $c > 0$
- Modular invariant partition function
- $c = 1/2$ Ising Model
- $c = 7/10$ Tricritical Ising Model
- Only examples, no unitary semiclassical models ($c \gg 1$)
[CGH⁺12]

Need an alternative theory with more allowed values of c

Generalization to Higher Spins

- Enlarge \mathfrak{sl}_2 to \mathfrak{sl}_N (or other gauge group containing \mathfrak{sl}_2)
- Choice of embedding $\mathfrak{sl}_2 \hookrightarrow \mathfrak{sl}_N$ determines other field content
- Spins of other field content given by weight under gravitational \mathfrak{sl}_2 action
- Number of embeddings grows exponentially with N
- Typical choice: Principal embedding, integer spins $2 \dots N$

$$\begin{aligned}
 g_{\mu\nu} &= \frac{1}{2} \text{tr} [(A - \bar{A})_\mu (A - \bar{A})_\nu] \\
 \phi_{\mu\nu\rho} &= \frac{1}{3!} \text{tr} [(A - \bar{A})_{(\mu} (A - \bar{A})_{\nu} (A - \bar{A})_{\rho)}] \\
 &\vdots
 \end{aligned}$$

Asymptotic Symmetry Algebra: Principal Embedding

- Boundary conditions

$$\hat{a}^{(0)} = L_0 d\rho + L_1 dx^+$$

$$a^{(0)} = \left(\mathcal{L}(x^+) L_{-1} + \sum_{n=2}^{N-1} \mathcal{W}_n(x^+) W_{-n}^n \right) dx^+$$

$$a^{(1)} = \mathcal{O}(e^{-\rho})$$

$$\hat{\bar{a}}^{(0)} = -L_0 d\rho - L_{-1} dx^-$$

$$\bar{a}^{(0)} = \left(\bar{\mathcal{L}}(x^-) L_1 + \sum_{n=2}^{N-1} \bar{\mathcal{W}}_n(x^-) W_n^n \right) dx^-$$

$$\bar{a}^{(1)} = \mathcal{O}(e^{-\rho})$$

- ASA: two copies of W_n algebra with central charges

$$c_L = c_R = 6k$$

Unitary Representations: Principal Embedding

- Possible unitary representations again (partially) classified by [CGH⁺12]
- $c = 4/5$ Potts Model
- $c = 6/7$ Tricritical Potts Model
- $c = 2\frac{N-1}{N+2}$, $N \in \{5, 6, 7, 8\}$ Parafermions

Again, no semi-classical limit $c \gg 1$ allowed. What about non-Principal embeddings?

No-Go Theorem

- No-Go Theorem for Unitary representations in the limit $k \rightarrow \infty$ [CHLJ12]
- All non-principal embeddings include a singlet, which leads to a Kac-Moody algebra as part of the ASA

$$[J_n, J_m] = \kappa n \delta_{n+m,0} + \cdots$$

- Unitarity requires $\kappa \geq 0$
- Unitary also requires central charge $c \geq 0$ for Virasoro algebra (from \mathfrak{sl}_2)

$$[L_n, L_m] = (n - m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0}$$

- In the limit $|c| \rightarrow \infty$, $\text{sign}(c) = -\text{sign}(\kappa)$.

No-Go Guides The Way

- Work at finite (but possibly large) central charge c
- Next-to-principal ($W_N^{(2)}$) higher spin gravity
- For a given value of N , allows a discrete spectrum of unitary representations
- Central charge c ranges from $\mathcal{O}(1)$ to $\mathcal{O}(N/4)$
- Asymptotic Symmetry Algebra is Feigin-Semikhatov algebra

Example: Polyakov-Bershadsky Algebra $W_3^{(2)}$

$$[L_n, L_m] = (n - m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0}$$

$$[J_n, J_m] = \kappa n \delta_{n+m,0}$$

$$[J_n, L_m] = nJ_{n+m} \quad [J_n, G_m^\pm] = \pm G_{n+m}^\pm$$

$$[G_n^+, G_m^-] = \frac{\lambda}{2}(n^2 - \frac{1}{4})\delta_{n+m,0} + \dots$$

$$[L_n, G_m^\pm] = \left(\frac{n}{2} - m\right) G_{n+m}^\pm$$

- Level $\kappa = \frac{2k+3}{3}$
- Central charge $c = 25 - \frac{24}{k+3} - 6(k+3)$
- G^\pm central term $\lambda = (k+1)(2k+3)$

Example: Polyakov-Bershadsky Algebra $W_3^{(2)}$

- Like $\mathcal{N} = 2$ Superconformal Algebra, but with commuting G^\pm
- Leads to negative norm descendents of the vacuum if G^\pm generate physical states
- G^\pm must be null for unitary theory $\Rightarrow \lambda = 0$
- No other restrictions on unitarity

Unitary Representations of $W_3^{(2)}$

$c = 0$ trivial theory

$c = 1$ theory of $\hat{u}(1)$ current algebra

Feigin-Semikhatov Algebra $W_N^{(2)}$

$$[L_n, L_m] = (n - m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0}$$

$$[J_n, J_m] = \kappa n \delta_{n+m,0}$$

$$[J_n, L_m] = nJ_{n+m} \quad [J_n, G_m^\pm] = \pm G_{n+m}^\pm$$

$$[G_n^+, G_m^-] = \lambda f(n) \delta_{n+m,0} + \dots$$

$$[L_n, G_m^\pm] = \left(n \left(\frac{N}{2} - 1 \right) - m \right) G_{n+m}^\pm$$

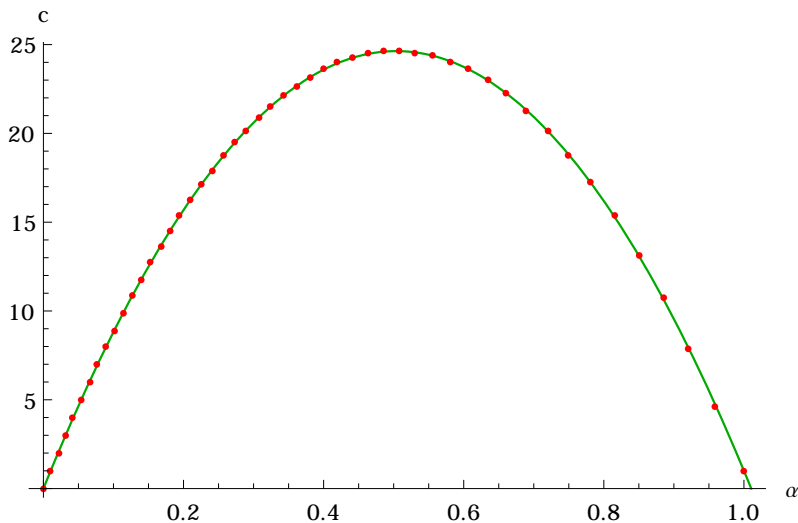
- G^\pm central term

$$\lambda = \prod_{m=1}^{N-1} (m(N + k - 1) - 1)$$

Unitarity requires $\lambda = 0$

Unitary Representations of $W_{100}^{(2)}$

Allowed values of central charge c for $N = 100$



Unitary Representations of $W_N^{(2)}$

- Demanding unitarity \rightarrow Newton's constant automatically quantized
- Critical values $\alpha = \frac{\hat{N}}{N - \hat{N} - 1}$ where $\hat{N} \in \mathbb{N}$, $\hat{N} \leq \frac{N-1}{2}$
- Allowed values of central charge (let $m = N - 2\hat{N} - 1$)

$$c(\hat{N}, m) - 1 = (\hat{N} - 1) \left(1 - \frac{\hat{N}(\hat{N} + 1)}{(m + \hat{N})(m + \hat{N} + 1)} \right)$$







- Exactly values of central charge $W_{\hat{N}}$ minimal models, up to shift by 1 due to $\hat{u}(1)$ current algebra
- Small α quantum regime with $c \sim \mathcal{O}(1)$
- Intermediate α semiclassical regime with $c \sim \mathcal{O}(\frac{N}{4})$
- $\alpha \sim \mathcal{O}(1)$ dual quantum regime with $c \sim \mathcal{O}(1)$

Open Issues

- Existence of modular invariant partition function
- $N \rightarrow \infty$ limit?
- Other non-principal embeddings
- Gauge invariance vs. Geometry

Thank You

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