Noncommutative geometry and differentiable stacks

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Why making geometry noncommutative?

Noncommutative geometry: Use algebra where geometry fails.

Example: orbits of particles moving freely (i.e., geodesically) on a torus $\Rightarrow$ Kronecker foliation of irrational slope λ.

$$\begin{array}{ccc} \text{leaf space} = \mathbb{C}/\mathbb{Z} & \xrightarrow{\text{complex functions}} & \text{trivial} = \mathbb{C} \\ \downarrow & & \\ \text{groupoid} = \mathbb{Z} \times \mathbb{C} & \xrightarrow{\text{convolution}} & \text{quantum torus: } \mathbb{C}\langle u, v\rangle / uv = e^{i\lambda} vu \end{array}$$

The propaganda minister says: The quantum torus is a substitute for the algebra of functions on the topologically trivial leaf space $S^1/\mathbb{Z}$. But $S^1/\mathbb{Z}$ is a **group**. So what ever happened to the group structure when going to the algebra side?

The missing Hopf structure on the noncommutative torus

Mystery (at least to me)

Why does the noncommutative torus algebra not have a Hopf structure? (In fact, it does not even admit a counit.)

Read the fine print: Association of algebra to “bad” leaf space only up to **Morita equivalence**. Therefore, we need a Morita invariant generalization of a Hopf structure and its geometric counterpart.

Fill in the blanks:

	Geometry	Algebra
isomorphism invariant	Lie group	Hopf algebra
Morita invariant	?	?

Groupoids

Example: group action of integers on complex plane:

$$\mathbb{Z} \times \mathbb{C} \longrightarrow \mathbb{C}\,, \qquad (k, z) \longmapsto k \cdot z := e^{i\lambda k} z\,.$$

Action non-free / non-proper $\Rightarrow$ quotient not differentiable.
Idea: Don't consider quotient but generalized equivalence relation:

Definition

A groupoid is a (small) category with inverses.

Equivalence relation:	Groupoid:
transitiviy, reflexivity, symmetry	multiplication, identities, inverses

Note: Two elements can now be equivalent in more than one way.

Action groupoid

Source and target maps:

$$l : G_1 = \mathbb{Z} \times \mathbb{C} \to G_0 = \mathbb{C}, \quad (k, z) \mapsto k \cdot z$$

$$r : G_1 = \mathbb{Z} \times \mathbb{C} \to G_0 = \mathbb{C}, \quad (k, z) \mapsto z$$

Multiplication: $(k,z)(k',z') = (k+k',z') \quad \text{if} \quad z = e^{i\lambda k'} z'$

Identity: $\mathrm{id}_z = (0,z)$

Inverse: $(k,z)^{-1} = (-k, e^{i\lambda k} z)$

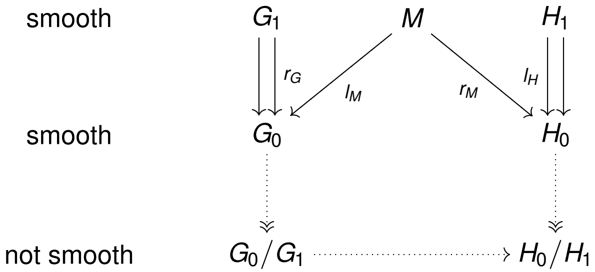
This is even a **Lie groupoid**.

Defintion

Lie groupoid: All structure maps are smooth.

Morphisms of generalized equivalence relations

Same idea for morphisms:



Require relation M to descend to function on quotients. This leads to:

Definition

A smooth G-H bibundle is a manifold with a left G action and a right H action which commute. It is right principal if l_M is a surjective submersion and the right H-action is free and transitive on the l_M-fibers.

Differentiable stacks

Price to be paid: Composition of bibundles is not associative. But it is associative up to isomorphism of groupoid bibundles, i.e., smooth biequivariant maps. We obtain a **weak 2-category**. The category of differentiable stacks is a strict 2-category.

	differentiable stacks	presentations
0-morph.	fibered cat. with glueing and atlas	Lie groupoids
1-morph.	fiber preserving functors	right principal bibundles
2-morph.	natural transformations	biequivariant maps

Theorem (CB, math.DG/0702399)

The 2-category of differentiable stacks and the weak 2-category of Lie groupoids, right principal bibundles, and biequivariant maps are equivalent.

The latter we call the 2-category of **stacky manifolds**.

Stacky Lie groups

Theorem (CB, math.DG/0702399)

The 2-category of stacky manifolds has finite products and a terminal object.

We need this in order to have the notion of a group:

Definition (CB, math.DG/0702399)

A **stacky Lie group** is a weak 2-group object in the 2-category of stacky manifolds.

Now we can fill in one of the blanks:

	Geometry	Algebra
isomorphism invariant	Lie group	Hopf algebra
Morita invariant	**stacky Lie group**	?

Dualization

The groupoid of the Kronecker foliation is a stacky Lie group. It acts faithfully on the groupoid $\mathbb{Z} \times \mathbb{C} \rightrightarrows \mathbb{C}$ of the initial example.

The convolution functor

groupoids	$\longrightarrow$	convolution algebras
bibundles	$\longrightarrow$	bimodules
biequivariant maps	$\longrightarrow$	bimodule homomorphisms

Applying this functor to the structure bibundles of the stacky group structure, we obtain the bimodule of the coproduct and the bimodule of the counit. We also have a bimodule of the antipode:

$$\boldsymbol{\Delta} \in {}_{\mathcal{A}\otimes\mathcal{A}}\mathrm{Mod}_{\mathcal{A}}\,, \quad \varepsilon \in {}_{\mathbb{C}}\mathrm{Mod}_{\mathcal{A}}\,, \quad \boldsymbol{S} \in {}_{\mathcal{A}^{\mathrm{op}}}\mathrm{Mod}_{\mathcal{A}}\,.$$

These structure bimodules satisfy by construction

$$(\mathcal{A} \otimes \mathbf{\Delta}) \otimes_{\mathcal{A}\otimes\mathcal{A}} \mathbf{\Delta} \cong (\mathbf{\Delta} \otimes \mathcal{A}) \otimes_{\mathcal{A}\otimes\mathcal{A}} \mathbf{\Delta}\,,$$
$$(\varepsilon \otimes \mathcal{A}) \otimes_{\mathcal{A}\otimes\mathcal{A}} \mathbf{\Delta} \cong \mathcal{A} \cong (\mathcal{A} \otimes \varepsilon) \otimes_{\mathcal{A}\otimes\mathcal{A}} \mathbf{\Delta}\,,$$

the axioms of coassociativity and counitality. There is an additional axiom for the antipode. What we thus obtain is called a **hopfish algebra** (CB/Tang/Weinstein/Zhu).

Now we can fill in the remaining blank:

	Geometry	Algebra
isomorphism invariant	Lie group	Hopf algebra
Morita invariant	**stacky Lie group**	**hopfish algebra**

Tensor representations

The hopfish coproduct can be used to define a tensor product of two right $\mathcal{A}$-modules $T, T' \in \mathrm{Mod}_{\mathcal{A}}$ by

$$T \otimes_{\boldsymbol{\Delta}} T' := (T \otimes T') \otimes_{\mathcal{A} \otimes \mathcal{A}} \boldsymbol{\Delta}\,.$$

The representations of the noncommutative torus are non-classifiable. We first have to choose a sub-class of representations for which we can try out the tensor product.

Denote by T^{α}_{pq} the irreducible representation generated by an eigenvector of the monomial $u^p v^q$,

$$u^p v^q |\psi\rangle = e^{i\alpha} |\psi\rangle\,,$$

for p and q relatively prime, $\alpha \in \mathbb{R}$.

Theorem (CB/Tang/Weinstein)

For $p_1 \neq 0$ or $p_2 \neq 0$ we have:

$$T^{\alpha_1}_{p_1 q_1} \otimes_{\mathbf{\Delta}} T^{\alpha_2}_{p_2 q_2} \cong \gcd(p_1, p_2)\, T^{\alpha}_{pq}\,,$$

where

$$p := \mathrm{lcm}(p_1, p_2)\,,\ q := \frac{p_1 q_2 + p_2 q_1}{\gcd(p_1, p_2)}\,,\ \alpha := \frac{\alpha_1 p_2 + \alpha_2 p_1}{\gcd(p_1, p_2)}\,.$$

For $p_1 = 0$ and $p_2 = 0$ we have:

$$T^{\alpha_1}_{0,q_1} \otimes_{\mathbf{\Delta}} T^{\alpha_2}_{0,q_2} \cong \begin{cases} T^{\alpha}_{0,q}\,, & \text{for } \frac{\alpha_1 q_2 - \alpha_2 q_1}{\lambda \gcd(q_1, q_2)} \in \mathbb{Z} \bmod \mathrm{lcm}(q_1, q_2)\frac{2\pi}{\lambda} \\ 0\,, & \text{otherwise} \end{cases}\,,$$

where

$$q := \gcd(q_1, q_2)\,, \quad \alpha := s_1\alpha_2 - s_2\alpha_1\,, \quad s_1, s_2 \in \mathbb{Z} : \quad \frac{s_1 q_2 - s_2 q_1}{\gcd(q_1, q_2)} = 1\,.$$

Observe that $q/p = q_1/p_1 + q_2/p_2$ and $\alpha/p = \alpha_1/p_1 + \alpha_1/p_2$.

There is more to be said...

- More stacky Lie groups:
  - String groups (Teichner)
  - Conjecture: Pseudogroup of local isometries of a Riemannian foliation (Haefliger)
  - Conjecture: A complex metaplectic group (Omori, Maeda et al.)
- C^*-version of hopfish algebras
- Tannaka-Krein type reconstruction theorem
- Lie theory on differentiable stacks
- Generalized quantization: Poisson manifold $\longrightarrow$ Lie algebroid (Dirac structure) $\longrightarrow$ groupoid $\longrightarrow$ convolution algebra.

References:

- CB, math.DG/0702399
- CB/Weinstein, math.SG/0701499 (to appear in Contemp. Math.)
- CB/Tang/Weinstein, math.QA/0604405 (to appear in Contemp. Math.)
- Tang/Weinstein/Zhu, math.QA/0510421 (Pac. J. Math.)