Bayrischzell Workshop 2007

On the algebraic foundation of perturbative quantum field theory and its roots in invariant theory

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CSI Bayrischzell

- QFT axiomatics
- Vertex operator algebras
- Representation rings of finite groups
- Algebraic group schemes
- Harmonic elements, cyclicity

crime scene do not enter : crime scene do not enter : crime scene do not enter : crime scene do not

Our DNA test (method of choice)
algebraic combinatorics

QFT

data:

- state space $\mathcal{H}$ (complex Hilbert space)
- vacuum vector $\Omega \in \mathcal{H}$
- unitary rep. of the Poincaré algebra (group)
  $(q, A) \longrightarrow U(q, A)$
- set of fields $\{\phi_a\}$; operator valued distribution
  $f \longmapsto \Phi_a(f)$ f test function on $\mathcal{M}$
  (g.h. manifold)

(Wightman) axioms:

W1: $U(q,\Lambda)\,\phi_a(f)\,U(q,\Lambda)^{-1} = \phi_a(f)$

$U(q,1) = e^{\sum a_i P_i}$ P_i selfadj. op. mutually com.

W2: stable vacuum
$U(q,\Lambda)|0\rangle = |0\rangle$ and spectrum of the P_a in the forward light cone (positivity)

W3: completeness $|0\rangle \in$ domain of each poly. funct. of the P_a and $\{P_a(\phi_a(f))\}$ is dense in $\mathcal{H}$

W4: locality
$\phi_a(f)\,\phi_b(g) = \phi_b(g)\,\phi_a(f)$ if f, g are space like separated $x \in \operatorname{supp} f$, $y \in \operatorname{supp} g$ $|x-y| > \sigma$ $\forall x, y$

4 Vertex algebras

data:

$V = V_{\bar 0} + V_{\bar 1}$ superspace, $|0\rangle$ vacuum vector,

state field correspondence $a \longrightarrow Y(a,z) = \sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1}$

$Y \in \mathrm{End}(V)[[z]]$

axioms:

V1: translation covariance

$[T, Y(a,z)] = \partial Y(a,z)$; $\qquad Y(a,z)|0\rangle = e^{zT}(a)$

$T \in \mathrm{End}(V)$; $\quad [T, a_{(n)}] = -n\, a_{(n-1)}$; $Ta = a_{(-2)}|0\rangle$

V2: vacuum

$Y(|0\rangle, z) = \mathrm{Id}_V \qquad Y(a,z)|0\rangle|_{z=0} = a$

$|0\rangle_{(n)} = \delta_{n,-1} \qquad a_{(n)}|0\rangle = 0 \ \forall n \geq 0 \quad a_{(-1)}|0\rangle = a$

V3: locality

for $N \gg 0 \quad (z-w)^N Y(a,z) Y(b,w) = (-1)^{\partial a \partial b} (z-w)^N Y(b,w) Y(a,z)$

recall

$$\delta(z-w) = z^{-1} \sum_{n \in \mathbb{Z}} \left(\frac{w}{z}\right)^n = w^{-1} \sum_{n \in \mathbb{Z}} \left(\frac{z}{w}\right)^n = \delta(w-z)$$

... normal ordering, modes

$$a(z) = a_-(z) + a_+(z) = \sum_{n\geq 0} a_{(n)} z^{-n-1} + \sum_{n<0} a_{(n)} z^{-n-1}$$

def: (normal order)

$$:a(z)b(w): \; = \; a_+(z)b(w) + b(w)a_-(z)(-1)^{\partial a \partial b}$$

OPE:

$$a(z)b(w) = \sum_{j=0}^{N-1} \frac{a(w)_{(j)} b(w)}{(z-w)^{j+1}} + :a(z)b(w):$$

with $a(w)_{(j)} b(w) = \mathrm{Res}_z [a(z), b(w)] (z-w)^j$

Examples: Current algebras, Witt, Virasoro, ...

let $[\alpha_m, \alpha_n] = m\,\delta_{m,-n} K \qquad [K, a_n] = 0$

$\hookrightarrow \; a(z) = \sum_{n\in\mathbb{Z}} \alpha_n z^{-n-1} \qquad \in V = V_{\bar{0}}$

$$[a(z), a(w)] = \partial_w \delta(z-w) K$$

$$a(z)a(w) \simeq \frac{K}{(z-w)^2} + \text{hdom. terms}$$

,free boson'

Irreducibles ($V = V_{\bar{0}}$)

V : look for a distinguished basis $\{V^\lambda\}$ of irreducibles (indecomposables) $V = \bigoplus_\lambda V^\lambda$

GL, U, S_n : Schur: "the irreps. are indexed by integer partitions $\lambda = (\lambda_1, \lambda_2, \lambda_3 \ldots)$"

characters: $ch(V^\lambda) = s_\lambda$ Schur function

$$s_\lambda = \sum_{T \in SSYT} x^T; \quad x^T = x_1^{t_1} \cdots x_n^{t_n}$$

$[1..n] \xrightarrow{T} \lambda$ 

$$\begin{array}{cccc} 11 & 12 & 12 & 13 \\ 2 & 2 & 3 & 2 \end{array} \quad \cdots$$

$$\downarrow \quad x_1^2 x_2 \quad x_1 x_2^2 \quad 2x_1x_2x_3 \quad \cdots$$

s_λ from Bernstein (Vertex) operators

$$H(u) = \sum_{n \geq 0} h_n u^n \qquad h_n = s_{(n)} \cong \boxed{\;\;\;\cdots\;}$$

$$E^\perp(-u^{-1}) = \sum_{n \geq 0} (-)^n e_n^\perp u^{-n} \qquad e_n = s_{(1^n)} \cong$$

$e_n^\perp$ 'remove n vertical boxes'

$f^\perp$ is a derivation

$$B(u) = H(u)\, E^\perp(-u^{-1}) = \sum_{n \in \mathbb{Z}} B_n u^n \in V \otimes V^*$$

$$B(u_1, \ldots, u_\ell) = H(u_1) \cdots H(u_\ell)\, E^\perp(-u_\ell^{-1}) \cdots E^\perp(-u_1^{-1})$$

$$B(u_1)B(u_2) = (1 - u_1/u_2)\, B(u_1, u_2)$$
$$= (1 - u_1/u_2) : B(u_1)\, B(u_2):$$

now

$$s_\lambda = [u^\lambda]\, B(u_1) \dots B(u_{\ell(\lambda)})\, 1$$
$$= [u^\lambda] \prod_{i<j} (1 - \tfrac{u_i}{u_j})\, H(u_1) \dots H(u_\ell)\, 1$$
$$= [u^\lambda] \prod_{i<j} (1 - \tfrac{u_i}{u_j}) \sum h_{n_1} h_{n_2} \dots h_{n_\ell} u_1^{n_1} u_2^{n_2} \dots u_\ell^{n_\ell}$$
$$= \det (h_{\lambda_i - i + j})$$

power sums

$$\mathbb{Q} \otimes V = \mathrm{Span}_{\mathbb{Q}} \{p_n\} \qquad p_k(z) = \sum_{i=1}^{n} z_i^k$$

$$s_\lambda = \sum_{\mu \vdash |\lambda|} \chi_\mu(\lambda)\, p_\mu \qquad \chi_\mu(\lambda) \text{ characters of } S_{|\lambda|}$$

$$h_n = \sum_{\lambda \vdash n} z_\lambda^{-1} p_\lambda$$

$$P_t = \sum_{n \geq 1} p_n t^n = t\, d_t \ln H(t)$$

$$p_n^\perp \cong n \frac{\partial}{\partial p_n} \quad \hookrightarrow \pi_0 = 1, \ \pi_n = p_n, \ \pi_{-n} = n \frac{\partial}{\partial p_n}$$
$$\hookrightarrow [\pi_n, \pi_{-m}] = \delta_{n,m}\, \pi_0$$

$$B(u) = \exp\left(\sum p_n u^n\right) \exp\left(\sum (-)^n \frac{p_n^\perp}{n} u^{-n}\right)$$

Tensor product (semi) ring structure

$$V \cong \mathrm{Span}\{V^\lambda\} \simeq \mathrm{Rep}^+(GL(N)) \qquad N \to \infty$$

$$\downarrow \mathrm{ch}$$

$$\Lambda \cong \bigoplus_n \bigoplus_{\lambda \vdash n} s_\lambda$$ ring of sym functions in infinitely many variables

$$V^\lambda \otimes V^\mu \xrightarrow{\mathrm{ch}} s_\lambda \cdot s_\mu = \sum_\nu c^\nu_{\lambda\mu}\, s_\nu$$

$c^\nu_{\lambda\mu} \in \mathbb{Z}_+$ Littelwood-Richardson

formally add <u>negative</u> weight spaces / characters

$\hookrightarrow \Lambda \simeq (\Lambda, \oplus, \otimes)$ character ring

Actually we find:

<u>Thm</u>: Λ is the (universal) commutative, cocommutativ, self dual (w.r.t. $\langle s_\lambda | s_\mu \rangle = \delta_{\lambda\mu}$) graded, connected Hopf algebra

$\hookrightarrow$ allows to write GL as an algebraic group scheme (via its characters ...)

Invariants

Ex plane conics (*) $ax^2+2bxy+cy^2+2dx+2ey+f=0$

$\vec{x}=(x,y,1)^t$ $\quad M=\begin{pmatrix} a & b & d\\ b & c & e\\ d & e & f\end{pmatrix}$ $\quad X=\begin{pmatrix} p & q & \ell\\ -q & p & m\\ 0 & 0 & 1\end{pmatrix}\in E(2)$

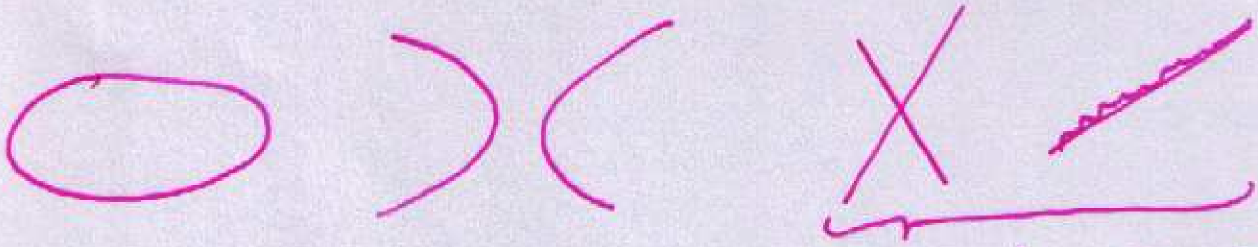
$\det X\neq 0,\ p^2+q^2=1$

(*) $\Leftrightarrow$ $\vec{x}^t M x = 0$

curves up to rigid motion:

$\mathbb{C}[a,b,c,d,e,f]^{E(2)} \simeq \mathbb{C}[D,T,E]$

$T=a+c \simeq \mathrm{tr}(O(2))$ $\quad E=ac-b^2\simeq \det(O(2))$ $\quad D=\mathrm{disc}\,X$



degenerate, $D=0$
and $\vec{x}^t M\vec{x}=0$
factorizes into 2 linear
factors

every point (D,T,E)
corresponds to a unique
plane conic (up to Euclidean
motion)

now allow scalings and remove degenerate cases :

$$X \to rX$$

$$D \to r^6 D, \quad E \to r^4 E, \quad T \to r^2 T$$

$$\mathbb{C}[a,b,c,d,e,f,\tfrac{1}{D}]^{\mathbb{R}\times E(2)} \sim \mathbb{C}[A,B,C]/\sim$$

$$A = \frac{E^3}{D^2}, \quad B = \frac{T^3}{D}, \quad C = \frac{ET}{D}$$

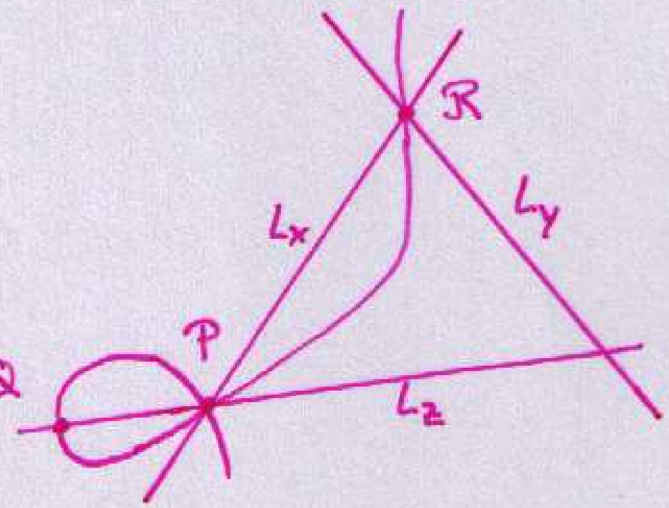
$$\boxed{AB - C^3 = 0}$$

$$\mathbb{C}[A,B,C]/\sim \; = \mathbb{C}[A,B] + \mathbb{C}[A,B]C + \mathbb{C}[A,B]C^2$$

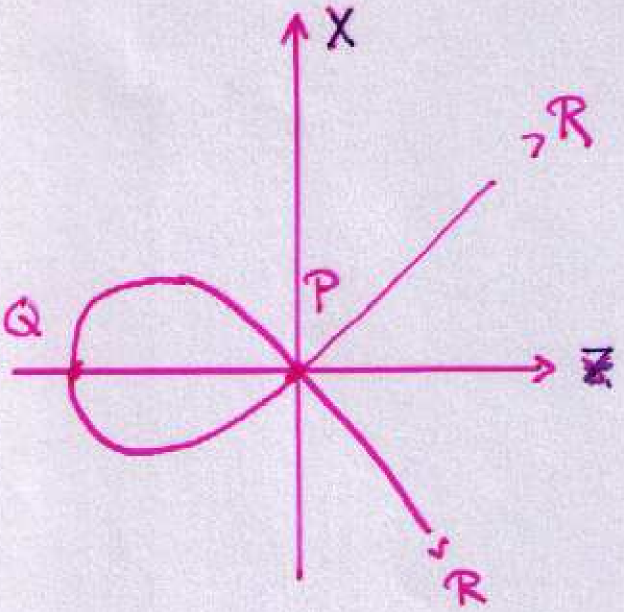
module over $\mathbb{C}[A,B]$

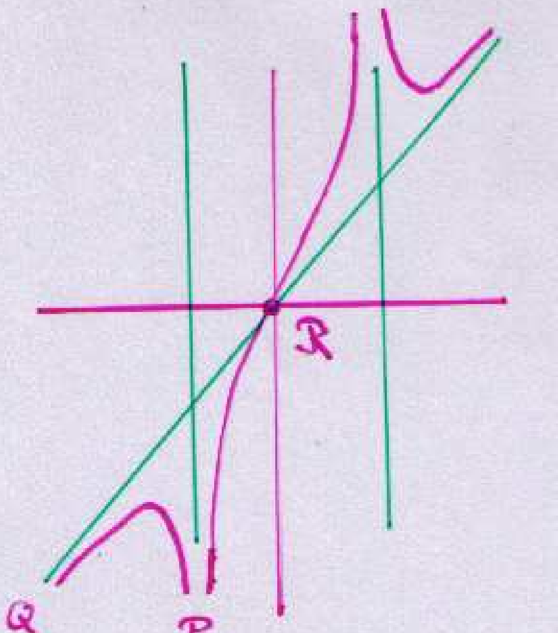
The <u>moduli space</u> for nondegenerate curves of degree 2 in the Euclidean plane is the affine surface in $\mathbb{A}^3$ defined by the equation $xz - y^3 = 0$

projective plane curves



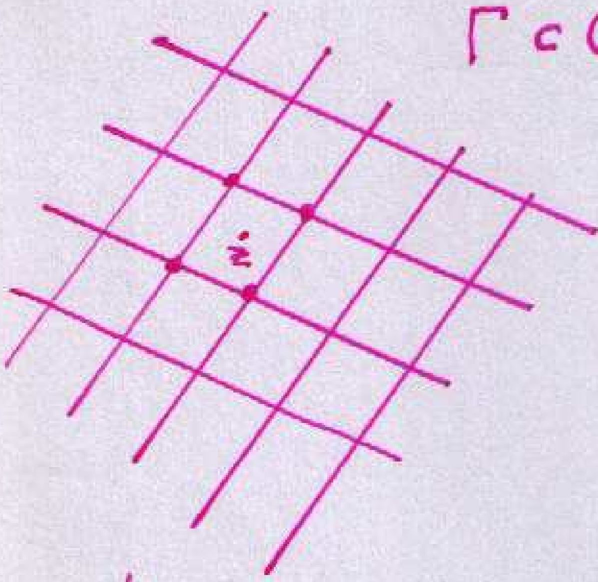
$L_z \Rightarrow \infty$ $\qquad$ $L_y \Rightarrow \infty$





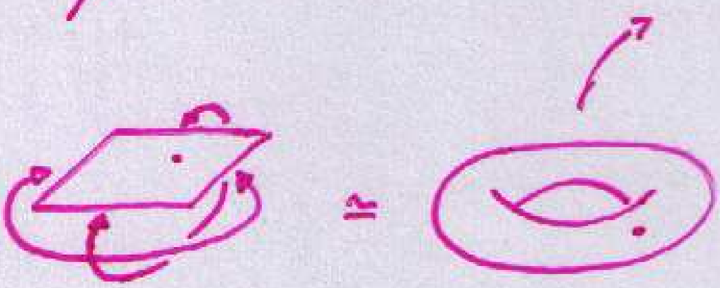
affine covers of the projective plane curve $y^2x - x^2z - x^3 = 0$

$\Gamma \subset \mathbb{C}$



$\mathbb{P}^2$

$\mathbb{C}/\Gamma$

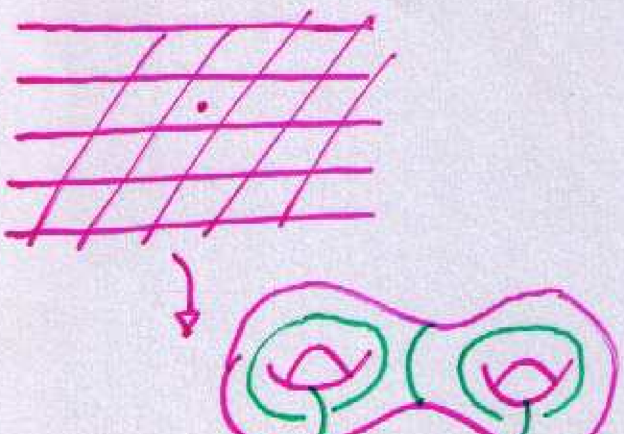


$\mathbb{C} \longrightarrow \mathbb{P}^2$

$z \longrightarrow (\wp(z), \wp'(z), 1) \simeq (X, Y, Z)$

$$Y^2 Z = 4X^3 - g_2 X Z - g_3 Z^3$$

Weierstrass
$\wp$-function
$g_n \sim$ Eisenstein



$\rightsquigarrow$ Dirichlet L-functions
(automorphic forms)

$\mathbb{C}/\Gamma' \subseteq_f \Gamma$

Invariants of Groups

$V \cong x_1 + x_2 + \sim + x_n \qquad f(x_1 \ldots x_n) \in \mathbb{C}[x_1 \ldots x_n]$

$f(x)$ is invariant w.r.t. $A \in End(V)$

$(g \ni G \xrightarrow{\varrho} GL \simeq End(V))$ if

i) $f(Ax) = f(\sum_i a_{1i} x_i, \ldots, \sum_i a_{ni} x_i)$

coordinate ~~function~~ transformation 'linear subst.'

(characters, Casimirs ...)

ii) $\mathcal{D}_A = \sum_{ij} a_{ij} x_i \frac{\partial}{\partial x_j}$ and $\mathcal{D}_A f(x) = 0$

invariant under derivation

(infinitesimal invariants $\to$ Lie alg)

$\mathbb{C}[x_1 \ldots x_n] \longrightarrow \mathbb{C}[x_1 \ldots x_n]^G$

,coordinates' ,invariants'

G 'nice' then

$\mathbb{C}[x_1 \ldots x_n] = \underbrace{\mathbb{C}[x_1 \ldots x_n]^G}_{\text{free, Cohen-Macaulay}} \otimes \mathcal{H}_G$,coinvariants'

standard example:

$S(V) = S^G(V) \otimes \mathcal{H}_G$

Example :

$S_k \subset GL(k)$ Let $W = V^{\otimes k}$

$GL(k \cdot n) \cdot W \simeq GL(n) \cdot V^{\otimes k} \cdot S_k$

$g = (h, \sigma) : W \to W$

$(h,\sigma)(v_1 \otimes \ldots \otimes v_k) = (hv_{\sigma(1)}, \ldots, hv_{\sigma(k)})$

$\to$ $GL(n)$ characters are described by Schur – Weyl duality via polynomial invariants under S_n

FFT of invariant theory

$\mathbb{Z}[x_1 \ldots x_n]^{S_n} \simeq \mathbb{Z}[\sigma_1, \ldots, \sigma_n]$ $\deg x_i = 1$

$\sigma_1 = \sum_i x_i$, $\sigma_2 = \sum_{i<j} x_i x_j$, $\ldots$, $\sigma_n = x_1 x_2 \ldots x_n$

$\deg \sigma_1 = 1$ $\deg \sigma_2 = 2$ $\deg \sigma_n = n$

S and S^G are graded rings

$S = \bigoplus_{d \geq 0} S_d$ $S^G = \bigoplus_{d \geq 0} S^G \cap S_d$

usefull information

$$\mathrm{Hilb}_t(S^G) = \sum_{d \geq 0} \dim (S^G \cap S_d)\, t^d \in \mathbb{Z}[[t]]$$

$|G|$ order of G (Molien)

$$\mathrm{Hilb}_t(S^G) = \frac{1}{|G|} \sum_{g \in G} \frac{1}{\det(1 - tg)} \cong \frac{\int dg \exp \operatorname{tr} \ln (1-tg)}{\int dg \exp \operatorname{tr} \ln (1-tg)|_{t=0}}$$

Facts about invariant rings

Thm: (Hilbert-Nöther) If $G \subset GL(V)$ is finite, then $S(V)^G$ is a finitely generated $\mathbb{F}$-algebra.

Def: A ring A is Cohen-Macaulay if there exists a polynomial algebra $\mathbb{F}(x_1 .. x_n) \subset A$ such that A is free and finite over $\mathbb{F}(x_1 .. x_n)$.
That is $\exists (\alpha_1, .., \alpha_s) \in A$ such that

$$A = \sum_{i=1}^{s} \mathbb{F}(x_1 .. x_n) \alpha_i$$

s is the transcendence degree

Thm: (Hochster-Egan) If $G \subset GL(V)$ is finite, nonmodular, then $S(V)^G$ is Cohen-Macaulay

$\Rightarrow$ $Av : S \to S \qquad Av(x) = \frac{1}{|G|} \sum_{g \in G} g \cdot x$

$\operatorname{Im} Av \cong S^G$

recall:

$$S^{S_n} \simeq \mathbb{Z}[x_1 \dots x_n]$$

$$S^{R \times E(2)} \simeq \bigoplus_{i=0}^{2} \mathbb{C}[x_1, x_2] x_3^i \qquad x_3^0 = 1$$

Cohen-Macauley

16 Example (Stanley)

$V = \mathbb{F}x \oplus \mathbb{F}y \qquad \mathbb{Z}/_{4\mathbb{Z}} = \{ \delta \mid \delta = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \}$

γ	$\det(1 - t\gamma)$
$1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$(1-t)^2$
$\delta = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$	$(1+t^2)$
$\delta^2 = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}$	$(1+t)^2$
$\delta^3 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$	$(1+t^2)$

$$\mathrm{Hilb}_t\, S(V)^{\mathbb{Z}/_{4\mathbb{Z}}} = \frac{1}{4}\left[\frac{1}{(1-t)^2} + \frac{2}{1+t^2} + \frac{1}{(1+t)^2} \right] = \frac{1+t^4}{(1-t^2)(1-t^4)}$$

$\hookrightarrow$ generators f_1, f_2, f_3 with degrees $2, 4, 4$

$$S(V)^{\mathbb{Z}/_{4\mathbb{Z}}} = \mathbb{F}[f_1 f_2] + \mathbb{F}[f_1 f_2] f_3 = \frac{\mathbb{F}[X, Y, Z]}{(Z^2 - X^2 Y + 4Y^2)}$$

$$f_1 = x^2 + y^2, \quad f_2 = x^2 y^2, \quad f_3 = x^3 y - x y^3$$

with syzygy: $f_3^2 = f_1^2 f_2 - 4 f_2^2$

$$\mathrm{Av}(x^2) = \frac{1}{4}(x^2 + + \cancel{\sigma} + x^2) \simeq x^2$$

$\vdots$

$\{f_i\}$ as reduced Gröbner basis

Thm: let $G \subset GL(V)$ finite, nonmodular and assume $S^G(V) = \mathbb{F}[x_1 .. x_n]$ with $\deg x_i = d_i$

i) $|G| = d_1 \cdots d_n$

ii) # of (pseudo) reflections is G is $\sum_{i=1}^{n}(d_i - 1)$

Thm: (Chevalley - Shepard - Todd - Bourbaki)
Let $G \subset GL(V)$ finite, nonmodular subgroup ~~of~~
then G is a (pseudo) reflection group iff S^G is a polynomial algebra.

(modular case: Serre S^G poly. $\to$ G refl. grp)

Every reflection defines an action on S via

$x \in V \quad s \cdot x = x + \lambda_x \alpha = x + \Delta(x)\alpha \qquad \Delta : V \to \mathbb{F}$

and $s \cdot (xy) = (s \cdot x)(s \cdot y)$

$\Rightarrow \Delta(xy) = \Delta(x) y + x \Delta(y) + \Delta(x)\Delta(y)\alpha$

,Skew derivation' (dual cousin of Rota-Baxter)

Extended invariants

$$V \longrightarrow S(V) \otimes E(V) \ni f(\underline{x})\, d\underline{x}$$

$$(S \otimes E)^G \simeq S^G \otimes E^G \qquad G\text{-inv. diff. forms}$$

$$\mathrm{Hilb}_{t,q}((S \otimes E)^G) = \frac{1}{|G|} \sum_{\varphi \in G} \frac{\det(1 + q\varphi)}{\det(1 - t\varphi)}$$

Ring of Covariants

$S(V)$, I = ideal in S generated by $\bigoplus_{d\geq 1} S_d$

(formal power series of sym. fct. with vanishing const. term)

$S_G = S/I$ ring of covariants

Thm. S_G gives the regular rep. of G ; $\dim_{\mathbb{F}} S_G = |G|$

Harmonics

(better use Hopf alg. here ...)

$S^*(V) \cong S(V^*)$ graded dual of $S(V)$

Def: $\mathcal{D}_\alpha : S \to S$ differential operator, induced from $S^* \otimes S \to S$ action

$$\mathcal{D}_\alpha(x) = \langle \alpha | x \rangle \quad \forall x \in V$$

$$\mathcal{D}_\alpha(xy) = \mathcal{D}_\alpha(x) y + x \mathcal{D}_\alpha(y)$$

$$\left(\mathcal{D}_\alpha(x) = \langle x_{(1)} | \alpha \rangle x_{(2)} \right)$$

~~Def~~: ~~...~~

$\mathcal{R}^*_+ = \sum_{i\geq 1} S^{*G}$ G-inv. diff. op. without const. term

Def: $x \in S$ is <u>harmonic</u> if $D_\alpha(x) = 0$ $\forall \alpha \in \mathcal{R}^*_+$. Denote $\mathcal{H} = \{x, D_\alpha(x) = 0\}$

For A_n, B_n, D_n type Lie groups this reduces to $\Delta(x) = 0$ where $\Delta = \sum_i \frac{\partial^2}{\partial x_i^2}$ 'Laplacian'

Thm: (Steinberg) char $\mathbb{F} = 0$, $G \subset GL(V)$, finite, grp.
G is a reflection group, iff H is a cyclic S^* module

$\Leftrightarrow \exists\ |0\rangle$ so that $\mathcal{H}$ is generated by (skew derivations of) reflections $\mathcal{H} = \mathrm{Span}\{\Delta_{s_1} \ldots \Delta_{s_k} |0\rangle\}$

$|0\rangle$ is given by the unique skew invariant Ω of highest weight.

Thm: If H is a cyclic S module then G is a (pseudo) reflection group.

$\Leftrightarrow S \cong S^G \otimes S_G = S^G \otimes \mathcal{H}$

... and noncommutativity?

$$\begin{tikzcd}
 & T(V) \simeq \mathbb{F}\langle x_1 .. x_n\rangle \arrow[dd, two heads, "\pi"] \\
V \arrow[ur, hook] \arrow[dr, hook] & \\
 & S(V) \simeq \mathbb{F}[x_1 .. x_n]
\end{tikzcd}$$

very general machinery of lax tensor functors

$K, \bar{K}$: Species $\longmapsto$ Hopf algebras   Aguiar, Mahajan

$$\begin{tikzcd}
 & MR \arrow[dd, two heads] \\
\text{[linear orders]} \arrow[ur, hook, "K"] \arrow[dr, hook, "\bar{K}"'] & \\
 & \Lambda
\end{tikzcd}$$

Malvenuto - Reutenauer HA of permutations

HA of symmetric funct.

$$\begin{tikzcd}
X^{\{\lambda\}} \arrow[d, two heads] \\
X^{\lambda}
\end{tikzcd}$$

NC variables indexed by set partitions

com. variables indexed by integer partitions

$$\begin{tikzcd}
\{\{126\}\{7\}\{345\}\} \arrow[d, two heads, "\dim"] \\
[3,1,3]
\end{tikzcd}$$

The HA structure lifts to NC vars, but not the geometry. However, one finds NCHA such that they contain the same information as the com. HA : Blessenohl-Schocker, NC character theory of the sym. grp.

Positivity / Combinatorics

recall: $V^\lambda \otimes V^\mu = \bigoplus_\nu c_{\lambda\mu}^\nu V^\nu$

with $c_{\mu\lambda}^\nu \in \mathbb{Z}_+$

Integrable models → (6-vertex, square ice, ...)
give Hall-Littlewood, Macdonald sym fct.

$\Lambda[t] \cong \mathbb{Z}[t]\Lambda \qquad \Lambda[q,t] \cong \mathbb{Z}[q,t]\Lambda$

where positivity is also available
(<u>very</u> diff. proof by M. Haiman)

- such things can be ‚counted'
  ↪ explicit combinatorial evaluations
  ↪ Littlewood-Richardson rule

- such things give dimensions of
  (quantum) holonomies and cohomologies
  of some Hilbert scheme ....

⇒ geometry ?!

Analogy: (SciFi)

pQFT $\longleftrightarrow$ rep. theory (comm. HA)

state space $\mathcal{H} = \mathrm{Span}\{\vert a\rangle\}$	$\mathcal{H}$ as cyclic S-module
vacuum	unique skew invariant highest weight $\Omega \in S_6$
state field corresp. $a \longrightarrow Y(a,z)$	Schur-Functors, Vertex operators $\in \mathrm{End}\,\Lambda[z]$
transl. covariance	Cohen-Macauley transcendence degree ?
locality $\longrightarrow$	positivity + ? allows to compute in $F(S)$ field of fractions, and come back to S
renormalization $\overset{?}{\longleftarrow}$	
‚path integral'	Hilbert series + Molien thm. (Frob. characteristics)

drawback:
Analogy moves to NC Hopf algebras but looses the geometric aspects

$\Rightarrow$ ALL ABOVE GIVEN THEOREMS ARE FALSE FOR NON FINITE GROUPS ....