MONOPOLES IN NONCOMMUTATIVE GAUGE THEORIES FOR SIMPLE GROUPS

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Monopoles?

Why monopoles, If ....

- in spite of extensive experimental search they have not been found yet

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- even in 1981 Dirac himself wrote in response to an invitation to a conference on 50th anniversary of his paper:

  "I am inclined now to believe that monopoles do not exist. So many years have gone by without any encouragement from the experimental side"?

Monopoles?

Well, there are a handful of reasons:

- Monopoles in SB gauge theories –GUTS, in particular– are predictions, not just possibilities ['t Hooft (NPB, 1974) and Polyakov (JETP Lett.,1974)]
- Actually, the need to explain why there is not an overabundance of monopoles surviving from the early universe led to the inflationary universe scenario [Guth (PRD, 1981)]
- In QCD, the confinement of colour can be explained ['t Hooft] as the effect of monopole condensation in the vaccum, as monopole degrees of freedom can be uncovered by means of the abelian projection
- BPS monopoles occur as single-particle states in quantum non-abelian gauge theories with extended supersymmetry. Seiberg and Witten (NPB, 1994 papers) showed that BPS monopoles are key in the understanding of the nonperturbative dynamics of supersymmetric gauge theories and string theories
- One can do beautiful and highbrow Mathematics with BPS monopoles
- .....

Monopoles in NC space-time?

Since the limit $h \to 0$ of NC space-time –defined by $[X^\mu, X^\nu] = ih\theta^{\mu\nu}$– yields formally ordinary space-time, it is natural to ask in view of what has been said above whether there exist (BPS) NC monopoles and, in particular,

- whether there are solutions to the NC field eqs. that "deform" the ordinary monopole fields in the the sense that, at least formally, the NC field goes to the ordinary monopole as $h \to 0$

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- whether these NC monopoles can be obtained by expanding in positive powers of $h\theta^{\mu\nu}$

BPS Monopoles for NC U(N)

For NC U(N) gauge theories:

- A NC U(2) BPS monopole that ”deforms" the ordinary SU(2) BPS monopole has been explicitly worked out up to $0(\theta^2)$[Bak(1999), Goto and Hata (PRD, 2000)]
- There exist NC Nahm's equations that lead to solutions of the NC BPS monopole eqs.[Bak (PLB, 1999)]
- NC U(N) BPS monopoles have been studied, for some N, by several groups:[Hashimoto and Hashimoto (JHEP, 1999)] [Hashimoto and Hirayama (NPB, 2000), Gross and Nekrasov (JHEP, 2000) Lechtenfeld and Popov (JHEP, 2004), ...]

But, what about NC monopoles when the gauge group is simple?

- Never discussed in the literature before our paper came along: U(N) $\nrightarrow$ SU(N)
- It should be recalled that NC gauge theory for SU(N) groups is rather different from NC U(N) gauge theories:

$A_\mu \in$ enveloping algebra of SU(N) $A_\mu \in$ Lie algebra of U(N)

AIM

To remedy this lack of results for simple gauge groups was the main aim of the work I shall report on:

- When a solution exists, compute up to $O(\theta^{\mu\nu})$ and for appropriate boundary conditions, the most general smooth solution of the SU(2) BPS eqs.
- Same for the SU(3) NC counterparts of the ordinary fundamental BPS monopoles and some two-monopole configurations
- Explicit computation up to $O(\theta^{\mu\nu})$ of the NC "deformation" of the ordinary family of solutions with a massless monopole that were analytically worked out for SO(5) [Weinberg (PLB, 1982)]
- We shall find that the NC BPS eqs. –for SU(2), SU(3) and SO(5)– do not have smooth solutions that "deform" at first order in $\theta^{\mu\nu}$ the "known" ordinary monopole configurations but for a specific type of Seiberg-Witten(SW) map. Then show –for arbitrary SW maps– the existence of smooth NC deformations of the ordinary monopole configurations by solving up to $0(\theta^{\mu\nu})$ the NC Yang-Mills-Higgs eqs. in the BPS limit.

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The model

The ordinary model:

- The BPS action, ie, no Higgs potential,reads

$$S_{ord} = \int d^4x \; -\tfrac{1}{2}\mathrm{Tr}\, f_{\mu\nu} f^{\mu\nu} + \mathrm{Tr}\,(D_\mu\phi)^\dagger D^\mu\phi$$
$$f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu - i[a_\mu, a_\nu], \quad D_\mu\phi = \partial_\mu\phi - i[a_\mu,\phi]$$

- The gauge field a_μ and the Higgs field $\phi \in$ Lie algebra of simple gauge group a G (SU(2), SU(3) and SO(5)in the fundamental rep.) and satisfy, in the $a_0 = 0$ gauge, the usual boundary conditions –$H < \infty$– in monopole physics:

$$\phi(t,\vec{x}) = g(t,\hat{x})\phi_0 g(t,\hat{x})^\dagger + O(\tfrac{1}{|\vec{x}|}) \quad \text{as} \quad |\vec{x}| \to \infty,$$
$$a_i(t,\vec{x}) \sim \tfrac{1}{|\vec{x}|} \quad \text{as} \quad |\vec{x}| \to \infty,$$
$$D_i\phi \sim \tfrac{1}{|\vec{x}|^2} \quad \text{as} \quad |\vec{x}| \to \infty,$$

where $\hat{x} = \frac{\vec{x}}{|\vec{x}|}$ and $\phi_0 = \phi(t,0,0,z=-\infty)$ and $v = 2\mathrm{Tr}\phi^2(t,|\vec{x}|=\infty)$

$$g(t,\hat{x}) : S_2 \to G/H \rightsquigarrow \pi_2(G/H)$$

G/H and H being respectively the broken and unbroken –the little group of ϕ_0– gauge groups.

The noncommutative model

The noncommutative model:

- Since the gauge group is simple, we must employ the formalism put forward and developed by [Madore, Schraml, Schupp and Wess (EPJC, 2000)], [Jurco, Möller, Schraml, Schupp and Wess (EPJC, 2001)], [Calmet, Jurco, Schupp, Wess and Wohlgenannt (EPJC, 2001)], [Aschieri, Jurco, Schupp and Wess (NPB, 2003)]
- The NC Yang-Mills-Higgs action in the BPS limit (no Higgs potential) reads

$$S = \int d^4x \, -\tfrac{1}{2}\mathrm{Tr}\, F_{\mu\nu} \star F^{\mu\nu} + \mathrm{Tr}\,(D_\mu\phi)^\dagger D^\mu \Phi$$
$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]_\star, \quad D_\mu\phi = \partial_\mu \Phi - i[A_\mu, \Phi]_\star$$
$$\star \rightsquigarrow (f \star g)(x) = f(x) e^{\frac{i}{2} h\theta^{\mu\nu} \overleftarrow{\partial}_\mu \overrightarrow{\partial}_\nu} g(x)$$

- The NC gauge field A_μ and the NC Higgs field Φ ϵ enveloping algebra of the Lie algebra of the gauge group G and are defined in terms of the ordinary fields a_μ and ϕ by means of the Seiberg-Witten map –see next:

The noncommutative model: the SW map

At first order in $h\theta^{\mu\nu}$, the most general SW map that yields hermitian A_μ and Φ and is a polynomial in the fields their derivatives and v reads

$$
\begin{aligned}
A_\mu &= a_\mu - \tfrac{h}{4}\theta^{\alpha\beta}\{a_\alpha, \partial_\beta a_\mu + f_{\beta\mu}\} + hD_\mu H + hS_\mu + O(h^2),\\
\Phi &= \phi - \tfrac{h}{4}\theta^{\alpha\beta}\{a_\alpha, 2D_\beta\phi + i[a_\beta,\phi]\} + ih[H,\phi] + hF + O(h^2),\\
H &= \mu_1\,\theta^{\alpha\beta} f_{\alpha\beta} + \mu_2\,\theta^{\alpha\beta}[a_\alpha, a_\beta],\\
S_\mu &= \kappa_1\,\theta^{\alpha\beta} D_\mu f_{\alpha\beta} + \kappa_2\,\theta_\mu{}^\beta\{D_\beta\phi,\phi\} + i\kappa_3\theta_\mu{}^\beta\,[D_\beta\phi,\phi] + k_4\,v\,\theta_\mu{}^\beta D_\beta\phi\\
&\qquad + w\,\theta_\mu{}^\rho D^\nu f_{\nu\rho},\\
F &= \lambda_1\theta^{\alpha\beta}\,\{f_{\alpha\beta},\phi\} + i\lambda_2\,\theta^{\alpha\beta}[f_{\alpha\beta},\phi] + \lambda_3\,v\,\theta^{\alpha\beta} f_{\alpha\beta}.
\end{aligned}
$$

$\mu_i,\ \kappa_i,\ \lambda_i$ and w dimensionless real constants.

SOME COMMENTS:

- When $\mu_i,\ \kappa_i,\ \lambda_i$ and $w = 0 \rightsquigarrow$ Standard SW map
- The monomials $\kappa_1\,\theta^{\alpha\beta} D_\mu f_{\alpha\beta}$, $\kappa_3\theta_\mu{}^\beta\,[D_\beta\phi,\phi]$, $w\,\theta_\mu{}^\rho D^\nu f_{\nu\rho}$, $i\lambda_2\,\theta^{\alpha\beta}[f_{\alpha\beta},\phi]$, $\kappa_4\,v\,\theta_\mu{}^\beta D_\beta\phi$ and $\lambda_3\,v\,\theta^{\alpha\beta} f_{\alpha\beta}$ $\in$ Lie algebra of the gauge group $\rightsquigarrow$ set to zero by redefining a_μ
- The monomials $\kappa_2\,\theta_\mu{}^\beta\{D_\beta\phi,\phi\}$ and $\lambda_1\theta^{\alpha\beta}\,\{f_{\alpha\beta},\phi\}\notin$ Lie algebra of the gauge group $\rightsquigarrow$ NOT set to zero by redefining a_μ –See below that κ_2 and λ_1 carry some Physics; not so for U(N)

The noncommutative model: the Hamiltonian

The Hamiltonian formalism:

- Use an elementary Hamiltonian formalism in the gauge $a_0 = 0$: one $p = \frac{\partial \mathcal{L}}{\partial \dot{q}}$ per $q = \{a_i, \phi\}$. Then,
- The action must not depend neither on the generalized accelerations nor on higher time derivatives:no $\partial_0^2 a_i,\ \partial_0^2 \phi$, etc. Thus,
- we set $\theta^{0i} = 0$ and $w = 0$ in the SW map:
  - No time derivatives in $\Phi[\phi, a_i],\ A_i[\phi, a_i]$
  - $a_0 = 0 \not\Rightarrow A_0 = 0$, but A_0 is linear in $\{\partial_0 a_i, \partial_0 \phi\}$
- $\mathcal{H} = \sum_i p_i \dot{q}_i - \mathcal{L} \quad \longmapsto$

$$\mathcal{H} = \int d^3\vec{x}\, \mathrm{Tr}\left(E_i E_i + B_i B_i + D_0\Phi D_0\Phi + D_i\Phi D_i\Phi\right),$$
$$E_i = F_{i0}, \quad B_i = \frac{1}{2}\epsilon_{ijk} F_{jk}$$

- Gauss's Law: $\mathrm{Tr}\frac{\delta A_0}{\delta a_0^a}\left(D_j E_j + i[D_0\Phi, \Phi]_\star\right) = 0$

U(N) case: $D_j E_j + i[D_0\Phi, \Phi]_\star = 0$

The Magnetic charge

- Asymptotic behaviour of A_μ and Φ:

$$\Phi(t,\vec{x}) \sim \phi(t,\vec{x}) + O(\tfrac{1}{|\vec{x}|}) \quad \text{as} \quad |\vec{x}| \to \infty,$$
$$A_\mu(t,\vec{x}) \sim a_\mu(t,\vec{x}) + O(\tfrac{1}{|\vec{x}|^2}) \quad \text{as} \quad |\vec{x}| \to \infty$$

- These boundary conditions $\Longrightarrow \mathcal{H} < \infty$ at each order in $h\theta^{\mu\nu}$
- (A_μ, Φ) are sorted out into equiv. classes by $\pi_2(G/H)$
- Let's introduce the magnetic charge Q_M:

$$Q_M = \frac{1}{2\pi v}\mathrm{Tr}\int_{S_2^\infty} dS_i\, B_i\Phi = \frac{1}{2\pi v}\mathrm{Tr}\int_{S_2^\infty} dS_i\, b_i\phi$$

$$S_2^\infty : \text{2sphere at } \infty,\ \ b_i = \tfrac{1}{2}\epsilon_{ijk} f_{jk}$$

- NC monopoles have the same magnetic charge as their ordinary counterparts: at very large $|\vec{x}|$ they look the same $\Longleftarrow \theta$-expanded SW map
- Q_M is a topological invariant

The Bogomol'nyi bound. The BPS EQS.

- The Bogomol'nyi trick:

$$\mathcal{H} = \int d^4x \,\mathrm{Tr}\left(D_0\Phi D_0\Phi + E_iE_i + (B_i \mp D_i\Phi)^2 \mp 4\pi\, v\, Q_M\right) \geq 4\pi\, v\, |Q_M|.$$

- For each Q_M, the absolute minima of $\mathcal{H}$ is reached iff:

$$B_i = \pm D_i\Phi, \quad D_0\Phi = 0, \quad E_i = 0$$

- If a_i, ϕ satisfying the previous eqs. are formal power series in $h\theta^{\mu\nu}$, the fact that for our class of SW maps

$$\begin{aligned} A_0 &= \textstyle\sum_{l>0} h^l L_0^{(l)i}[\theta, a_k, \phi, \partial_k]\dot{a}_i + \sum_{l>0} h^l M_0^{(l)}[\theta, a_k, \phi, \partial_k]\dot{\phi} \\ F_{0i} &= \dot{a}_i + \textstyle\sum_{l>0} h^l P_{0i}^{(l)j}[\theta, a_k, \phi, \partial_k]\dot{a}_j + \sum_{l>0} h^l Q_{0i}^{(l)}[\theta, a_k, \phi, \partial_k]\dot{\phi} \\ D_0\Phi &= \dot{\phi} + \textstyle\sum_{l>0} h^l S_0^{(l)j}[\theta, a_k, \phi, \partial_k]\dot{a}_j + \sum_{l>0} h^l T_0^{(l)}[\theta, a_k, \phi, \partial_k]\dot{\phi}, \end{aligned}$$

leads to $E_i = 0$ and $D_0\Phi = 0 \iff \dot{a}_i = 0$ and $\dot{\phi} = 0$

- We thus end with the NC BPS EQS.: $B_i = \pm D_i\Phi$,
to be satisfied by stationary a_i, ϕ that are power series in $h\theta^{\mu\nu}$

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NC SU(2) BPS Monopoles

- Let us look for solutions to $B_i = D_i\Phi$ that are of the type

$$a_i = a_i^{(0)} + \sum_{l>0} h^l a_i^{(l)}, \quad \phi = \phi^0 + \sum_{l>0} h^l \phi^{(l)}$$

  $a_i^{(l)}$ and $\phi^{(l)}$ ϵ Lie algebra of SU(2) in the fundamental rep. and are homogenous polynomials in $h\theta^{\mu\nu}$ of degree l

- Up to order $O(h\theta^{\mu\nu})$ the BPS eqs. read

$$O(h^0) \to b_i^{(0)} = (D_i\phi)^{(0)}$$
$$O(h^1) \to (f_{ij}^{(1)}[a_i^{(1)}] + F_{ij}^{(1,0)}[a_i^{(0)}, \phi^{(0)}]) = \epsilon_{ijk}[(D_k\phi)^{(1)} + \mathcal{O}_k^{(1,0)}[a_i^{(0)}, \phi^{(0)}]]$$

-

$$O(h^0) \quad \to \begin{cases} \phi^{(0)} = \frac{x^a}{r} H(r)\, \frac{\sigma^a}{2}, & H(r) = \pm(\frac{1}{r} - \lambda \coth \lambda r) \\ a_i^{(0)} = [1 - K(r)]\,\epsilon_{ial}\, \frac{x^l}{r^2}\, \frac{\sigma^a}{2}, & K(r) = 2 - \frac{\lambda r}{\sinh \lambda r}, \end{cases} \qquad \lambda = v$$

NC SU(2) BPS Monopoles: $O(\theta)$ eqs.

- $$O(h^1) \quad \Longleftrightarrow \begin{cases} a) & \mathrm{Tr}\big[(f_{ij}^{(1)} + F_{ij}^{(1,0)})\big] = \epsilon_{ijk}\mathrm{Tr}\big[(D_k\phi)^{(1)} + \mathcal{O}_k^{(1,0)}\big], \\ b) & \mathrm{Tr}\big[\frac{\sigma^a}{2}(f_{ij}^{(1)} + F_{ij}^{(1,0)})\big] = \epsilon_{ijk}\mathrm{Tr}\big[\frac{\sigma^a}{2}((D_k\phi)^{(1)} + \mathcal{O}_k^{(1,0)})\big] \end{cases}$$

- $a)$ $\Longleftrightarrow$ an eq. with no unkowns–ie, it is a constraint rather than a dynamical eq.:

$$\sum_a \tfrac{1}{2}[(f_{12}^{(0),\,a})^2 + (f_{13}^{(0),\,a})^2 + (f_{23}^{(0),\,a})^2]\theta_{ij} = \kappa_2\theta_{jk}\,\partial_i(\partial_k\phi^{(0),\,a}\phi^{(0),\,a}) - (i\leftrightarrow j) \pm \lambda_1\epsilon_{ijk}\,\partial_k\,[\,\theta^{kl} f_{kl}^{(0),\,a}\phi^{(0),\,a}].$$

- Substitution of $\{a_i^{(0)}, \phi^{(0)}\}$ –the ordinary monopole– in the previous eq. yields

$$\kappa_2 = -\frac{1}{2}, \quad \lambda_1 = \frac{1}{4}$$

- Only if the prevoius eqs. hold there exist NC BPS monopoles for SU(2). However, we are not through yet $\longmapsto$

NC SU(2) BPS Monopoles: zero mode $O(\theta)$ eqs.

- For the previous values of κ_2 and λ_1, eq. $b)$ $\Longleftrightarrow$

$$D_i^{(0)}(a_j') - D_j^{(0)}(a_i') = \epsilon_{ijk}\left(D_k^{(0)}\phi' - i[a_k', \phi^{(0)}]\right),$$

where

$$a_i' = a_i^{(1)} + \kappa_1\,\theta^{kl}\,D_i^{(0)} f_{kl}^{(0)} + i\kappa_3\,\theta_i{}^l\,[(D_l\phi)^{(0)}, \phi^{(0)}] + \kappa_4\,v\theta_i{}^j D_j^{(0)}\phi^{(0)},$$
$$\phi' = \phi^{(1)} + i\lambda_2\,\theta^{kl}\,[f_{ij}^{(0)}, \phi^{(0)}] + \lambda_3\,v\,\theta^{ij} f_{ij}^{(0)}.$$

- Hence, a_i' and ϕ' verify the zero mode eq. of the ordinary BPS monopole, so that
- the $O(h\theta^{\mu\nu})$ $-\{a_i^{(1)}, \phi^{(1)}\}-$ corrections to the ordinary SU(2) BPS monopole are given, modulo field redefinitions, by $\theta-$dependent linear combinations of the zero modes of that ordinary monopole

NC SU(2) BPS Monopoles: Conclusions

- If $\kappa_2 \neq -\frac{1}{2}$, or $\lambda_1 \neq \frac{1}{4}$, then, no NC BPS monopoles for SU(2). Hence, the value of κ_2 and λ_1 are not physically irrelevant –unlike the U(2) case.
- When they exist, the $O(h\theta^{\mu\nu})$ BPS corrections to the ordinary SU(2) BPS monopoles are given, modulo field redefinitions, by linear combinations with coefficients that depend linearly on $h\theta^{\mu\nu}$
- The dimension of the SU(2) BPS NC moduli space $=$ 4:
  - 3 moduli $\longleftrightarrow$ translations
  - 1 modulus $\longleftrightarrow$ U(1) transf.: $e^{i\chi\frac{\phi(\vec{x})}{v}}$, $0 \leq \chi < 2\pi$

Zero mode eqs.:

$\delta z = (\delta a_i, \delta\phi)$, $\delta z = \sum_{l\geq 0} h^l \delta z^{(l)}$ $=$ zero mode:

$$L^{(0)}\delta z^{(0)} = 0, \quad L^{(0)}\delta z^{(l)} = f^{(l)}[a_i^{(m)}, \phi^{(p)}, \delta z^{(q)}],\ f^{(l)}$$
$$L^{(0)} = \text{ordinary zero mode op., doesn't depend on } \delta z^{(l)}$$

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Fundamental monopoles

In ordinary BPS Yang-Mills-Higgs theories with simple higher-rank group G broken down to H:

- Fundamental monopole: BPS monopole obtained by embedding the SU(2) BPS monopole in G along a simple root of G. As E. Weinberg showed (NPB, 1980) they are building blocks since
- when there is maximal symmetry breaking –H is the maximal torus– all monopole and multi-monopole solutions can be thought of as objects made out of suitable numbers of different types of fundamental monopoles. In particular,

$$\text{Mass} = \sum_i n_i \, \text{Mass}_{\text{fund},\, i}$$

- It is natural to ask what are the NC BPS deformations –if any– of these fundamental monopoles: we shall consider the simplest choice SU(3) in the fundamental rep. $\longrightarrow$

SU(3) Fundamental monopoles: some group theory

- We shall take $G = SU(3)$ broken down to $U(1)\times U(1)$ by

$$\phi_0 \equiv \phi(0,0,z\to\infty) = v\,\vec{h}\cdot\vec{H}.$$

where

$$\vec{H} = (H_1, H_2),\ H_1, H_2 \text{ generators of the Cartan subalgebra of SU(3)}$$
$$\vec{h} = (h_1, h_2)\quad \backslash \vec{h}\cdot\vec{h} = 1,\ \vec{h}\cdot\vec{\alpha} \neq 0\ \ \forall \text{ roots } \vec{\alpha} \text{ of SU(3)}$$

- Let $\{\beta_1, \beta_2\}$ be the unique set of SU(3) simple roots such that $\vec{h}\cdot\beta_i > 0$ and $T^a_{\beta_i}$, $a = 1,2,3$ and $i = 1,2$ be the generators of $SU(2) \subset SU(3)$ defined by the simple root $\vec{\beta}_i$:

$$T^1_{\beta_i} = \frac{1}{\sqrt{2\beta_i^2}}(E_{\vec{\beta}_i} + E_{-\vec{\beta}_i}),\quad T^2_{\beta_i} = \frac{-i}{\sqrt{2\beta_i^2}}(E_{\vec{\beta}_i} - E_{-\vec{\beta}_i}),\quad T^3_{\beta_i} = \frac{1}{\beta_i^2}\vec{\beta}_i\cdot\vec{H},$$

SU(3) Fundamental monopoles: field configurations.

- By embedding the SU(2) monopole in the SU(2) subgroups of SU(3) defined by $\vec{\beta}_k,\ k=1,2$ one obtains the two fundamental SU(3) monopoles:

$$\phi^{(0)}_{\beta_k} = \sum_{a=1,2,3} \phi^{(0)\,a}\, T^a_{\beta_k} + v\vec{h}\cdot\vec{H} - v\vec{h}\cdot\vec{\beta}_k\, T^3_{\beta_k}$$
$$a^{(0)}_{i\,\beta_k} = \sum_{a=1,2,3} a^{(0)\,a}_i\, T^a_{\beta_k}, \quad k=1,2,$$

where $(a^{(0)}_i, \phi^{(0)})$ are coordinates –for $\lambda = v\vec{h}\cdot\vec{\beta}_i$– of the SU(2) monopole in the Pauli matrices basis

NC def. of SU(3) fundamental monopoles

- Let us look for solutions to the SU(3) NC BPS eqs. that are of the form
$$a_i = a^{(0)}_{i\,\beta_k} + h a_i^{(1)} + O(h^2), \quad \phi = \phi^{(0)}_{\beta_k} + h\phi^{(1)} + O(h^2), \quad k = 1, 2$$
$(a^{(0)}_{i\,\beta_k}, \phi^{(0)}_{\beta_k})$ are the ordinary fundamental monopoles in the previous slide
- Substituting (a_i, ϕ) above in the NC BPS eqs. and, then, taking traces, one obtains that $(a_i^{(1)}, \phi^{(1)})$ must verify the following eqs. –analogous to those for SU(2):
$$\begin{array}{ll} a) & \mathrm{Tr}\big[(f_{ij}^{(1)} + F_{ij}^{(1,0)})\big] = \epsilon_{ijk}\mathrm{Tr}\big[(D_k\phi)^{(1)} + \mathcal{O}_k^{(1,0)}\big], \\ b) & \mathrm{Tr}\big[\frac{\lambda^a}{2}(f_{ij}^{(1)} + F_{ij}^{(1,0)})\big] = \epsilon_{ijk}\mathrm{Tr}\big[\frac{\lambda^a}{2}\big((D_k\phi)^{(1)} + \mathcal{O}_k^{(1,0)}\big)\big] \end{array}$$
- Eq. $a)$ above only involves $(a_i^{(0)}, \phi^{(0)})$ –$\mathrm{Tr}\big[f_{ij}^{(1)}\big] = 0$, $\mathrm{Tr}\big[(D_k\phi)^{(1)}\big]$– so it is a no-go eq. rather than a dynamical eq. It holds iff
$$\kappa_2 = -1/2, \quad \lambda_1 = 1/4$$
- Hence, no NC deformations of the fundamental monopoles for arbitrary SW maps $\longrightarrow$

NC def. of SU(3) fund. monopoles: solutions

- For these values of κ_2 and λ_1 one shows after a lengthy algebra that Eq. *b*) above is equivalent to the zero mode eqs.

$$D_i^{(0)}(a_j') - D_j^{(0)}(a_i') = \pm\epsilon_{ijk}\left(D_k^{(0)}\phi' - i[a_k', \phi^{(0)}]\right),$$

where a_j' and ϕ' are defined in terms of $a_i^{(1)}$ and $\phi^{(1)}$ by the following identities:

$$a_i^{(1)} = a_i' - \kappa_1\,\theta^{kl}\,D_i^{(0)} f_{kl}^{(0)} - i\kappa_3\,\theta_i{}^l\,[D_l^{(0)}\phi^{(0)}, \phi^{(0)}] - \kappa_4\,v\,\theta_i^j D_j^{(0)}\phi^{(0)} + \underline{\frac{\phi^s}{2\sqrt{3}}\theta_i{}^j\,D_j^{(0)}\phi^{(0)}},$$

$$\phi^{(1)} = \phi' - i\lambda_2\,\theta^{kl}\,[f_{ij}^{(0)}, \phi^{(0)}] - \lambda_3\,v\,\theta^{ij} f_{ij}^{(0)} - \underline{\frac{\phi^s}{4\sqrt{3}}\,\theta^{ij}\,f_{ij}^{(0)}},$$

respectively

- $\phi^s = 2v\mathrm{Tr}(T^s_{\beta_k}\vec{h}\cdot\vec{H})$, with $T^s_{\beta_k}$ singlet under the action of the SU(2) associated to $\vec{\beta}_k$

NC deformation of an SU(3) two-monopole config.

A NC two-monopole deformation:

- Let us consider the embedding of the the ordinary SU(2) BPS monopole along the positive root $\vec{\beta}_3 = \vec{\beta}_1 + \vec{\beta}_2$. This is a field configuration with vector charge $(1, 1)$ and $\text{mass} = \text{mass}_{\beta_1} + \text{mass}_{\beta_2}$. It is a two-monopole configuration.
- For the appropriate SW map, there is, of course, a NC "deformation" of the ordinary configuration that is obtained from the expressions above by replacing $\vec{\beta}_k$ with $\vec{\beta}_3$
- Since the two-monopole is BPS its NC mass equals is ordinary mass:

$$\text{mass}_{\beta_3} = \text{mass}_{\beta_1} + \text{mass}_{\beta_2}, \quad \text{mass}_{\beta_k} = v\vec{h}\cdot\vec{\beta}_k,\ k = 1, 2$$

NC SU(3) fund. BPS Monopoles: Conclusions

When there is maximal symmetry breaking –SU(3)–>U(1)×U(1):

- If $\kappa_2 \neq -\frac{1}{2}$, or $\lambda_1 \neq \frac{1}{4}$, then, no NC deformations of SU(3) BPS monopoles obtained by embedding the SU(2) BPS monopole along roots. Hence, the value of κ_2 and λ_1 are not physically irrelevant –unlike the U(3) case.
- When they exist, the $O(h\theta^{\mu\nu})$ BPS corrections to the ordinary SU(3) fundamental monopoles are given, modulo field redefinitions, by linear combinations with coefficients that depend linearly on $\theta^{\mu\nu}$
- In the $O(h\theta^{\mu\nu})$ contributions there are field redefinitions that were not present in the SU(2) case:

$$
\begin{aligned}
a_i^{(1)} &\longrightarrow \frac{\phi^s}{2\sqrt{3}}\theta_i^{\ j}\, D_j^{(0)}\phi^{(0)} \\
\phi^{(1)} &\longrightarrow -\frac{\phi^s}{4\sqrt{3}}\,\theta^{ij}\, f_{ij}^{(0)}
\end{aligned}
$$

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SO(5): non-maximal breaking

- Let us assume that SO(5)–>SU(2)×U(1), ie, non-maximal SB
- A family of solutions to the BPS eqs. was found [Weinberg (PLB 1982)] that carry non-abelian magnetic charges:

$$
\begin{array}{ll}
a_i(1)^a = \epsilon_{aim}\, A(r)\, \frac{x_m}{r}\,, & \phi(1)^a = H(r)\, \frac{x_a}{r}\,, \\
a_i(2)^a = \epsilon_{aim}\, G(r,b)\, \frac{x_m}{r}\,, & \phi(2)^a = G(r,b)\, \frac{x_a}{r}\,, \\
a_i(3) = \sigma_i\, F(r,b), & \phi(3) = -iIF(r,b) \\
A(r) = \frac{1}{r} - \frac{v}{\sinh(r)} & H(r) = \frac{1}{r} - v\coth(vr) \\
F(r,b) = \frac{v}{\sqrt{8}\cosh(vr/2)}\, L(r,b)^{1/2} & G(r,b) = A(r)L(r,b) \\
L(r,b) = \left[1 + \frac{r}{b}\, \coth(\frac{vr}{2})\right]^{-1} & b > 0
\end{array}
$$

- The previous field configuration contains a massless monopole and does not correspond to an SU(2) embedding, but
  - there exist NC BPS deformations of it iff $\kappa_2 = -1/2$ and $\lambda_1 = 1/4$ and
  - the $O(h\theta^{\mu\nu})$ BPS deformations are linear combinations of the zero modes of the ordinary configuration + field redefinitions

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Purpose

- We shall look for static solutions –for $a_0 = 0$– to eqs. of motion that "deform" the ordinary BPS monopoles –$(a_i^{(0)}, \phi^{(0)})$– considered so far, for an arbitrary SW map:

$$a_i = a_i^{(0)} + h\, a_i^{(0)} + 0(h^2), \quad \phi = \phi^{(0)} + h\,\phi^{(1)}$$

- We will have to tackle the dreaded Yang-Mills-Higgs eqs.:

$$\int d^4x \left\{ \mathrm{Tr}\left[\frac{\delta A_\nu(x)}{\delta a^a_\mu(y)} \{ D_\rho F^{\rho\nu}(x) - i[\Phi, D^\nu \Phi]_\star(x) \} \right] - \mathrm{Tr}\left[\frac{\delta \Phi(x)}{\delta a^a_\mu(y)} \{ D_\rho D^\rho \Phi(x) \} \right] \right\} = 0$$
$$\int d^4x \left\{ \mathrm{Tr}\left[\frac{\delta A_\nu(x)}{\delta \phi^a(y)} \{ D_\rho F^{\rho\nu}(x) - i[\Phi, D^\nu \Phi]_\star(x) \} \right] - \mathrm{Tr}\left[\frac{\delta \Phi(x)}{\delta \phi^a(y)} \{ D_\rho D^\rho \Phi(x) \} \right] \right\} = 0$$

- BUT... IN THE BPS LIMIT: NO HIGGS POTENTIAL

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BPS YMH SU(2)

- For SU(2), the corrections, $(a_i^{(1)}, \phi^{(1)})$, linear in $h\theta^{\mu\nu}$ must verify the following eqs.:

$$
\begin{array}{l}
I)\quad D_i D_i \phi' - iD_i[a_i', \phi] - i[a_i', D_i\phi] = 0, \\
II)\quad D_i(D_i a_j' - D_j a_i') - i[a_i', f_{ij}] + i[\phi', D_j\phi] + i[\phi, D_j\phi' - i[a_j', \phi]], \\
\quad a_j' = a_j^{(1)} + \kappa_1\, \theta^{kl}\, D_j f_{kl} + i\kappa_3\, \theta_j{}^l\, [(D_l\phi), \phi] + v\kappa_4\, D_j\phi, \\
\quad \phi' = \phi^{(1)} + i\lambda_2\, \theta^{kl}\, [f_{ij}, \phi] + v\,\lambda_3\, \theta^{ij} f_{ij}
\end{array}
$$

- We shall look for (a_i', ϕ') such that

$$
a_i'(\vec{x}) \sim 1/|\vec{x}|^2 \quad \text{and} \quad \phi'(\vec{x}) \sim 1/|\vec{x}| \quad \text{as} \quad |\vec{x}| \to \infty.
$$

- Introducing $\bar{a}'_\mu = (a_i', \phi')$ and $\bar{a}_\mu = (a_i^{(0)}, \phi^{(0)})$, one may cast eq. $I)$ in the form

$$
\bar{D}_\mu \bar{X}_{\mu 4} = 0, \quad \bar{X}_{\mu 4} = \bar{D}_\mu \bar{a}'_4 - \bar{D}_4 \bar{a}'_\mu \mp \epsilon_{\mu 4 \rho\sigma} \bar{D}_\rho \bar{a}'_\sigma
$$

- One may show that the only normalizable $\bar{X}_{\mu 4}$ are those with $\bar{X}_{i4} = 0 \longrightarrow$

BPS YMH SU(2): the sol.

- Now $\bar{X}_{i4} = 0 \Leftrightarrow D_i\phi' - i[a'_i, \phi'] \pm \epsilon_{ijk} D_j a'_k = 0$
- Eq. *II*) is verified by (a'_i, ϕ') satisfying the previous eq.
- Hence, (a'_i, ϕ') are solutions to the zero mode eq. of the ordinary SU(2) BPS monopole
- In summary, for any SW map of our class the 1st order deformation is made out of the zero modes of the ordinary monopole plus field redefinitions

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BPS YMH SU(3): the sol

- By solving at first order in $\theta^{\mu\nu}$ the YMH eqs. in the BPS limit, we have found that the smooth deformation of an ordinary BPS monopole obtained by embedding the SU(2) BPS monopole along a root $\vec{\beta}_i,\ i=1,2,3$, is equal to

$$\phi^{(1)} = \delta\phi^{(0)} + (1-4\lambda_1)\theta^{ij}\epsilon_{ijk}x^k g(r) T^s_\beta - \frac{\lambda_1\,\phi^s}{\sqrt{3}}\theta^{ij}\,f_{ij} - i\lambda_2\,\theta^{ij}\,[f_{ij},\phi] - \lambda_3\,v\,\theta^{ij}\,f_{ij}$$
$$a_i^{(1)} = \delta a_i^{(0)} + (4\kappa_2+2)\theta^j{}_i x^j g(r) T^s_\beta - \kappa_1\,\theta^{kl}\,D_i f_{kl} - \frac{\kappa_2\phi^s}{\sqrt{3}}\,\theta_i{}^j D_j\phi - i\kappa_3\theta_i{}^j[D_j\phi,\phi]$$
$$-\,\kappa_4\,v\,\theta_i{}^j\,D_j\phi.$$

$\delta\phi^{(0)}$ and $\delta a_i^{(0)}$ are zero modes linear in $\theta^{\mu\nu}$

- The contributions in red have sharp physical consequences:

$$M_\beta = M_{\rm ordinary} + 0.10274\ h^2\theta^{ij}\theta^{ij}\,\lambda^5\left[\left(\kappa_2+\tfrac{1}{2}\right)^2 + 2\left(\lambda_1-\tfrac{1}{4}\right)^2\right] + O(h^2\theta^2)$$
$$M_{\rm ordinary} = 4\pi\lambda,\qquad \lambda = v\,\vec{\beta}\cdot\vec{h}$$

Unstable solution

- One may show that
  $$M_{\beta_3} - (M_{\beta_1} + M_{\beta_2}) = (\text{+ve no.})\left[\left(\kappa_2 + \frac{1}{2}\right)^2 + 2\left(\lambda_1 - \frac{1}{4}\right)^2\right] > 0$$
- Hence, unless we are dealing with a NC BPS field, the NC two-monopole configuration would not be stable and its constituents will move apart

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NC SO(5) BPS YMH sol.

- Ordinary family with massless monopole for SO(5)–>SU(2)$\times$U(1) leads to
- Results anologous to those obtained for SU(2)

Summary

We have shown that for Seiberg-Witten maps that do not give rise to higher time derivatives and for SU(2), SU(3) and SO(5) gauge groups

- there exist NC BPS monopoles that are ”smooth” deformations

$$a_i = a_i^{\text{ordinary}} + h a_i^{(1)} + ..., \quad \phi_i = \phi^{\text{ordinary}} + h\phi^{(1)} + ...$$

  of the ”basic" ordinary BPS monopole iff

$$\kappa_2 = -\frac{1}{2}, \quad \lambda_1 = \frac{1}{4}$$

- that when these BPS deformations exist they are given by the zero modes of the corresponding ordinary monopole plus field redefinitions

Summary cont.

- We have also shown that there exist NC NonBPS ”smooth" deformations of the ”basic" ordinary BPS monopole whatever the value of our huge family of Seiberg-Witten maps
- that for SU(2) and SO(5) these NC NonBPS deformations are given by the zero modes of the appropriate BPS monopole plus field redefinitions, but for the SU(3) monopole obtained by embedding the SU(2) monopole along a given root there contributions that are not of this type and give rise to NC corrections of mass of the ordinary monopole. The-two monopole configuration associated to $\vec{\beta}_3$ seems to be unstable.

Outlook

- Why the conditions on κ_2 and λ_1? Are they general (other groups, other reps, other config.)?
- Are the conditions on κ_2 and λ_1 demanded by extented SUSY
- Compute the order θ^2 contributions. Unlike the instanton case, Derrick´s th. does not precludes their existence –this we have shown