On the Phase Diagram of Fuzzy Scalar Field Theory

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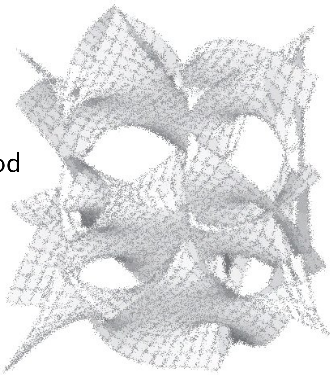
Based on:

- hep-th/0705.????, Denjoe O'Connor and CS.

Outline



Fuzzy Geometry as a Regulator

Studying fuzzy geometry is well motivated.

Planck-Scale Structure of Spacetime

Smooth structure of spacetime probably not to arbitrary scales.
The most prominent modifications: SUSY and Noncommutativity.
Fuzzy Geometry: NC on compact symplectic Riemannian spaces arise naturally in string theory

Regularization of Field Theories

Field theories on fuzzy spaces: finite-dimensional matrix models.
QFTs are finite and path integrals well-defined.
Advantages over lattice approach:
Isometries preserved, no fermion doubling, analytical handle
Numerical Simulations are easily done

Taking the commutative limit does not reproduce scalar ϕ^4-theory.
UV/IR-mixing distorts the picture.

[Vaidya, Chu, Madore, Steinacker]

Modifications of the naïve model, however, could cure this problem.

[Dolan, O'Connor, Presnajder]

To do:
Obtain an analytical handle on fuzzy ϕ^4-theory, in particular its phase diagram and study the effect of the proposed modifications.

(Gauge theory on the fuzzy sphere has recently been solved

[Steinacker, Szabo, hep-th/0701041]

scalar field theory appears to be simpler.)

Quantization of the sphere:

As usually, do not quantize space itself, but algebra of functions.

Basis: Spherical harmonics Y_{lm} with $l = 0, .., \infty, m = -l, ..., l$.

Quantization: Truncate angular momentum $l \leq L$
Multiplication will not close any more: $Y_{l_1 \ldots} Y_{l_2 \ldots} = Y_{l_1+l_2 \ldots} + \ldots$
However, deforming the product to the star product

$$[x^i \overset{\star}{,} x^j] \sim \mathrm{i}\varepsilon^{ijk} x^k \ ,$$

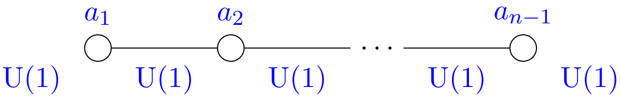
where $x^i \in \mathbb{R}^3 \supset S^2$ yields a closed, truncated algebra.

The Fuzzy Sphere

(a) Explicite construction from coherent states/group theory.

$$S^2 \cong \mathrm{SU}(2)/\mathrm{U}(1)$$

Consider irreducible representation ρ of $\mathrm{SU}(n)$, extended to $\mathrm{U}(n)$:



$n^2 - n$ "simple" raising and lowering operators $E^{\pm}_{\vec{\alpha}_i}$.
a_i: nontrivial actions $(E^{-}_{\vec{\alpha}_i})^{a_i}|\mu\rangle \neq 0 = (E^{-}_{\vec{\alpha}_i})^{a_i+1}|\mu\rangle$
n Cartan generators, isotropy group of $|\mu\rangle$: $H \supset U(1)^{\times n}$

One-To-One Correspondence:

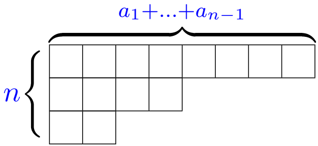
Coherent States $|p\rangle \in \rho \leftrightarrow p \in$ Coset $\mathrm{U}(n)/(U(m_1) \times ... \times U(m_k))$

$\Rightarrow$ Fuzzy Flag (Super) Manifolds [S. Murray, CS, hep-th/0611328]

The Fuzzy Sphere

Young diagrams yield a Fock space construction of NC functions on fuzzy geometries.

Representation $\rho = (a_1, \ldots, a_{n-1})$ corresponds to Young diagram



Fuzzy sphere: $\rho = (a_1) = L$ of $\mathrm{SU}(2)$:

$$\overbrace{\square\square\square\square}^{L} \cong \mathrm{span}(\ \hat{a}^\dagger_{\alpha_1} \ldots \hat{a}^\dagger_{\alpha_L} |0\rangle\ )\ , \quad \hat{\mathcal{L}}_i = \hat{a}^\dagger_\alpha \sigma^i_{\alpha\beta} \hat{a}_\beta$$

Isometry-preserving quantization of functions on S^2 via the rule:

$$f(p) = \langle p | \hat{f} | p \rangle\ ,$$

$$\hat{f} \in \overbrace{\square\square\square\square}^{L} \otimes \overbrace{\square\square\square\square}^{L} \cong \mathrm{span}(\ \hat{a}^\dagger_{\alpha_1} \ldots \hat{a}^\dagger_{\alpha_L} |0\rangle\langle 0| \hat{a}_{\beta_1} \ldots \hat{a}_{\beta_L}\ )$$

$$S^2 \cong \mathbb{C}P^1$$

The spherical harmonics $Y_{lm}, l \leq L$ can be written in terms of homogeneous coordinates z_α ($x^i \sim \bar{z}_\alpha \sigma^i_{\alpha\beta} z_\beta$) on $\mathbb{C}P^1$ in terms of

$$z_{\alpha_1} \dots z_{\alpha_L} \bar{z}_{\beta_1} \dots \bar{z}_{\beta_L}$$

with $\alpha_i, \beta_i = 1, 2$ due to the Hopf fibration

$$0 \rightarrow \mathrm{U}(1) \rightarrow S^3 \rightarrow \mathbb{C}P^1 \rightarrow 0 \ .$$

Quantization as in flat case, $(z_\alpha, \bar{z}_\beta) \rightarrow (\hat{a}^\dagger_\alpha, \hat{a}_\beta)$:

$$\hat{a}^\dagger_{\alpha_1} \dots \hat{a}^\dagger_{\alpha_L} |0\rangle\langle 0| \hat{a}_{\beta_1} \dots \hat{a}_{\beta_L}$$

In this way, also fuzzy versions of $X \hookrightarrow \mathbb{C}P^n$. [CS, hep-th/0612124]

$$S^2 \cong (\mathbb{C}P^1, \omega)$$

Take the line bundle $\mathscr{L} := \mathcal{O}(1)$ as the quantum line bundel.
Toeplitz quantization (Geometric quantization):

$$T^{(L)}: \quad C^\infty(M) \to \mathsf{End}\left(\Gamma(M, \mathscr{L}^{\otimes L})\right).$$

The set of sections $\Gamma(\mathbb{C}P^1, \mathscr{L}^{\otimes L})$ is spanned by

$$z_{\alpha_1} \ldots z_{\alpha_L}$$

The quantized algebra of functions is thus spanned by

$$z_{\alpha_1} \ldots z_{\alpha_L} \frac{\partial}{\partial z_{\beta_1}} \ldots \frac{\partial}{\partial z_{\beta_L}}$$

or, equivalently, by

$$\hat{a}^\dagger_{\alpha_1} \ldots \hat{a}^\dagger_{\alpha_L} |0\rangle\langle 0| \hat{a}_{\beta_1} \ldots \hat{a}_{\beta_L}$$

Consider the product of base elements

$$\hat{a}^\dagger_{\alpha_1}\dots\hat{a}^\dagger_{\alpha_L}|0\rangle\langle 0|\hat{a}_{\beta_1}\dots\hat{a}_{\beta_L} \quad \cdot \quad \hat{a}^\dagger_{\gamma_1}\dots\hat{a}^\dagger_{\gamma_L}|0\rangle\langle 0|\hat{a}_{\delta_1}\dots\hat{a}_{\delta_L}$$

This translates into

$$z_{\alpha_1}\dots z_{\alpha_L}\bar{z}_{\beta_1}\dots\bar{z}_{\beta_L} \quad \star \quad z_{\gamma_1}\dots z_{\gamma_L}\bar{z}_{\delta_1}\dots\bar{z}_{\delta_L}$$

With $f\star g := \mu(\mathcal{F} f\otimes g)$, the twist element reads as

$$\mathcal{F} = \left(\frac{1}{L!}\frac{\partial}{\partial\bar{z}^{\alpha_1}}\dots\frac{\partial}{\partial\bar{z}^{\alpha_L}}\right)\otimes\left(\frac{1}{L!}\frac{\partial}{\partial z^{\alpha_1}}\dots\frac{\partial}{\partial z^{\alpha_L}}\right)$$

This twist element does not have a left-inverse, however, a right-inverse can be defined [S. Kürkçüoğlu, CS, hep-th/0606197].

Fuzzy Scalar Field Theory: Definition.

Scalar field theory on the fuzzy sphere is a finite hermitian matrix model

Quantized algebra of functions on the fuzzy sphere:

$$\mathrm{span}(\ \hat{a}^\dagger_{\alpha_1}\ldots\hat{a}^\dagger_{\alpha_L}|0\rangle\langle 0|\hat{a}_{\beta_1}\ldots\hat{a}_{\beta_L}\ ) \cong \mathrm{Mat}(L+1)$$

Integration and Laplacian on the fuzzy sphere:

$$\int_{S^2} \mathrm{d}A\ f\ \rightarrow\ \frac{4\pi R^2}{N}\,\mathrm{tr}\,(\hat{f}) \qquad \mathcal{L}_i = \mathrm{i}\varepsilon_{ijk}x^j\partial_k \rightarrow [L_i,\cdot]\ ,\quad \Delta \rightarrow C_2$$

The action of real scalar field theory on the fuzzy sphere:

$$S\ \sim\ \mathrm{tr}\,\left(a[L_i,\Phi][L_i,\Phi] + b\,\Phi^2 + c\,\Phi^4\right)$$

We define the partition function

$$Z\ =\ \int \mathrm{d}\mu_D(\Phi)\ \mathrm{e}^{-\,\mathrm{tr}\,\left(a[L_i,\Phi][L_i,\Phi]+b\,\Phi^2+c\,\Phi^4\right)}$$

with the Dyson measure

$$\mathrm{d}\mu_D(\Phi)\ =\ \prod_{i\leq j}\mathrm{d}\Re(\Phi_{ij})\prod_{i>j}\mathrm{d}\Im(\Phi_{ij})$$

Fuzzy Scalar Field Theory vs. HMMs

Fuzzy scalar field theory is significantly harder than matrix models usually considered.

Fuzzy scalar field theory: $Z = \int \mathrm{d}\mu_D(\Phi)\ \mathrm{e}^{-\operatorname{tr}\left(a[L_i,\Phi][L_i,\Phi]+b\,\Phi^2+c\,\Phi^4\right)}$

First example: One-Hermitian Matrix Model

$$Z \;=\; \int \mathrm{d}\mu_D(\Phi)\ \mathrm{e}^{-\operatorname{tr}\left(b\,\Phi^2+c\,\Phi^4\right)}$$

Solution: splitting $\Phi = \Omega\Lambda\Omega^\dagger$, $\Lambda \;=\; \mathrm{diag}(\lambda_1,\ldots,\lambda_N)$ as well as $\int \mathrm{d}\mu_D(\Phi) \;=\; \int \prod_{i=1}^N \mathrm{d}\lambda_i \Delta^2(\Lambda) \int \mathrm{d}\mu_H(\Omega)$ yields

$$Z \;=\; \int \prod_{i=1}^N \mathrm{d}\lambda_i\ \mathrm{e}^{-2\sum_{i>j}\ln|\lambda_i-\lambda_j| - b\sum_i \lambda_i^2 + c\sum_i \lambda_i^4}$$

From here: saddle point, orthogonal polynomials, etc.

Fuzzy Scalar Field Theory vs. HMMs

Fuzzy scalar field theory is significantly harder than matrix models usually considered.

Fuzzy scalar field theory: $Z = \int \mathrm{d}\mu_D(\Phi)\ \mathrm{e}^{-\,\mathrm{tr}\left(a[L_i,\Phi][L_i,\Phi]+b\,\Phi^2+c\,\Phi^4\right)}$

Second example: Hermitian matrix model with one external matrix

$$Z \ = \ \int \mathrm{d}\mu_D(\Phi)\ \mathrm{e}^{-\,\mathrm{tr}\left(V(A\Phi)+b\,\Phi^2+c\,\Phi^4\right)}$$

Solution: splitting $\Phi = \Omega\Lambda\Omega^\dagger$, as well as character expansion

$$\exp(\,\mathrm{tr}\,(V(A\Phi))) = \sum_\rho f_\rho \chi_\rho(A\Phi)$$

Orthogonality relation: $\int \mathrm{d}\mu_H(\Omega)\chi_\rho(A\Omega^\dagger\Lambda\Omega) = \frac{1}{\dim(\rho)}\chi_\rho(A)\chi_\rho(\Lambda)$

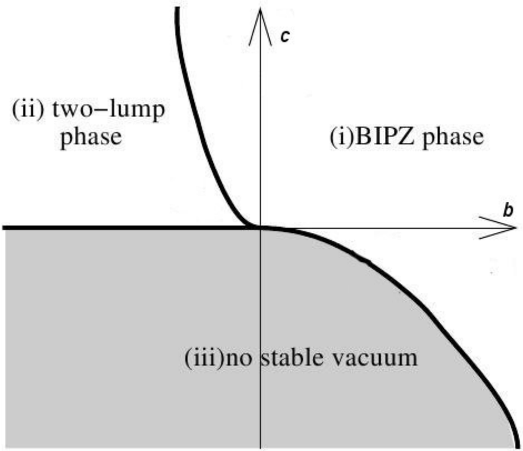
Formula by Itzykson and Di Francesco [hep-th/9212108]:

$$Z = \sum_{h_1<\ldots<h_N} \frac{\Pi(h^e-1)!!h^o!!}{\Pi(h^e-h^o)}\chi_\rho(A)\chi_\rho(t)$$

Known Results: Matrix Model Phase Diagram

The matrix model phase diagram suggests two phases.

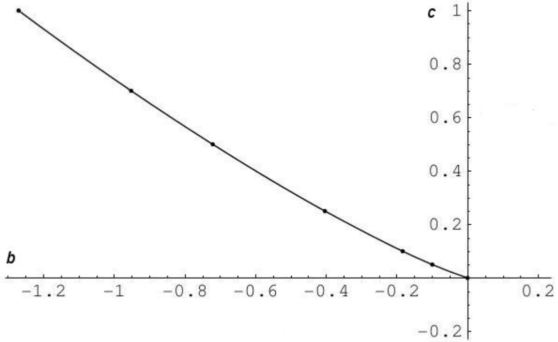
$$Z = \int \mathrm{d}\mu_D(\Phi)\ \mathrm{e}^{-\operatorname{tr}\left(b\,\Phi^2 + c\,\Phi^4\right)}\ , \quad 0 < \frac{\mathrm{d}x}{\mathrm{d}\lambda} < \infty$$



Known Results: Phase Diagram for ϕ^4-Theory on $\mathbb{R}^2$

The (lattice) model has two different phases.

$$Z = \int \mathscr{D}\phi \; \mathrm{e}^{-\int \mathrm{d}^2x \frac{1}{2}(\nabla\phi)^2 + b\phi^2 + c\phi^4} \;, \quad \langle\phi\rangle = 0 \;, \quad \frac{c}{b} = \text{const.}$$

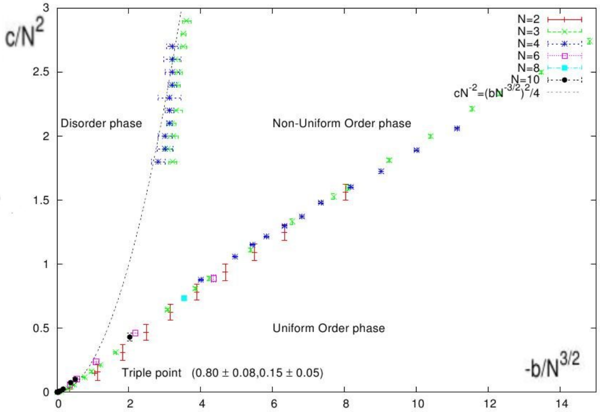


proof of existence: [Glimm, Jaffe, Spencer, 1974/1975]
exact shape numerically: [Loinaz, Willey, hep-lat/9712008],
confirmed by [Lee, hep-th/9811117]

Known Results: Numerical Simulations

The phase diagram suggests a combination of the phases.

$$Z = \int \mathrm{d}\mu_D(\Phi)\ \mathrm{e}^{-\beta\,\mathrm{tr}\left(a[L_i,\Phi][L_i,\Phi]+b\,\Phi^2+c\,\Phi^4\right)}\ , \quad \chi = \frac{\partial}{\partial\beta^2}\log(Z)$$



[Flores, O'Connor, Martin, hep-th/0601012, Panero, hep-th/0608202]

Known Results: ϕ^4-Theory on $\mathbb{R}^2_\theta$

The fuzzy model corresponds actually to regularized NC ϕ^4 theory.

- In ϕ^4-theory on $\mathbb{R}^2_\theta$:
  New phase predicted [Gubser, Sondhi, hep-th/0006119], analytically confirmed [Chen, Wu, hep-th/0110134] and numerically confirmed [Ambjorn, Catterall, hep-th/0209106].
- Indications for the new phase also found by regularization ϕ^4-theory with fuzzy spaces by [Steinacker, hep-th/0501174].
- Removal of new phase would be an indicator of successful regularization of commutative ϕ^4-theory.

$\Rightarrow$ Better analytical handle on fuzzy ϕ^4 theory necessary.

Fuzzy Scalar Field Theory: Simplifications

Symmetry arguments yield simplifications.

Fuzzy scalar field theory: $Z = \int \mathrm{d}\mu_D(\Phi)\ \mathrm{e}^{-\operatorname{tr}\left(a[L_i,\Phi][L_i,\Phi]+b\,\Phi^2+c\,\Phi^4\right)}$

$L = 1:\ Z = \int \mathrm{d}\lambda_1 \mathrm{d}\lambda_2\ (\lambda_1 - \lambda_2)^2\ \mathrm{e}^{-\left(a(\lambda_1-\lambda_2)^2+b(\lambda_1^2+\lambda_2^2)+c(\lambda_1^4+\lambda_2^4)\right)}$

Symmetries:

1. $\mathrm{d}\mu_D(\Phi) = \mathrm{d}\mu_D(\Omega\Phi\Omega^\dagger) \Rightarrow \int \mathrm{d}\mu_D(\Phi)\ \mathrm{e}^{-S} = \int \mathrm{d}\mu_D(\Phi)\ \mathrm{e}^{-S_0}$

$$S_0 = \sum_n s_n \operatorname{tr}(\Phi^n) + \sum_{n,m} s_{nm} \operatorname{tr}(\Phi^n) \operatorname{tr}(\Phi^m) + \ldots$$

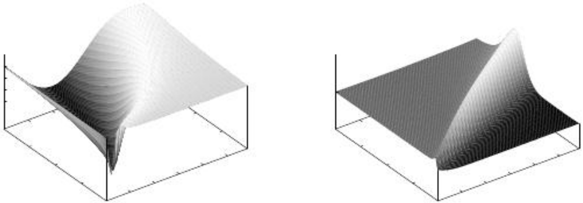
2. $\mathrm{d}\mu_D(\Phi)\ f(\Phi) \sim \mathrm{d}^{N^2}\Phi^\mu\ f(\Phi^\mu \tau_\mu) \Rightarrow S_0 = \sum_n s_n \left(\operatorname{tr}(\Phi^2)\right)^n$

3. $[L_i, \mathbb{1}] = 0, \lambda \leftrightarrow -\lambda \Rightarrow \operatorname{tr}([L_i,\Phi]^2) \sim \left(\sum_{i>j} (\lambda_i - \lambda_j)^{2m_k}\right)^{n_l}$

Idea: Treat the kinetic term perturbatively.

Motivation:

- Hopping parameter expansion successfully used on the lattice.
- Specific heat up to $\mathcal{O}(a^8)$ for $L = 1$:



- Group theoretical considerations allow everything else to be treated exactly.

Perturbative expansion: Principles

The angular variables can be integrated out in the perturbative series.

Introduce $K_{ab} := \operatorname{tr}([L_i, \tau_a][L_i, \tau_b])$, $\Phi^a = \operatorname{tr}(\tau^a \Omega \Lambda \Omega^\dagger)$. Then:

$$\mathrm{e}^{a\Phi^a K_{ab}\Phi^b} = 1 + a\Phi^a K_{ab}\Phi^b + \frac{a^2}{2}\Phi^a K_{ab}\Phi^b\, \Phi^c K_{cd}\Phi^d + \dots$$

To integrate over $\mathrm{d}\mu_H(\Omega)$ we need to compute terms like

$$\int \mathrm{d}\mu_H(\Omega) K_{ab} \operatorname{tr}(\tau^a \Omega \Lambda \Omega^\dagger) \operatorname{tr}(\tau^b \Omega \Lambda \Omega^\dagger)$$

Recall: $$\int \mathrm{d}\mu_H(\Omega)\ \ [\rho(\Omega)]_{ij}\ [\rho^\dagger(\Omega)]_{kl} = \frac{1}{\dim(\rho)}\delta_{il}\delta_{jk}$$

$$\operatorname{tr}\big((\tau^a \Omega \Lambda \Omega^\dagger) \otimes (\tau^b \Omega \Lambda \Omega^\dagger)\big) = \operatorname{tr}\big((\tau^a \otimes \tau^b)(\Omega \otimes \Omega)(\Lambda \otimes \Lambda)(\Omega^\dagger \otimes \Omega^\dagger)\big)$$

Thus: $$\int \mathrm{d}\mu_H(\Omega) K_{ab}\Phi^a\Phi^b = K_{ab}\sum_\rho \tfrac{1}{\dim(\rho)} \operatorname{tr}_\rho(\tau^a \otimes \tau^b) \operatorname{tr}_\rho(\Lambda \otimes \Lambda)$$

Perturbative expansion: Results

Up to $\mathcal{O}(a^2)$, the perturbative expansion is easily doable.

After some group theory and algebra, we obtain:

$$\int \mathrm{d}\mu_H(\Omega) K_{ab}\Phi^a\Phi^b = \frac{1}{2}N\sum_{i>j}(\lambda_i-\lambda_j)^2$$

$$\int \mathrm{d}\mu_H(\Omega) K_{ab}\Phi^a\Phi^b K_{cd}\Phi^c\,\Phi^d = \frac{(2\,\mathrm{tr}\,K^2+(\,\mathrm{tr}\,K)^2)}{N^2(N^4-10N^2+9)}(\alpha_1 A_1+\alpha_2 A_2) + \frac{1}{N(-36+N^2(-7+N^2)^2)}(\beta_1 A_1+\beta_2 A_2)K\urcorner K\ ,$$

where

$$A_1 = \sum_{i>j}(\lambda_i-\lambda_j)^4 \quad \text{and} \quad A_2 = \left(\sum_{i>j}(\lambda_i-\lambda_j)^2\right)^2$$

(Confirmation: Structure correct, limit $L=1$ is valid.)

In the large N limit, we have:

$$\int \mathrm{d}\mu_H(\Omega) K_{ab}\Phi^a\Phi^b = \frac{1}{2} N \sum_{i>j} (\lambda_i - \lambda_j)^2$$

$$\int \mathrm{d}\mu_H(\Omega) (K_{ab}\Phi^a\Phi^b)^2 = -\tfrac{N^2}{2} \sum_{i>j} (\lambda_i - \lambda_j)^4 + \tfrac{N^2}{4} \left(\sum_{i>j} (\lambda_i - \lambda_j)^2 \right)^2$$

After re-exponentiating the terms (still exact to $\mathcal{O}(a^2)$)

$$S = \sum_i \left(b\lambda_i^2 + c\lambda_i^4 \right) + \sum_{i>j} \Big(-\tfrac{a}{2} N (\lambda_i - \lambda_j)^2 + \tfrac{a^2}{4} N^2 (\lambda_i - \lambda_j)^4 - 2 \ln |\lambda_i - \lambda_j| \Big)$$

Saddle point approximation

The saddle point approximation gives a rought picture of what is going on.

Rewrite: $\lambda_i \to \lambda(\frac{i}{N}) = \lambda(x)$, $0 < x < 1$, $\sum_{i=0}^{N} \to N\int_0^1 \mathrm{d}x$

Rescale: $a = N^{\theta_a}\tilde{a},\ b = N^{\theta_b}\tilde{b},\ c = N^{\theta_c}\tilde{c},\ \lambda(x) = N^{\theta_\lambda}\tilde{\lambda}(x)$

Partition function: $Z = \displaystyle\int \mathscr{D}\lambda\ \exp(-N^2\tilde{S})$

Action:

$$\tilde{S} = \int_0^1 \mathrm{d}x\Big(\tilde{b}\tilde{\lambda}^2(x) + \tilde{c}\tilde{\lambda}^4(x) + \int_0^1 \mathrm{d}y\Big(-\tfrac{\tilde{a}}{4}(\tilde{\lambda}(x) - \tilde{\lambda}(y))^2 + \tfrac{\tilde{a}^2}{8}(\tilde{\lambda}(x) - \tilde{\lambda}(y))^4 - \ln|\tilde{\lambda}(x) - \tilde{\lambda}(y)|\Big)\Big)$$

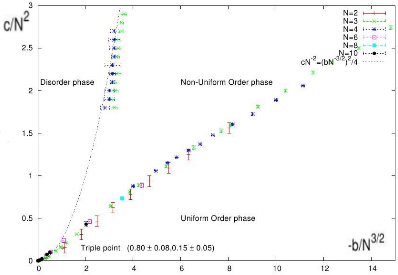
Saddle point solution (one symmetric cut $[-\delta, \delta]$): $u(\tilde{\lambda}) =$

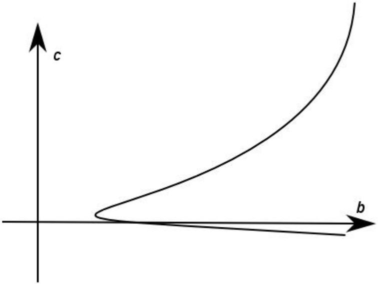
$$\left(4\tilde{b} - \tilde{a} + 12\pi\tilde{a}^2 c_2 + 4\left(\tilde{c} + \tfrac{\pi\tilde{a}^2}{2}\right)\delta^2 + 8\left(\tilde{c} + \tfrac{\pi\tilde{a}^2}{2}\right)\tilde{\lambda}^2\right)\sqrt{\delta^2 - \tilde{\lambda}^2}$$

Saddle point approximation

New phase lies within the region of the two-cut solution.

The boundary of the region of validity of the one-cut solution is consistent with the data.





Numerics, fuzzy ϕ^4-theory   Analytical results, multi-trace MM

$\Rightarrow$ "New phase" must already be a feature of the matrix model.

Modification of the model

The proposed modification of the model moves the triple point in the right direction.

Modify the action, assuming momentum-dependent wave function regularization $\mathcal{Z}_L(C_2) \approx 1 + \kappa C_2$:

$$\tilde{S} = \operatorname{tr}\left(a\Phi(C_2 + \kappa C_2 C_2)\Phi + b\,\Phi^2 + c\,\Phi^4\right)$$

This implies the following modification in our analysis

$$K_{ab} \to \check{K}_{ab} := K_{ab} + \kappa K_{ac} K_{cb} \quad \text{and} \quad \check{a} = a(1 + \tfrac{2}{3}\kappa(N^2 - 1))$$

Rescaling of κ to keep highest order term yields $\tilde{\check{a}} = a(1 + \frac{2}{3}\tilde{\kappa})$.

$\Rightarrow$ The triple point moves off to infinity for increasing $\tilde{\kappa}$.

We achieved the following:

- formulated a generalized character expansion technique
- reformulated fuzzy ϕ^4-theory as multi-trace matrix model
- preliminary analysis of the approximation looks promising

Future directions:

- Examine all possible one- and two-cut solutions; this should yield a (full) explanation of the phase diagram
- Examine modifications of the model for extended solutions
- Extract more information from the expansion
- Study other models (other fuzzy spaces) with this technique
- Relation to $c > 1$ string theories?

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