Quantum field theory on projective modules

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Introduction

- Classical field theories for fundamental interactions (electroweak, strong, gravitational) of geometrical origin
- Quantum field theory for standard model (electroweak+strong) is renormalisable
- Gravity is not renormalisable

Renormalisation group interpretation

- space-time being smooth manifold $\Rightarrow$ gravity scaled away
- weakness of gravity determines Planck scale where geometry is something different

promising approach: noncommutative geometry
(unifies standard model with gravity [as classical field theories])

Scalar fields and projective modules

- classical picture: scalar fields on (space-time) manifold M are sections of some vector bundle $\mathcal{V}$ over M

Serre-Swan theorem (M – compact)

space of sections of $\mathcal{V}$ $\Leftrightarrow$ finitely generated projective module $\mathcal{E}$ over algebra $C(M)$ of continous functions

- $\{U_i\}_{i=1,\dots,N}$ – open cover of M
  $|f_i|^2$ – partition of unity, g_{ij} – transition functions on $U_i \cap U_j$
- $e_{ij} := f_i^* g_{ij} f_j \in C(M)$ satisfy $\sum_{j=1}^N e_{ij} e_{jk} = e_{ik}$
  $\mathcal{E} = e(C(M))^N$ projective module over $C(M)$
- generalisation: A - noncommutative C^*-algebra
  $e \in M_N(A)$ projection, $\rightarrow \mathcal{E} = eA^N$

Connections

- pass to smooth level: $\mathcal{A}$ – Fréchet pre-C^*-algebra of A
- differential algebra $\Omega(\mathcal{A}) = \bigoplus_{n\in\mathbb{N}} \Omega^n(\mathcal{A})$, $\quad \Omega^0(\mathcal{A}) = \mathcal{A}$
  – differential $d : \Omega^n(\mathcal{A}) \to \Omega^{n+1}(\mathcal{A})$
  – scalar product $\langle\ ,\ \rangle_n$ on $\Omega^n(\mathcal{A})$
  (e.g. obtained from spectral triple and Dixmier trace)
- hermitian structure $(\ ,\ )_{\mathcal{A}} : \mathcal{E} \times \mathcal{E} \to \mathcal{A}$
  – yields scalar product $\langle\ ,\ \rangle_{\mathcal{A},n}$ on $\mathcal{E} \otimes_{\mathcal{A}} \Omega^n(\mathcal{A})$
- connection $\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega^1(\mathcal{A})$
  – fulfilling $d(\psi,\eta)_{\mathcal{A}} = (\nabla\psi,\eta)_{\mathcal{A}} + (\psi,\nabla\eta)_{\mathcal{A}} \in \Omega^1(\mathcal{A})$

Noncommutative torus

$$\mathcal{A}^d_\Theta := \Big\{ a = \sum_{\gamma\in\mathbb{Z}^d} a_\gamma U_\gamma \;:\; U_\gamma \text{ - unitary}, \; U_\gamma U_{\gamma'} = e^{-i\pi\Theta(\gamma,\gamma')} U_{\gamma+\gamma'}, \; \{a_\gamma\} \in \mathcal{S}(\mathbb{Z}^d) \Big\}$$

$\Theta = -\Theta^t \in M_d(\mathbb{R})$ defines 2-cocycle on $\mathbb{Z}^d \subset \mathbb{R}^d$

$\Theta_{ij} \in \mathbb{Z}$: $\mathcal{A}^d_\Theta$ = algebra of functions on ordinary torus
$\Theta_{ij} \in \mathbb{Q}$: $\mathcal{A}^d_\Theta$ = bundle of matrix algebras over ordinary torus
$\Theta_{ij} \notin \mathbb{Q}$: $\mathcal{A}^d_\Theta$ = truly noncommutative space

- Fréchet semi-norms $p_n(a) := \sup_{\gamma\in\mathbb{Z}^d} (1+\|\gamma\|^2)^n |a_\gamma|$
- derivations $\delta_\mu(U_\gamma) = 2i\pi\gamma_\mu U_\gamma \qquad \gamma = (\gamma_1,\dots,\gamma_d) \in \mathbb{Z}^d$
- $\Omega^n(\mathcal{A}^d_\Theta) = \{(\omega_{\mu_1,\dots,\mu_n}) \in \Lambda^n\big((\mathcal{A}^d_\Theta)^d\big)$ – completely antisymm.$\}$
- trace $\mathrm{Tr}_{\mathcal{A}^d_\Theta}\Big(\sum_{\gamma\in\mathbb{Z}^d} a_\gamma U_\gamma\Big) := a_0$

irrational $\mathcal{A}^d_\Theta$ is nc manifold without boundary, $\mathrm{Tr}_{\mathcal{A}^d_\Theta}(\delta_\mu U_\gamma) = 0$

Heisenberg modules

- $G = \mathbb{R}^p \times \mathbb{Z}^q \times F$ – abelian group, $\widehat{G}$ – dual of G
- Heisenberg group = central extension of $G \times \widehat{G} \ni (g, \mu)$, acts on Hilbert space $L^2(G, dg) \ni \psi$ by
  $(T_{g,\mu}\psi)(x) := \mu(g)^{\frac{1}{2}}\mu(x)\psi(x+g)$
- $\Gamma \simeq \mathbb{Z}^d$ lattice in $G \times \widehat{G}$; restriction of $G \times \widehat{G}$ to $\Gamma \ni \gamma = (g, \mu)$ yields right action of $\mathcal{A}^d_\Theta$ on $L^2(G, dg)$:
  $$\boxed{\psi U_\gamma := T_{g,\mu}\psi}$$
  Heisenberg cocycle $e^{-2i\pi\Theta(\gamma,\gamma')} := \mu(g')\mu'(g)^{-1}$
- $(G \times \widehat{G})/\Gamma$ compact (enforces d even)
  $\Rightarrow \mathcal{E}_H := \mathcal{S}(G)$ projective module (Heisenberg module)

  hermitian structure $(\psi, \chi)_{\mathcal{A}_\Theta} := \sum_{\gamma\in\Gamma}\langle\psi, \chi U_\gamma\rangle_{L^2(G,dg)} U_{-\gamma}$

  connection from infinitesimal action of $G \times \widehat{G}$

Simplest example: Schwartz module $\mathcal{E}_S = \mathcal{S}(\mathbb{R})$ over $\mathcal{A}_\theta^2$

i.e. $d = 2$ and $G = \widehat{G} = \mathbb{R}$, $\Theta = \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix}$

- $(\phi U_\gamma)(x) := e^{i\pi\theta mn} e^{2i\pi nx} \phi(x+m\theta)$, $\gamma = (\theta m, 2\pi n) \in \Gamma$
- hermitian structure
  $$(\phi, \chi)_{\mathcal{A}_\theta^2} = \sum_{\gamma\in\Gamma} \Big(e^{i\pi\theta mn} \int_{\mathbb{R}} dx\ \bar{\phi}(x) e^{2i\pi nx} \chi(x+\theta m) \Big) U_{-\gamma}$$
- covariant derivatives compatible with hermitian structure:
  $$(\nabla_1 \phi)(x) = -\frac{2i\pi x}{\theta} \phi(x) \qquad (\nabla_2 \phi)(x) = \phi'(x)$$

extended to $\mathcal{S}(\mathbb{R}^{\frac{d}{2}})$ over $\mathcal{A}_\theta^d := (\mathcal{A}_\theta^2)^{\frac{d}{2}}$, interesting case is $d = 4$

Bargmann module

Bargmann space $\mathcal{H}_{\mathcal{B}} = L^2_{hol}(\mathbb{C}, d\mu)$ of holomorphic functions

- scalar product $\langle \phi, \chi \rangle_B := \int d\mu(z,\bar{z})\ \bar{\phi}(\bar{z})\chi(z)$

  measure $d\mu(z,\bar{z}) = \frac{\omega}{\pi} e^{-\omega|z|^2} d\Re(z) d\Im(z) \qquad \omega = \frac{2\pi}{\theta}$

- projective repr. $(T_v\phi)(z) := e^{-\frac{\omega|v|^2}{2} - \omega\bar{v}z}\phi(z+v)$ of $v \in \mathbb{C}$ satisfies $T_v T_w = e^{\frac{\omega}{2}(\overline{v}w - \overline{w}v)} T_{v+w}$

  yields right action $(\phi U_\gamma)(z) := (T_{\tilde{\gamma}}\phi)(z)$ of $\mathcal{A}^2_\theta$ if $\tilde{\gamma} = \frac{\theta}{\sqrt{2}}(m + in)$ for $\gamma = (\theta m, 2\pi n) \in \Gamma$

- Bargmann transform $B : \mathcal{H}_{\mathcal{S}} \to \mathcal{H}_{\mathcal{B}}$

  $(B\chi)(z) := \left(\frac{\omega}{\pi}\right)^{\frac{1}{2}} \int_{\mathbb{R}} dx\ e^{-\frac{\omega}{2}(x^2 - 2\sqrt{2}zx + z^2)}\chi(x)$

  $(B^{-1}\phi)(x) := \left(\frac{\omega}{\pi}\right)^{\frac{1}{2}} \int_{\mathbb{C}} d\mu(z,\bar{z})\ e^{-\frac{\omega}{2}(x^2 - 2\sqrt{2}\bar{z}x + \bar{z}^2)}\phi(z)$

  transports structures from Schwartz to Bargmann module

Scalar field theory

$\phi \in \mathcal{E} = \mathcal{S}(\mathbb{R}^2)$ $\qquad \phi^\dagger \in \mathcal{E}^* = \{\mathcal{A}_\theta^4\text{-linear forms on } \mathcal{E}\}$

$\phi^\dagger(\chi) := (\phi, \chi)_{\mathcal{A}_\theta^4}$

Action functional on 2-dim. Schwartz module over $\mathcal{A}_\theta^4$

$$
\begin{aligned}
&S[\phi, \phi^\dagger] \\
&:= \langle \nabla\phi, \nabla\phi \rangle_{\mathcal{A}_\theta^4, 1} + \mathrm{Tr}_{\mathcal{A}_\theta^4}\Big(\mu^2 (\phi,\phi)_{\mathcal{A}_\theta^4} + \frac{\lambda}{2} (\phi,\phi)^2_{\mathcal{A}_\theta^4}\Big) \\
&= \int_{\mathbb{R}^2} dx\, \bar{\phi}(x) \Big(-\frac{\partial^2}{\partial x_i \partial x_i} + \frac{4\pi^2}{\theta^2} x_i x_i + \mu^2 \Big) \phi(x) \\
&+ \frac{\lambda}{2} \int_{\mathbb{R}^2} dx \sum_{m,n \in \mathbb{Z}^2} \bar{\phi}(x+n+\theta m)\, \phi(x+n)\, \bar{\phi}(x)\, \phi(x+\theta m)
\end{aligned}
$$

- non-local interaction
- harmonic oscillator term $x^2|\phi(x)|^2$ appears automatically (ensures renormalisation of scalar models on Moyal plane)

Relation to matrix models ($d = 2$)

interaction term $\displaystyle\int_{\mathbb{R}} dx\, dy\, dz\, dt\; V(x,y,z,t)\,\bar\phi(x)\,\phi(y)\,\bar\phi(z)\,\phi(t)$

with $\displaystyle V(x,y,z,t) = \frac{\lambda}{2}\sum_{m,n\in\mathbb{Z}} \delta(y-x-m\theta)\,\delta(z-x-m\theta-n)\,\delta(t-x-n)$

local interaction on quotient space $\mathbb{R}/(\mathbb{Z}+\theta\mathbb{Z})$

meaningful for $\theta=\frac{p}{q} \in \mathbb{Q} \Rightarrow$ models of $p\times q$ rectangular matrices

- F =vector bundle over 2-torus $\mathbb{T}^2$ of radius $\frac{1}{q}$

$$\Gamma^\infty(F) := \{\mathcal{M} : [0,\tfrac{1}{q}]\times[0,\tfrac{1}{q}] \to M(p\times q,\mathbb{C}) \text{ smooth },$$
$$\mathcal{M}(x,y+\tfrac{1}{q}) = \mathcal{M}(x,y)$$
$$\mathcal{M}(x+\tfrac{1}{q},y) = (\Omega_p)^a(qy)\mathcal{M}(x,y)(\Omega_q)^{-b}(-qy)\}$$

$$aq+bp=1, \qquad \Omega_N(y) = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ e^{2i\pi y} & & & 0 \end{pmatrix} \in M(N\times N,\mathbb{C})$$

relation to Schwartz module:

$\rho : \mathcal{S}(\mathbb{R}) \to \Gamma^\infty(F)$, $\phi \mapsto \mathcal{M}_{kl}(x,y) := \sum_{n\in\mathbb{Z}} \phi\big(x+\frac{kq+lp+npq}{q}\big)\, e^{-2i\pi nqy}$

inverse $\rho^* : \Gamma^\infty(F) \to \mathcal{S}(\mathbb{R})$, $\mathcal{M}(x,y) \mapsto \phi(x) := q\int_0^{\frac{1}{q}} dy\, \mathcal{M}_{00}(x,y)$

- scalar product

$$\langle \mathcal{M},\mathcal{N}\rangle_F := q\int_0^{\frac{1}{q}} dx \int_0^{\frac{1}{q}} dy\, \mathrm{Tr}(\mathcal{M}^\dagger(x,y)\mathcal{N}(x,y)) = \langle \rho^*\mathcal{M}, \rho^*\mathcal{N}\rangle_{L^2(\mathbb{R})}$$

- induces connection on $\Gamma^\infty(F)$:

  $\nabla_1 = \frac{\partial}{\partial y} - \frac{2i\pi px}{q} + L(A) + R(B)$, $\qquad \nabla_2 = \frac{\partial}{\partial x}$

  with diagonal matrices $A_{kk} = -2i\pi pk$, $B_{ll} = -\frac{2i\pi l}{p}$

Action functional

$$S[\rho^*\mathcal{M}, (\rho^*\mathcal{M})^\dagger]$$
$$= q\int_0^{\frac{1}{q}} dy \int_0^{\frac{1}{q}} dx\, \mathrm{Tr}\Big((\nabla_i\mathcal{M})^\dagger(\nabla_i\mathcal{M}) + \mu^2\mathcal{M}^\dagger\mathcal{M} + \frac{\lambda}{2}(\mathcal{M}^\dagger\mathcal{M})^2\Big)(x,y)$$

Quantum field theory

Correlation functions = distributions on $\mathcal{E}^{*\otimes N} \otimes \mathcal{E}^{\otimes N}$

$$G_{2N}(\chi_1^\dagger,\dots,\chi_N^\dagger,\psi_1,\dots,\psi_N) = \frac{\int [D\phi][D\phi^\dagger]\,\mathrm{Tr}_{\mathcal{A}_\theta^d}\big((\chi_1,\phi)_{\mathcal{A}_\theta^d}\big)\dots\mathrm{Tr}_{\mathcal{A}_\theta^d}\big((\phi,\psi_N)_{\mathcal{A}_\theta^d}\big)e^{-S[\phi,\phi^\dagger]}}{\int [D\phi][D\phi^\dagger]e^{-S[\phi,\phi^\dagger]}}$$

- generating functional (of general correlation functions)
$$Z[J,J^\dagger] = \frac{\int [D\phi][D\phi^\dagger]\, e^{-S[\phi,\phi^\dagger]+\mathrm{Tr}_{\mathcal{A}_\theta^d}((J,\phi)_{\mathcal{A}_\theta^d})+\mathrm{Tr}_{\mathcal{A}_\theta^d}((\phi,J)_{\mathcal{A}_\theta^d})}}{\int [D\phi][D\phi^\dagger]e^{-S[\phi,\phi^\dagger]}}$$
- connected correlations functions: $W[J,J^\dagger] = \log Z[J,J^\dagger]$
- one-particle irreducible correlations functions:
$\Gamma[\varphi,\varphi^\dagger] = \mathrm{Tr}_{\mathcal{A}_\theta^d}\big((J,\varphi)_{\mathcal{A}_\theta^d}\big) + \mathrm{Tr}_{\mathcal{A}_\theta^d}\big((\varphi,J)_{\mathcal{A}_\theta^d}\big) - W[J,J^\dagger]$
where $\varphi = \frac{\delta W}{\delta J^\dagger}, \quad \varphi^\dagger = \frac{\delta W}{\delta J}$

Feynman rules for Bargmann module

Action functional for 2-dim. Bargmann module ($z \in \mathbb{C}^2$)

$$S[\phi,\bar{\phi}] = \int d\mu\, \bar{\phi}(\bar{z})(H\phi)(z) + \int d\mu_1 \dots d\mu_4 V(z_1,\overline{z_2},z_3,\overline{z_4})\bar{\phi}(\overline{z_1})\phi(z_2)\bar{\phi}(\overline{z_3})\phi(z_4)$$

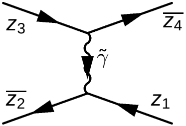
$$H = 2\omega\Big(z_i \frac{\partial}{\partial z_i} + \frac{d}{4}\Big) + \mu^2$$

$$V(z_1,\overline{z_2},z_3,\overline{z_4}) = \frac{\lambda}{2}\sum_{\tilde{\gamma}} e^{-\omega|\tilde{\gamma}|^2+\omega((\overline{z_2}-\overline{z_4})\tilde{\gamma}+\bar{\tilde{\gamma}}(z_3-z_1)+\overline{z_2}z_1+\overline{z_4}z_3)}$$

- regularised propagator

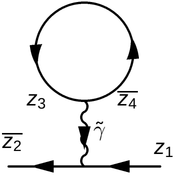
$$= H_\epsilon^{-1}(z_1,\bar{z}_2) = \frac{1}{2}\int_\epsilon^\infty d\beta\, e^{-\frac{\beta\mu^2}{2}+\omega\bar{z}_1 z_2 e^{-\beta\omega}}$$

- vertex

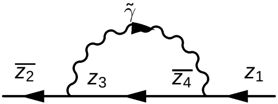


$$= V(z_1,\overline{z}_2,z_3,\overline{z_4})$$

One-loop two-point function



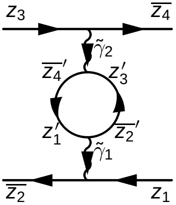
$$= \frac{\lambda}{2\omega}\mathrm{Tr}_{\mathcal{A}^4_\theta}\big[(\varphi_1,\varphi_2)_{\mathcal{A}^4_\theta}\big] \times\Big(\frac{1}{\epsilon\omega}+\Big(1-\frac{\mu^2}{2\omega}\Big)\ln\frac{1}{\epsilon}\Big)+\mathcal{O}(1)$$



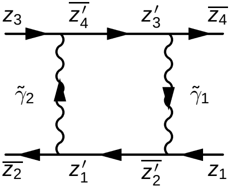
$$= \frac{\lambda}{2\theta^2\omega}\mathrm{Tr}_{\mathcal{A}^4_\theta}\big[(\varphi_1,\varphi_2)_{\mathcal{A}^4_\theta}\big] \times\Big(\frac{1}{\epsilon\omega}+\Big(1-\frac{\mu^2}{2\omega}\Big)\ln\frac{1}{\epsilon}\Big)+\mathcal{O}(1)$$

- the graphs are dual to each other (Poisson resummation)
- corresponds to local mass renormalisation for any $\theta \in \mathbb{R}$
- no wave function renormalisation

One-loop four-point function: planar sector



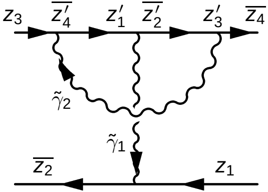
$$= -\frac{\lambda^2}{8\omega^2}\mathrm{Tr}_{\mathcal{A}^4_\theta}\Big[(\varphi_1,\varphi_2)_{\mathcal{A}^4_\theta}(\varphi_3,\varphi_4)_{\mathcal{A}^4_\theta}\Big]\ \ln\frac{1}{\epsilon} + \mathcal{O}(1)$$



$$= -\frac{\lambda^2}{8\theta^2\omega^2}\mathrm{Tr}_{\mathcal{A}^4_\theta}\Big[(\varphi_1,\varphi_2)_{\mathcal{A}^4_\theta}(\varphi_3,\varphi_4)_{\mathcal{A}^4_\theta}\Big]\ \ln\frac{1}{\epsilon} + \mathcal{O}(1)$$

- the graphs are dual to each other
- corresponds to local coupling constant renormalisation for any $\theta \in \mathbb{R}$

One-loop four-point function: non-planar sector



$$= -\frac{\lambda^2}{4\theta^2} \sum_{\tilde\gamma_1,\tilde\gamma_2^*} e^{\omega(\overline{z_2}z_1+\overline{z_4}z_3+i\overline{\tilde\gamma_2^*}z_3+\overline{z_4}i\tilde\gamma_2^*-\overline{\tilde\gamma_1}z_1+\overline{z_2}\tilde\gamma_1)}$$

$$\times \int_\epsilon^\infty d\beta_1 d\beta_2 \frac{e^{-\frac{\mu^2}{2}(\beta_1+\beta_2)}}{(1-e^{-(\beta_1+\beta_2)\omega})^2}$$

$$\times e^{-\frac{\omega}{1-e^{-(\beta_1+\beta_2)\omega}}|\tilde\gamma_1+i\tilde\gamma_2^*|^2}$$

$$\times e^{-\frac{\omega}{1-e^{-(\beta_1+\beta_2)\omega}}(-\overline{\tilde\gamma_1}i\tilde\gamma_2^*(1-e^{-\beta_2\omega})+i\overline{\tilde\gamma_2^*}\tilde\gamma_1(1-e^{-\beta_1\omega}))}$$

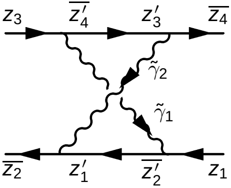
- $\tilde\gamma_2^*$ appears after Poisson resummation in $\tilde\gamma_2$
  $\tilde\gamma_1 = \frac{\theta}{\sqrt{2}}(m_1+in_1)$, $\tilde\gamma_2^* = \frac{1}{\sqrt{2}}(m_2+in_2)$ $\qquad m_{1,2}, n_{1,2} \in \mathbb{Z}$
- β-integral divergent for $\tilde\gamma_1 + i\tilde\gamma_2^* = 0$, solutions depend on θ

Number-theoretical aspect for $\tilde{\gamma}_1 + i\tilde{\gamma}_2^* = 0$

- $\theta \in \mathbb{Z}$ $\Rightarrow$ divergence for all $\tilde{\gamma}_1$
  local counterterm
- $\theta = \frac{p}{q} \in \mathbb{Q}$ $\Rightarrow$ divergence for some $\tilde{\gamma}_1$
  translation to matrices:
  counterterm $q^2 \int_{[0,\frac{1}{q}]^4} d^2y\, d^2x \left(\mathrm{Tr}(\mathcal{M}^\dagger \mathcal{M})(x,y)\right)^2$
  unfamiliar, but local on 4-torus
- $\theta \notin \mathbb{Q}$ $\Rightarrow$ only solution is $\tilde{\gamma}_1 = \tilde{\gamma}_2^* = 0$
  additionally: finiteness of integrals for $\tilde{\gamma}_1 + i\tilde{\gamma}_2^* \neq 0$ requires Diophantine condition on θ:
  $\forall n \in \mathbb{Z} \setminus \{0\}\ \exists C, \delta > 0$ s.t. $\inf_{m\in\mathbb{Z}} |n\theta - m| \geq C|n|^{-(1+\delta)}$

  non-local counterterm $\left(\mathrm{Tr}_{\mathcal{A}_\theta^4}\left[(\varphi,\varphi)_{\mathcal{A}_\theta^4}\right]\right)^2$

similar discussion for second non-planar graph



(proof of finiteness under Diophantine condition much more complicated)

Result

The scalar field theory on the Schwartz module $\mathcal{S}(\mathbb{R}^2) \ni \phi$ over the 4-dimensional noncommutative torus $\mathcal{A} = \mathcal{A}_\theta^4$ (θ irrational and Diophantine), defined by the action

$$S[\phi, \phi^\dagger] = \langle \nabla\phi, \nabla\phi \rangle_{\mathcal{A},1} + \mathrm{Tr}_{\mathcal{A}}\Big(\mu^2(\phi,\phi)_{\mathcal{A}} + \frac{\lambda}{2}(\phi,\phi)^2_{\mathcal{A}}\Big) + \frac{\lambda'}{2}\Big(\mathrm{Tr}_{\mathcal{A}}\big((\phi,\phi)_{\mathcal{A}}\big)\Big)^2$$

is one-loop renormalisable

Outlook: What about higher loop order?

Bargmann module inappropriate (no positivity)
work directly with Schwartz module $\mathcal{S}(\mathbb{R}^2)$

- propagator becomes 2-dim. Mehler kernel
$$H^{-1}(x,y) = \frac{\omega^2}{4\pi^2}\int_0^\infty \frac{d\beta}{\sinh(2\omega\beta)} e^{-\frac{\omega}{4}\left(\coth(\omega\beta)|x-y|^2+\tanh(\omega\beta)|x+y|^2\right)}$$
(cf. 4-dim. Mehler kernel for renormalisable Moyal model)
- vertex reads after Poisson resummation $(x,y,z,t \in \mathbb{R}^2)$
$$V(x,y,z,t) = \frac{\lambda}{(2\theta)^4}\sum_{m,n\in\mathbb{Z}^2} \delta(x-y+z-t)\, e^{\frac{2i\pi}{\theta}m(x-y)+2i\pi n(x-t)}$$
- compare with Moyal vertex $(x,y,z,t \in \mathbb{R}^4)$
$$V_\star(x,y,z,t) = \frac{\lambda}{(\pi\theta)^4}\,\delta(x-y+z-t)\, e^{2i\theta^{-1}(x,y)+2i\theta^{-1}(z,t)}$$

Mehler + Moyal in x-space well understood [Orsay group]
$\rightarrow$ extend this analysis by the new γ-x-interaction