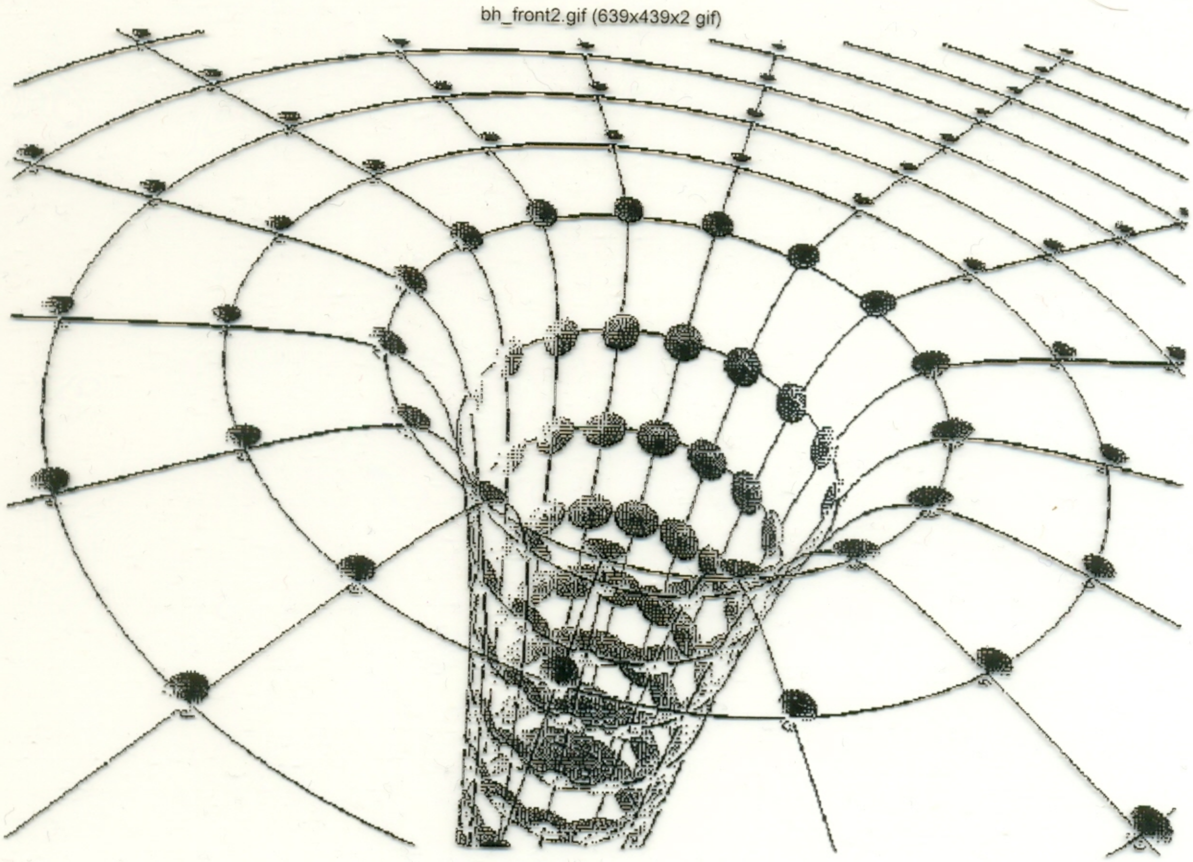
DESTINATION:

NONCOMMUTATIVE SCHWARZSCHILD

(VIA FRAMES)



1. Noncommutative frames & constraints
2. 4d example
3. 5d models

1. Frames

- Classical gravity, or more generally, Riemann-Cartan geometry, can be (very efficiently) described in terms of the frames (tetrad, Vielbein)

- A frame is, roughly, a set of orthonormal 1-forms:

$$\theta^\alpha = \theta^\alpha{}_\mu(x)\, dx^\mu$$

$$g^{\alpha\beta} = g(\theta^\alpha \otimes \theta^\beta) = \eta^{\alpha\beta}$$

- For example, for the 2d Schwarzschild line element

$$ds^2 = -f(r)\, dt^2 + \frac{1}{f(r)}\, dr^2$$

with $f(r) = 1 - \frac{2m}{r}$, we choose

$$\theta^0 = \sqrt{f}\, dt \quad , \quad \theta^1 = \frac{1}{\sqrt{f}}\, dr$$

so that

$$ds^2 = -(\theta^0)^2 + (\theta^1)^2$$

1. Frames

- One can choose differently too, e.g. define the null-frame:

$$ds^2 = -f\left(dt^2 - \frac{1}{f^2}dr^2\right) = -f\left(dt^2 - dr_*^2\right)$$

where r_* is the tortoise coordinate, $r_* = \int \frac{dr}{f}$. Then with

$$\theta^+ = \sqrt{\frac{f}{2}}\,(dt + dr_*) \ , \ \theta^- = \sqrt{\frac{f}{2}}\,(dt - dr_*)$$

we have

$$ds^2 = -2\theta^+\theta^-$$

- Metric elements are constant in the frame basis; θ^α are, however, not full differentials

$$d\theta^\alpha = -\frac{1}{2}C^\alpha{}_{\beta\gamma}\,\theta^\beta\theta^\gamma$$

- If we assume that torsion vanishes, $C^\alpha{}_{\beta\gamma}$ determine the connection

$$\omega_{\alpha\beta} = -\frac{1}{2}\left(C_{\alpha\beta\gamma} - C_{\gamma\alpha\beta} + C_{\beta\gamma\alpha}\right)\theta^\gamma$$

and the connection defines the curvature

$$\Omega^\alpha{}_\beta = d\omega^\alpha{}_\beta + \omega^\alpha{}_\gamma\,\omega^\gamma{}_\beta$$

1. Frames

- In the noncommutative case, in order to describe gravity geometrically, we need to define a noncommutative space = an algebra

$$[x^\mu, x^\nu] = i\hbar J^{\mu\nu}(x)$$

with a compatible differential structure.

- Usually one assumes that the differential obeys the Leibniz rule: if we are in a formal algebra this is a constraint

- Differential on NC space can be defined using a noncommutative frame θ^α as

$$d\varphi = e_\alpha \varphi \cdot \theta^\alpha = [p_\alpha, \varphi]\, \theta^\alpha,$$

to have constant $g^{\alpha\beta}$, it is sufficient to impose that θ^α commute with the algebra

$$[\theta^\alpha, x^\mu] = 0 .$$

- p_α are the momenta; usually we take them to be in the NC space

- To summarize : if we denote

$$dx^\mu = e^\mu{}_\alpha(x)\,\theta^\alpha \quad , \quad \theta^\alpha = \theta^\alpha{}_\mu(x)\,dx^\mu ,$$

we have the following consistency constraints

- Leibniz $\quad i\hbar\, e_\alpha J^{\mu\nu} = [e^\mu{}_\alpha, x^\nu] - [e^\nu{}_\alpha, x^\mu]$
- Jacobi $\quad [x^\lambda, J^{\mu\nu}] + [x^\mu, J^{\nu\lambda}] + [x^\nu, J^{\lambda\mu}] = 0$
- from $d[\theta^\alpha, x^\mu] = 0$ we get

$$[x^\mu, C^{[\alpha}{}_{\gamma\delta}]\,\theta^{\beta]}{}_\mu = 0 .$$

- To proceed : we restrict to almost-commutative approximation and work to first order in $\hbar$. This is not necessary (eg., fuzzy sphere). Then we have

$$[x^\mu, \varphi] = i\hbar\, J^{\mu\nu}\partial_\nu \varphi + O(\hbar^2) .$$

- We wish to find a smooth NC space, i.e. solve the given constraints. The initial step is to define spherically symmetric Ansatz.

2. 4d example

- Assume we have 4 NC coordinates, t and x^i, $i = 1,2,3$. The radial coordinate r is defined by $r^2 = x^i x^i$

- A possible choice of commutators, a solution of the Jacobi constraints, is

$$J^{0i} = e x^i \,, \quad J^{ij} = 0 \,.$$

Then also $[t, r] = i\hbar\, e \cdot r$

- Assume further Schwarzschild-like, diagonal Ansatz for the frame

$$\theta^0 = F(r)\, dt \,, \qquad \theta^i = \frac{1}{F(r)}\, dx^i$$

- The Leibniz constraint then gives

$$J^{0r} \frac{dF}{dr} = eF$$

i.e. $F(r) = \mu \cdot r$

2.4d example

- Thus the corresponding limiting classical metric is fixed

$$ds^2 = -\mu^2 r^2 dt^2 + \frac{1}{\mu^2 r^2} dx^i dx^i$$

$$= -\mu^2 r^2 dt^2 + \frac{1}{\mu^2 r^2} dr^2 + \frac{1}{\mu^2} d\Omega^2$$

- This solution is not really satisfactory, as metrically, the space is a direct product of a (t, r) part and the angular part

- Still, this solution is quite interesting: if we define electromagnetic field tensor as

$$F_{0i} = (-g)^{1/4} J_{0i}$$

we obtain that

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \text{const} \cdot T_{\mu\nu}$$

for this space, $T_{\mu\nu}$ is the energy-momentum of the electromagnetic field. Noncommutativity behaves (is) as a source of gravity

5d models

- Maybe, in the previous example, the spherical symmetry was not represented in a proper way. Idea: use the fuzzy sphere construction as it contains representation of rotations (as momenta). Deform?

- For the fuzzy sphere: given SO(3) generators J^i

$$[J^i, J^j] = i\varepsilon^{ijk} J^k$$

the coordinates are defined as

$$x^i = \frac{\hbar}{r} J^i,$$

$$[x^i, x^j] = i\hbar \frac{1}{r} \varepsilon^{ijk} x^k.$$

r - radius - is a constant. The momenta, which define the differential calculus, are

$$p^i = \frac{1}{i\hbar} x^i$$

and obviously also generate rotations.

5d models

- Ansatz : choose 5 coordinates: t, r, x^i.
  x^i are proportional to J^i, except that now r is not a constant. As quantization we require that the area of the sphere is

$$4\pi r^2 = 2\pi\hbar \cdot n$$

From this we deduce that the relation between the coordinates should be

$$x^2 = x^i x^i = r^2\left(1 - \frac{c\hbar^2}{r^4}\right) = \frac{r^4 - a^4}{r^2} \qquad (*)$$

It reduces the number of coordinates on the manifold to 4.

- We assume further

$$[x^i, r] = 0$$

$$[t, r] = i\hbar\, \beta(r)$$

$$[t, x^i] = i\hbar\, \frac{\gamma(r)}{r}\, x^i$$

The consistency with the constraint $*$ gives

$$\frac{\beta}{\gamma} = \frac{r^4 - a^4}{r^4 + a^4}$$

5d models

- The remaining commutator is $[x^i, x^j]$. Assuming

$$[x^i, x^j] = i\hbar\, \varepsilon^{ijk} \frac{\alpha(r)}{r} x^k$$

and imposing the Jacobi identity

$$[t,[x^i,x^j]]+[x^j,[t,x^i]]+[x^i[x^j,t]]=0$$

we can determine the function $\alpha(r)$:

$$\log\frac{\alpha}{r} = \int \frac{r^4+a^4}{r^4-a^4}\frac{dr}{r} = \frac{1}{2}\log\sinh\frac{2r}{a} + \text{const}$$

- This completes the algebra of coordinates. It has one undetermined (or arbitrary) function $\beta(r)$.

5d models

- Choosing momenta, in the frame formalism, we fix the differential calculus. This is an additional Ansatz. We take

$$p_i = \frac{1}{i\hbar} \sigma(r)\, x_i$$

$$p_r = \frac{1}{i\hbar} t$$

$$p_t = \frac{1}{i\hbar} g(r)$$

- Note that then

$$[p_i, x_j] = \varepsilon_{ijk} \frac{\sigma \alpha}{r} x^k$$

$$[p_i, p_j] = \varepsilon_{ijk} \frac{\sigma \alpha}{r} p^k \, ;$$

- p_i are not exactly the generators of rotations

5d models

- For the differentials we get

$$dx^i = \varepsilon^i{}_{jk} \frac{\sigma\alpha}{r} x^j \theta^k + \frac{\gamma}{r} x^i \theta^1$$

$$dr = \beta \theta^1$$

$$dt = -\sigma\beta\left(\frac{\sigma'}{\sigma} + \frac{\gamma}{\beta r}\right) x^i \theta^i - \beta g' \theta^0$$

where $\theta^0, \theta^1, \theta^i$ is the NC frame.

- From the fuzzy sphere we know that the first relation can not be inverted easily; one has to introduce, in principle, a 'monopole potential'

$$\theta^i = \tau(r)\, \varepsilon^i{}_{jk} x^j dx^k + x^i A$$

$$= \tau(r)\, \varepsilon^i{}_{jk} x^j dx^k - i\hbar \frac{\tau\alpha\gamma}{r^2} x^i \theta^1 + x^i A'$$

- We obtain

$$\left(1 + \hbar^2 \frac{\tau\sigma\alpha^3}{r^3}\right) x^i \theta^i = x^2 A'$$

5d models

- The momentum algebra is

$$[p_t, p_i] = 0$$

$$[p_r, p_i] = \left(\frac{\sigma'}{\sigma}\beta + \frac{\gamma}{r}\right) p_i$$

$$[p_r, p_t] = \frac{1}{i\hbar} g'\beta$$

but from the general properties of the NC frame formalism we know it should be quadratic. This gives the following constraints on the unknown functions $\beta(r)$, $\sigma(r)$, $g(r)$:

$$\beta\left(\log \frac{d\sigma}{r}\right)' = Eg + F$$

$$\beta g' = Ag^2 + Bg + C + D\sigma^2 x^2$$

$A, B, C, \ldots F$ are constants, arbitrary.

5d models

- One can now analyze the solutions of the given structure, in particular, the classical limit

- A promising one seems to be $A' = 0$.

  Then we have

$$dx^i = \varepsilon^i{}_{jk} \frac{\sigma\alpha}{r} x^j \theta^k + \frac{\gamma}{r} x^i \theta^1$$

$$dr = \beta \theta^1$$

$$dt = -\beta g' \theta^0$$

and $x^i \theta^i = 0$. It is a static metric.

Calculating $\theta^i \theta^i$ we obtain, additionaly, the function $\tau(r)$

$$\frac{\tau\sigma\alpha}{r} x^2 = 1 .$$

- The line element in the limit is

$$ds^2 = -(\theta^0)^2 + (\theta^1)^2 + \theta^i\theta^i$$

$$= -\frac{1}{(\beta g')^2} dt^2 + \frac{1}{\beta^2} dr^2 + r^2 d\Omega^2$$

assuming that $\theta^i \theta^i = r^2 d\Omega^2$

5d models

- This solution can accomodate Schwarzschild.

Comparing it with

$$ds^2 = -f(r)\,dt^2 + \frac{1}{f(r)}\,dr^2 + r^2 d\Omega^2$$

we see that we need

$$\beta = \sqrt{f} \quad , \quad g = \int \frac{dr}{f} = r_*$$

- This is possible, as β, g and σ are related by 2 equations. The choice of β and g necessary for the Schwarzschild metric, through the equations & for $D=0$, fixes $\sigma(r)$:

$$\left(\frac{d\sigma}{r}\right)^c = d + c r_*$$

for some fixed constants c and d.

(I.e. : if there is no further restriction beyond the horizon!!)