DYNAMICAL NONCOMMUTATIVITY AND NOETHER THEOREM IN TWISTED FIELD THEORY

L. CASTELLANI

U.P.O. - ALESSANDRIA

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- Why noncommutative geometry at small scales ?

- Euristically:

General relativity → Existence of a minimal length ← Quantum mechanics

$$L_P = \sqrt{\frac{hG}{2\pi c^3}} \approx 10^{-33}\, cm$$

L_P: Planck scale

- To "measure" geodetics in a gravitational field: use freely falling particles with mass ***m***.

- How precisely can we measure geometry?

- Uncertainty in position of the particle:

$$\Delta q \approx \frac{\hbar}{mc}$$

corresponding to $\Delta p \sim mc$ (pair creation threshold)

- To decrease Δq, increase ***m*** $\rightarrow$ then also the curvature of spacetime increases until …

curvature radius $\approx \Delta q$

- what is then the order of Δq ?

$$-R = \frac{8\pi G}{c^4} T \rightarrow 1/\Delta q^2 \approx \frac{8\pi G}{c^4} T^0{}_0 \approx \frac{8\pi G}{c^2} \frac{m}{\Delta q^3}$$

and substituting $m \sim h / (c\,\Delta q)$

$$\Delta q \approx (hG / c^3)^{1/2}$$

$\approx$ Planck length

It is therefore impossible to observe phenomena (or spacetime structure) under the Planck scale L_P

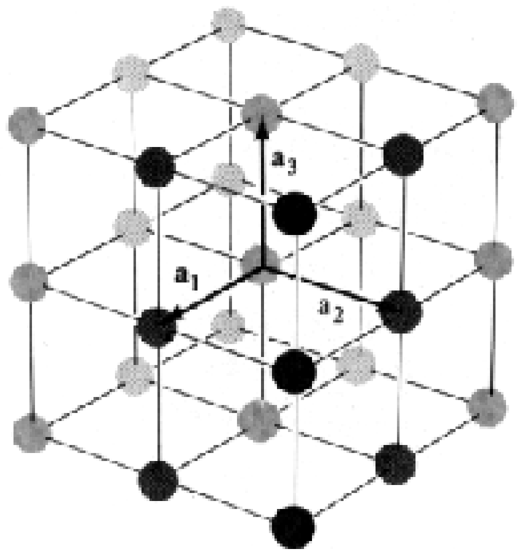
- This indetermination emerges automatically if the coordinates are noncommutative :

$$x\,y - y\,x = \alpha (L_P)^2 \qquad\qquad x \quad y \quad (L_P)^2$$

or in general:

$$[x^i, x^j] = i\,\vartheta^{ij}$$

with $\vartheta^{ij} =$ antisymmetric tensor (constant)



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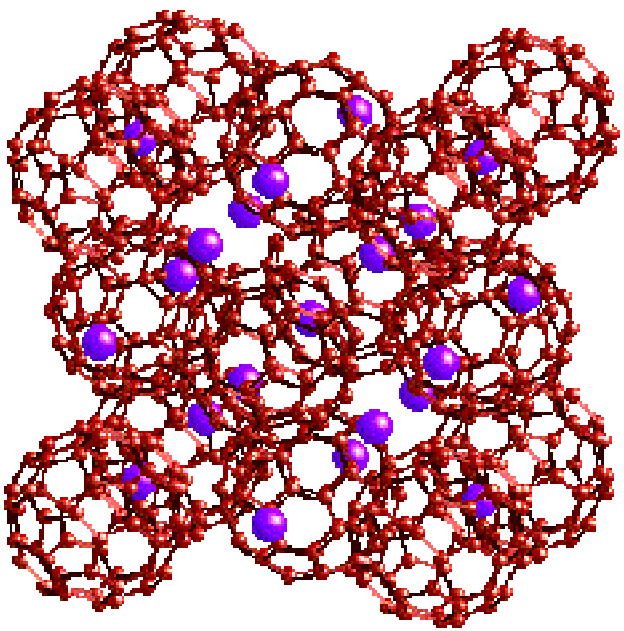


NCG is a recurrent theme in physics:

- Well-known example: phase space of (nonrelativistic) quantum mechanics

$$[x^i, p^j] = -i\hbar\,\delta^{ij}$$

$\Rightarrow$ Heisenberg indetermination principle: $x \quad p \quad \approx \hbar$

First discussion on the "quantum" differential geometry of phase space are due to Dirac (1926).

- from the algebra of operators x^i, p^j $\rightarrow$ idea of considering also spacetime coordinates as noncommuting operators.

Heisenberg, in a letter to Peierls, suggests that an indetermination principle on space(time) coordinates could resolve the problem of UV divergences.
Peierls ③ Pauli ③ Oppenheimer ③ Snyder (1947)

Noncommutativity of coordinates *x, y* of an electron in a strong magnetic field (Peierls 1933) orthogonal to *xy* plane

$$L = \tfrac{1}{2} m v^2 + \frac{e}{c} \mathbf{A} \cdot \mathbf{v} - V(x, y)$$

In the gauge **A** = *(0, bx)* where *b* is the modulus of the magnetic field **b** :

$$L = \tfrac{1}{2} m v^2 + \frac{eb}{c} x\dot{y} - V(x, y)$$

With *V(x,y)=0* , the quantized theory predicts the well known Landau energy levels, with separations of the order of *b/m* .
The limit of high *b* (equivalent to the limit *m* ③ *0*) projects the system on the lowest Landau level. Taking *m = 0* in the Lagrangian:

$$L_0 = \frac{eb}{c} x\dot{y} - V(x, y)$$

$\left(\frac{eb}{c} x, y \right)$ are conjugated variables

$$\Rightarrow \quad \left\{ x^i, x^j \right\} = \frac{c}{eb} \varepsilon^{ij}$$

NB: *x, y* become noncommuting operators only after quantizing the theory, i.e. noncommutativity is a *quantum effect.*

Some appearances of NCG in physics:

- in other problems of condensed matter physics, for ex. in the quantum Hall effect (Belissard et al, 1993)

- in Connes' "reconstruction" of the standard model, where the NCG of a suitable algebra of operators is used (Connes, 1991)

- in field theories on products (Minkowski) × (discrete spaces) (Connes 1991, Cocquereaux et al 1991, Mueller-Hoissen et al 1993, Madore et al 1996, O'Raifeartaigh et al 1998, LC 1999,…) Higgs fields

- in Yang-Mills and gravity theories on q-group "manifolds" (LC 1992, Volovich 1993, Majid 1993,…) , in q-deformations of group structures and their NCG (for ex k-Minkowski , Florence group since 1991…), in q-deformations of (single particle) quantum mechanics (Wess et al 1994,…)

- Noncommutative structures in string field theory (Witten, 1986, NCG and string field theory)

- Fuzzy spaces (Madore, Presnajder, Grosse, Steinacker…)

More recently:

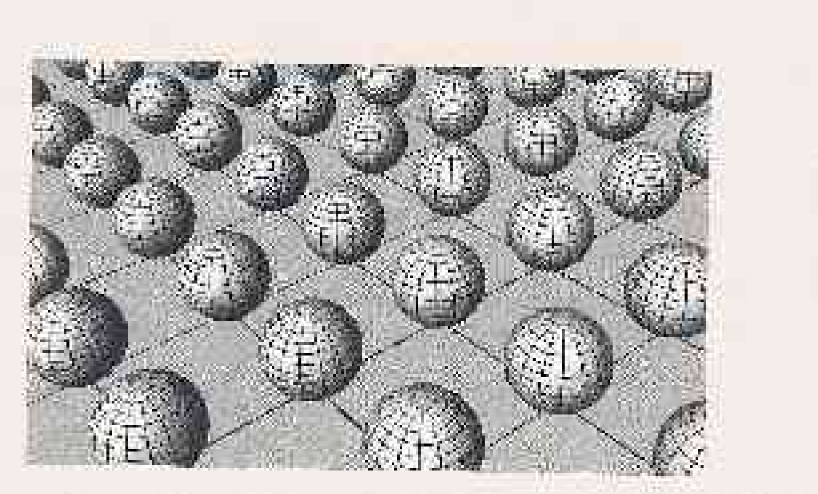
- Fuzzy coset space dimensional reduction (Aschieri, Madore, Manousselis, Steinacker, Zoupanos…)

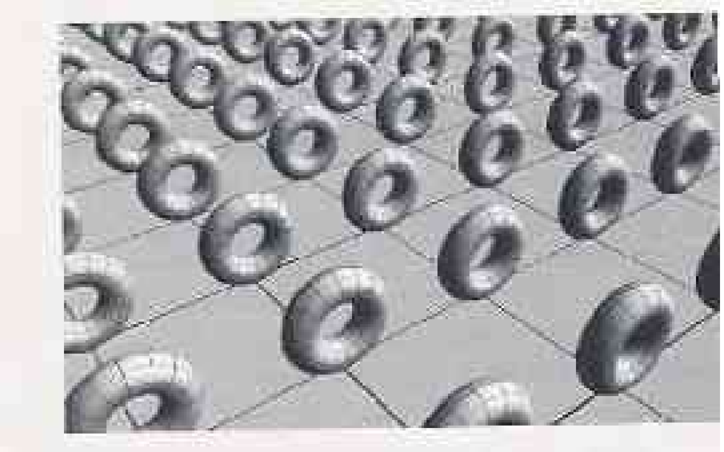
- Yang-Mills theories on noncommutative spaces emerge in the context of M theory compactified on a torus, with a background 3-form field (Connes, Douglas e Schwarz, 1997), or as low energy limit of open strings with background 2-form field B, a limit that captures the dynamics of *D-branes* (nonperturbative extended objects on which open strings can enc
(Douglas,Hull 1998,…Seiberg,Witten, 1999)

- Field theories on noncommutative spacetime: realized via
Groenewold-Moyal-Weyl product (and

- - deformed (twisted) field theories, twisted Yang-Mills
  Study of their quantum behaviour. (Minwalla, van Raamsdonk, Seiberg; Hayakawa; Martin, Sanchez-Ruiz; Wulkenhaar; Grosse, Wohlgenannt; Burić, Latas, Radovanović; Steinacker; Aschieri, Dimitrijević, Meyer, Schraml, Wess; Alvarez-Gaumé, Vazquez-Mozo; Vassilievich...)

  - Twisted diffeomorphisms (gravity) (Aschieri, Blohmann, Dimitrijević Meyer, Schupp, Wess...)





FIELD THEORY ON NONCOMMUTATIVE $\mathbb{R}^d$

- ALGEBRA OF FUNCTIONS ON NC $\mathbb{R}^d$ ~
  ALGEBRA OF FUNCTIONS ON USUAL $\mathbb{R}^d$ WITH
  DEFORMED PRODUCT $*$

$$A(x) * B(x) \equiv A(x)\, e^{\frac{i}{2} \overleftarrow{\partial}_\mu \theta^{\mu\nu} \overrightarrow{\partial}_\nu} B(x)$$

- $\Rightarrow$ STUDY OF $*$ DEFORMED THEORIES

$$S = \int d^d x\, \mathcal{L}[\Phi]$$

  WHERE FIELDS IN $\mathcal{L}$ ARE MULTIPLIED USING $*$

- $x^\mu * x^\nu - x^\nu * x^\mu = i\,\theta^{\mu\nu}$

...A PARTICULAR CASE OF DEFORMATION QUANTIZATION

- SHORT FOR: DEFORMATION OF PRODUCT DUE TO QUANTIZATION
- ARISES IN THE NC STRUCTURE OF QUANTUM MECH.
- CONSIDER THE WEYL QUANTIZATION RULE W

$$\text{CLASSICAL PHASE-SPACE FUNCTIONS} \xrightarrow{W} \text{QUANTUM OPERATORS}$$

$$q^m p^m \to W(q^m p^m) = \frac{1}{2^m} \sum_{k=0}^{m} \binom{m}{k} \hat{p}^{m-k} \hat{q}^m \hat{p}^k$$

FOR EX: $W(q p^2) = \frac{1}{4}\left(\hat{p}^2\hat{q} + 2\hat{p}\hat{q}\hat{p} + \hat{q}\hat{p}^2\right)$

(SUM ON ALL PERM. OF $\hat{p}$, $\hat{q}$ CONSIDERED AS DIFFERENT OBJECTS)

- YIELDS HERMITEAN OPERATOR
- CAN BE RESTATED AS :

$$W(q^m p^n) = \left[\exp\left(-\frac{i\hbar}{2}\frac{\partial^2}{\partial q\,\partial p}\right) q^m p^n\right]_{\substack{q \to \hat{q} \\ p \to \hat{p}}}$$

(q ORDERED TO <u>LEFT</u>)

- CAN BE CHECKED ON BASIC MONOMIALS $q^r p^s$
- EXTENDS TO ANY PHASE SPACE FUNCTION $A(q,p)$

$$W[A(q,p)] = :\exp\left[-\frac{i\hbar}{2}\frac{\partial^2}{\partial q\,\partial p}\right] A(q,p):$$

$: \; :$ NORMAL ORDERING

- THE MAP $\mathcal{W}$ IS INVERTIBLE $\Rightarrow$ 1-1 CORRESPONDANCE BETWEEN QUANTUM OPERATORS AND FUNCTIONS ON PHASE SPACE

- THIS IS THE CORE OF THE MOYAL FORMALISM; ENABLES STUDY OF QUANTUM SYSTEMS WITHIN THE CLASSICAL ARENA OF PHASE-SPACE VIA THE INVERSE MAP $\mathcal{W}^{-1}$

- $$A \star B \equiv \mathcal{W}^{-1}\left(\mathcal{W}(A)\,\mathcal{W}(B)\right)$$

VON NEUMANN 1931
GROENEWOLD 1946
MOYAL 1949

EXPLICITLY :

$$A * B = A(q,p)\, e^{\frac{i\hbar}{2}\Delta}\, B(q,p)$$

$$\Delta \equiv \frac{\overleftarrow{\partial}}{\partial q}\frac{\overrightarrow{\partial}}{\partial p} - \frac{\overleftarrow{\partial}}{\partial p}\frac{\overrightarrow{\partial}}{\partial q}$$

NB

$$A \Delta B = \{A, B\}_{PB}$$

THE MOYAL PRODUCT * INHERITS THE PROPERTIES OF THE OPERATOR PRODUCT

ASSOCIATIVE , NONCOMMUTATIVE

KONTSEVICH, FEDOSOV: GIVEN A POISSON STRUCTURE

$\{A,B\} = \theta^{ij}(x)\,\partial_i A\,\partial_j B$ ON A MANIFOLD X

THERE IS ESSENTIALLY ONE * PRODUCT

$$A * B = AB + \frac{i\hbar}{2}\{A,B\} + O(\hbar^2)$$

UP TO LINEAR REDEF. OF A, B

$$A \to D(\hbar)A = A + \hbar D_1(A) + \hbar^2 D_2(A) + \dots$$

(D_i : $\mathrm{Fun}(X) \to \mathrm{Fun}(X)$ differential operators)

- RELATION TO OTHER QUANTIZATION RULES (LC 1978)

SYMMETRIES OF $*$ DEFORMED THEORIES

- TYPICALLY INVARIANT UNDER $*$ DEFORMATIONS OF THE CORRESPONDING CLASSICAL SYMMETRIES

- FOR EX $U(m)$ $*$ YANG-MILLS IS INVARIANT UNDER

$$\delta_\epsilon A_\mu = \partial_\mu \epsilon - i (A_\mu * \epsilon - \epsilon * A_\mu)$$

- SPACETIME SYMMETRIES ARE LIKEWISE DEFORMED
    - INVARIANCE UNDER RIGID TRANSLATIONS
    - DEFORMED INVARIANCE UNDER RIGID ROTATIONS
- $*$-NOETHER ?

DYNAMICALLY TWISTED ϕ^{*4} THEORY

- A WAY TO RESTORE UNDEFORMED LORENTZ SYMM. IN κ-DEFORMED INTERACTING SCALAR THEORY
  $\rightarrow$ CONSERVED CHARGE

- GENERALIZATION OF GROENEWOLD-MOYAL PRODUCT

1) $f * g = \exp\left(-\frac{i}{2}\theta^{ab} X_a \otimes X_b\right)(f \otimes g)$

2) $X_a = e_a{}^{\mu}(x)\partial_\mu$

3) $[X_a, X_b] = 0$ (ASSOCIATIVITY OF $*$)

$$\Rightarrow \quad e_{[a}{}^{\nu}\partial_{\nu}e_{b]}{}^{\mu} = 0$$

SUPPOSING THAT THE SQUARE MATRIX $e_a{}^{\nu}$ HAS AN INVERSE $e_{\nu}{}^{a}$ EVERYWHERE

$$\Rightarrow \quad \partial_{[\mu}e_{\nu]}{}^{a} = 0$$

SOLVED BY

$$e_{\nu}{}^{a}(x) = \partial_{\nu}\phi^{a}(x)$$

* PRODUCT DETERMINED BY D SCALAR FIELDS SUBJECT TO THE CONDITION: $\partial_{\mu}\phi^{a}$ INVERTIBLE

- $x^\mu \star x^\nu - x^\nu \star x^\mu = \underbrace{i\theta^{ab} e_a{}^\mu(x) e_b{}^\nu(x)}_{\Theta^{\mu\nu}(x)}$

$\Rightarrow$ X-DEPENDENT NONCOMM.

- $F = e^{-\frac{i}{2}\theta^{ab} X_a \otimes X_b}$ IS A <u>TWIST</u>

$\Rightarrow$ USE OF TWIST MACHINERY TO CONSTRUCT DIFF CALCULUS AND GEOMETRY

- $\Theta^{\mu\nu}(x)$ TRANSFORMS AS A <u>TENSOR</u> UNDER LORENTZ ROTATIONS $x^\mu \to \Lambda^\mu{}_\nu x^\nu$ SINCE $e_a{}^\mu$ TRANSFORMS AS A VECTOR

- THEREFORE : $\star$ PRODUCT IS INVARIANT UNDER LORENTZ ROTATIONS ON COORD. x^μ

- LEIBNIZ RULE $X_a(f \star g) = (X_a f) \star g + f \star (X_a g)$

THE ACTION

OF ϕ^{*4} THEORY COUPLED TO ϕ^c

$$S[\phi,\phi^c] = \int \left(\frac{1}{2}\partial_\mu\phi * \partial^\mu\phi - \frac{m^2}{2}\phi*\phi - \frac{\lambda}{4!}\phi*\phi*\phi*\phi + \frac{1}{2}\partial_\mu\phi^c * \partial^\mu\phi_c \right) d^Dx$$

NOTE : THE ABOVE INTEGRAL IS NOT CYCLIC

$$\int (f*g)\, d^Dx \neq \int (g*f)\, d^Dx$$

SINCE $f*g = g*f + X_a(G^0)$

WITH $G^a = 2\left(\frac{\sinh \Delta}{\Delta} (f, \theta^{ab} X_b g) \right)$

$$\Delta(f,g) \equiv \frac{1}{2} \theta^{ab} (X_a f)(X_b g)$$

- BUT USING THE MEASURE $e\, d^D x \equiv \det(e_\mu{}^a)\, d^D x$

$$\int (f * g)\, e\, d^D x = \int (g * f)\, e\, d^D x \quad \Rightarrow \text{CYCLICITY}$$

UP TO BDY TERMS SINCE $e X_a (G^a) = \partial_\mu (e\, e_a{}^\mu G^a)$

$$S[\phi,\phi^c] = \int \Big[\Big(\frac{1}{2}\partial_\mu\phi \star \partial^\mu\phi - \frac{m^2}{2}\phi\star\phi - \frac{\lambda}{4!}\phi^{\star 4} + \frac{1}{2}\partial_\mu\phi^c \star \partial^\mu\phi_c \Big) \star e^{-1} \Big] e\, d^Dx$$

NB world-index contractions with MINKOWSKI METRIC $\eta^{\mu\nu}$ $\Rightarrow$ S IS INVARIANT UNDER LORENTZ.

- NOW WE ARE READY FOR VARYING S AND FINDING FIELD EQS AND CONSERVED CHARGES

VARIATION OF S

1) $$\delta_\phi S = \delta_\phi \int (\mathcal{L} * e^{-1}) e\, d^D x = \int (\delta\phi\, E_\phi + \partial_\mu K^\mu)\, d^D x$$

$$\boxed{E_\phi = \frac{1}{2}\partial_\mu \left(e\{\partial^\mu \phi \overset{*}{,} e^{-1}\}\right) + \frac{m^2}{2} e\{\phi \overset{*}{,} e^{-1}\} + \frac{\lambda}{4!} e\{\phi * \phi \overset{*}{,} \{\phi \overset{*}{,} e^{-1}\}\} = 0}$$

$$\xrightarrow[\text{COMM. LIMIT}]{} \quad \Box\phi + m^2\phi + \frac{\lambda}{3!}\phi^3 = 0$$

2) $$\delta_{\phi^c} S = \int \left(-\delta\phi^c (X_c \phi) E_\phi + \delta\phi_c E_{\phi^c} + \partial_\mu J^\mu\right) d^D x$$

$$\boxed{-(X_c\phi) E_\phi + E_{\phi^c} = 0} \quad \xrightarrow[\text{COMM. LIMIT}]{} \quad \Box\phi^c = 0$$

$$E_{\phi^c} = -\frac{1}{2}\partial_\mu\left(e\{\partial^\mu\phi_c \overset{*}{,} e^{-1}\}\right) + \frac{1}{2}(X_c\phi)\partial_\mu\left(e\{\partial^\mu\phi \overset{*}{,} e^{-1}\}\right)$$

$$+ \frac{e}{2}(X_c\partial_\mu\phi^a)\{\partial^\mu\phi_a \overset{*}{,} e^{-1}\} + \frac{e}{2}(X_c\partial_\mu\phi)\{\partial^\mu\phi \overset{*}{,} e^{-1}\} - X_c\mathcal{L}$$

- THE FIELD EQS ADMIT THE VACUUM SOLUTION

$$\phi = 0 \ , \quad e_\mu{}^a = \partial_\mu \phi^a = \delta_\mu^a$$

CORRESPONDING TO THE USUAL MOYAL PRODUCT

- THE FIELD ϕ ACTS AS A SOURCE FOR THE NONCOMMUTATIVITY FIELD ϕ^c

SYMMETRIES AND CONSERVED CURRENTS

- UNDER A FUNCTIONAL VARIATION OF ϕ, ϕ^c AND A COORD. CHANGE

$$\phi'(x) = \phi(x) + \delta\phi(x)$$

$$\phi'^c(x) = \phi^c(x) + \delta\phi^c(x)$$

$$x'^\mu = x^\mu + \varepsilon^\mu(x)$$

THE ON-SHELL VARIATION OF S (INTEGRATED ON AN ARBITRARY MANIFOLD) IS

$$\delta S = \int_M \partial_\mu \left[K^\mu + J^\mu + \varepsilon^\mu (d * e^{-1}) e \right] d^D x$$

• S IS INVARIANT UNDER GLOBAL TRANSLATIONS

$$\delta\phi = -\varepsilon^\nu \partial_\nu \phi \,, \quad \delta\phi^c = -\varepsilon^\nu \partial_\nu \phi^c \,, \quad \varepsilon^\nu = \text{CONST.}$$

$$0 = \delta S = \int_M \varepsilon^\nu \partial_\mu T^\mu{}_\nu \, d^D x$$

$$\Rightarrow T^\mu{}_\nu = -\frac{e}{2} (\partial_\nu \phi) \{ \partial^\mu \phi \overset{*}{,} e^{-1} \} - \frac{e}{2} (\partial_\nu \phi^c) \{ \partial^\mu \phi^c \overset{*}{,} e^{-1} \}$$

$$+ e\, e_a{}^\mu \Big(\alpha * (e^{-1} \partial_\nu \phi^a) - \frac{\exp(\Delta) - 1}{\Delta} \big(X_c \alpha, \theta^{ab} X_b (e^{-1} \partial_\nu \phi^c) \big) \Big)$$

IS THE CONSERVED ENERGY-MOMENTUM

(CHECK: COMPUTE $\partial_\mu T^\mu{}_\nu$ ON SHELL)

S IS INVARIANT UNDER GLOBAL ROTATIONS

$\delta\phi = -\varepsilon^{\nu\rho} x_\rho \partial_\nu \phi$, $\delta\phi^c = -\varepsilon^{\nu\rho} x_\rho \partial_\nu \phi^c$, $\varepsilon^\nu = \varepsilon^{\nu\rho} x_\rho$

$$0 = \delta S = \int_M \varepsilon^{\nu\rho} \partial_\mu M^\mu{}_{\nu\rho} \, d^D x$$

- $M^\mu{}_{\nu\rho}$ IS CONSERVED (FOR EXPLICIT EXPRESSION SEE HEP-TH 0803.4325)

- CAN BE CHECKED BY COMPUTING $\partial_\mu M^\mu{}_{\nu\rho}$ ON SHELL

$$M^{\mu}{}_{\nu\rho} = \frac{e}{2}\, x_{[\nu}\partial_{\rho]}\phi \,\{\partial^{\mu}\phi \,\overset{*}{,}\, e^{-1}\} + \frac{e}{2}\, x_{[\nu}\partial_{\rho]}\phi^{c} \,\{\partial^{\mu}\phi^{c} \,\overset{*}{,}\, e^{-1}\}$$

$$- e e_{a}{}^{\mu} \left(\mathcal{L} * \left(e^{-1} x_{[\nu}\partial_{\rho]}\phi^{a} \right) \right)$$

$$+ e e_{a}{}^{\mu} \Big[T(\Delta)\left(X_{c}\mathcal{L} \,,\, \theta^{ab} x_{b} \left(e^{-1} x_{[\nu}\partial_{\rho]}\phi^{c} \right) \right)$$

$$- T(\Delta)\left(\partial_{[\nu}\phi \,,\, \frac{1}{2}\theta^{ab} X_{b} \left(\{ \partial_{\rho]}\phi \,\overset{*}{,}\, e^{-1} \} \right) \right)$$

$$- T(\Delta)\left(\partial_{[\nu}\phi^{d} \,,\, \frac{1}{2}\theta^{ab} X_{b} \left(\{ \partial_{\rho]}\phi_{d} \,\overset{*}{,}\, e^{-1} \} \right) \right)$$

$$+ S(\Delta)\left(\partial_{[\nu}\phi \,,\, \theta^{ab} X_{b} \left(\partial_{\rho]}\phi * e^{-1} \right) \right)$$

$$+ S(\Delta)\left(\partial_{[\nu}\phi^{d} \,,\, \theta^{ab} X_{b} \left(\partial_{\rho]}\phi_{d} * e^{-1} \right) \right)$$

$$T(\Delta) \equiv \frac{\exp(\Delta) - 1}{\Delta} \quad , \quad S(\Delta) \equiv \frac{\sinh \Delta}{\Delta}$$

CONCLUSIONS

- BY MEANS OF AN EXTENSION OF THE MOYAL PRODUCT, WE HAVE IMPLEMENTED DYNAMICAL NONCOMMUTATIVITY IN ϕ^{*4} THEORY

- SIMULTANEOUSLY RESTORED GLOBAL LORENTZ SYMMETRY

- THIS HAS BEEN ACHIEVED BY INTRODUCING X-DEPENDENCE IN THE DEF. OF * PRODUCT IN A FACTORIZED WAY:

$$\theta^{ab} \, e_a^{\mu}(x) \, e_b^{\nu}(x)$$