Kerr-Schild ansatz in Einstein-Gauss-Bonnet gravity

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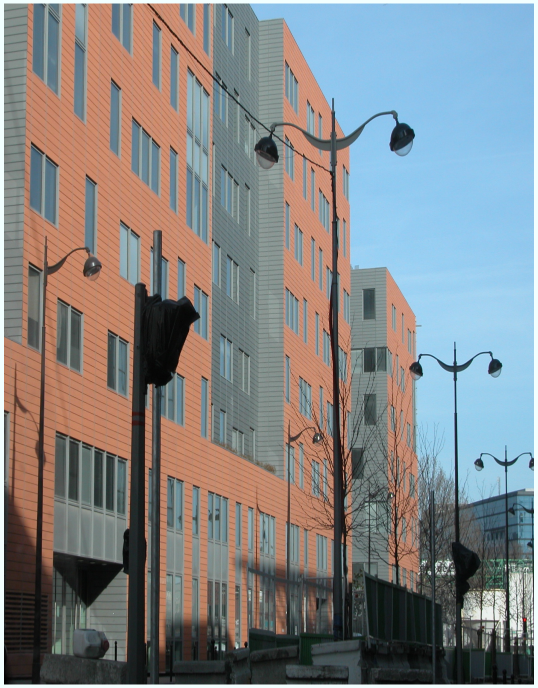

A talk dedicated to

John Madore

John and Einstein-Gauss-Bonnet gravity

"Kaluza-Klein Theory With The Lanczos Lagrangian", J. Madore (Toronto U.) Print-85-0340 (TORONTO), Apr 1985, 8pp, Phys.Lett.A110:289,1985. (followed by another four 1985-1987, plus one in 2003)

One of the very first papers (perhaps THE first)
using the "Lanczos" Lagrangian ($\mathrm{Riemann}^2 - 4\mathrm{Ricci}^2 + R^2$)

Lanczos, 1938 ; Chern, 1943 ; Lovelock, 1971
Boulware-Deser, 1985 ; Mueller-Hoissen, 1985 ; Zumino, 1986

Then John moved to non-commutative geometries :

"Kaluza-Klein aspects of noncommutative geometry", J. Madore (Orsay, LPT), In "Chester 1988, Proceedings, Differential geometric methods in theoretical physics", p 243-252

Einstein-Gauss-Bonnet gravity in brief

• The metric variation of $L_2 = R_{ijkl}R^{ijkl} - 4R^{ij}R_{ij} + R^2$ yields a tensor which is identically zero in 4 dimensions (Lanczos, 1938)

• Hilbert lagrangian: $R = \frac{1}{2}\delta^{i_1 i_2}_{j_1 j_2} R^{j_1 j_2}_{i_1 i_2}$. Einstein's tensor: $G^i_j = \frac{1}{2}\delta^{i i_1 i_2}_{j j_1 j_2} R^{j_1 j_2}_{i_1 i_2}$

Similarly : $L_2 = \frac{1}{4}\delta^{i_1 i_2 i_3 i_4}_{j_1 j_2 j_3 j_4} R^{j_3 j_4}_{i_1 i_2} R^{j_1 j_2}_{i_3 i_4}$ etc, (Lovelock, 1971)

Hence $\delta L_2 \equiv 0$ in $D = 4$ AND second order tensor in $D > 4$

• $\mathcal{L}_{(p)} = (\Omega^p)^{I_1 \cdots I_{2p}} \theta^*_{I_1 \cdots I_{2p}}$ where $\theta^*_{I_1 \ldots I_p} = \frac{1}{(D-p)!}\epsilon_{I_1 \ldots I_D}\theta^{I_{p+1}} \ldots \theta^{I_D}$ proportional to the Euler characteristic in $D = 2p$, Chern, 1943

hence the name "Dimensionally continued Euler forms"

(JM, 1985, Mueller-Hoissen, 1985, Teitelboim-Zanelli, 1987,...)

Some applications

• 80': Stability of Kaluza-Klein ground states ; FRW cosmologies as attractors of Lovelock cosmologies ; inflation ; structure of singularity...

• 00's: Randall-Sundrum model and "Brane cosmologies" (BDL, 2000)

Generalisation of the Israel junction conditions

On shell : $\delta\left[\int_{\mathcal{M}} d^D x \mathcal{L}_p - \int_{\partial\mathcal{M}} \mathcal{C}_p\right] = \int_{\partial\mathcal{M}} \delta\gamma_{\mu\nu}\mathcal{C}_p^{\mu\nu}$

where $\mathcal{C}_p$ is a Chern form: $C_1 = 2K$, $C_2 = 2\delta^{i_1 i_2 i_3}_{j_1 j_2 j_3} K^{i_1}_{j_1}(R^{j_2 j_3}_{i_2 i_3} - \frac{2}{3}K^{j_2}_{i_2}K^{j_3}_{i_3})$

and where $C^i_{(1)j} = K^i_j - \delta^i_j K$ and : $C^i_{(2)j} = 2\delta^{i i_1 i_2 i_3}_{j j_1 j_2 j_3} K^{i_1}_{j_1}(R^{j_2 j_3}_{i_2 i_3} - \frac{2}{3}K^{j_2}_{i_2}K^{j_3}_{i_3})$

ND Dolezel, 2000; Davis, 2002; Gravanis-Willison, 2002; Myers, 1987; Troncoso-Zanelli et al, 1999...

Gravity on a Einstein-Gauss-Bonnet brane

Randall-Sundrum : Newton's law recovered for scales $\gg \mathcal{L}$

EGB : Newton's law recovered for all scales (ND, Sasaki, 2003)

Numerous cosmological brane models (including CMB anisotropies)

Conservation laws in EGB gravity (Deser-Tekin)

Mass and angular momenta of EGB black holes ($T\,dS = dM - \Omega\,dJ$.)

ND Katz Morisawa Ogushi : $M = \int_S d^{D-2}x \hat{J}_t^{[01]} \quad , \quad J_i = \int_S d^{D-2}x \hat{J}_i^{[01]}$

$$\hat{J}^{[\mu\nu]} \equiv \hat{J}_E^{[\mu\nu]} + \alpha\, \hat{J}_{GB}^{[\mu\nu]}$$

$$-8\pi \hat{J}_E^{[\mu\nu]} \equiv D^{[\mu}\hat{\xi}^{\nu]} - \overline{D^{[\mu}\hat{\xi}^{\nu]}} + \hat{\xi}^{[\mu}k_E^{\nu]}\,.$$

$$-8\pi \hat{J}_{GB}^{[\mu\nu]} \equiv 2\left[P^{\mu\nu\alpha\beta}D_{[\alpha}\hat{\xi}_{\beta]} - \overline{P^{\mu\nu\alpha\beta}D_{[\alpha}\hat{\xi}_{\beta]}}\right] + \hat{\xi}^{[\mu}k_{GB}^{\nu]}\,.$$

Kerr-Schild ansatz in EGB gravity: Outline

- As is well-known, Kerr-Schild metrics linearize the Einstein tensor.
- They also simplify the Gauss-Bonnet tensor, which turns out to be only quadratic in the arbitrary Kerr-Schild function f.
- We give its analytical expression for any function f when the background is 5-dimensional Minkowski spacetime in spheroidal coordinates and equal rotation coefficients.
- This result may be of some use in the quest for Einstein-Gauss-Bonnet rotating black hole solutions.
- In particular we show that there is no such Kerr-Schild solution of the Einstein-Maxwell-Gauss-Bonnet field equations.

Introduction

Kerr-Schild metrics

$$g_{\mu\nu} = \overline{g}_{\mu\nu} + h_{\mu\nu} \qquad \text{with} \qquad h_{\mu\nu} = f\, h_\mu h_\nu$$

$$\overline{g}^{\mu\nu} h_\mu h_\nu = 0 \qquad \text{and} \qquad h^\mu \overline{D}_\mu h^\rho = 0\,.$$

INCLUDE

The whole Kerr-Newman family of the four dimensional black holes, solutions of Einstein's equations (with or without a cosmological constant)

The D-dimensional generalizations of (anti-de-Sitter) Kerr black holes (Einstein's theory) [Gibbons et al 2004]

The spherically symmetric (charged) Einstein-Gauss-Bonnet black hole solutions [Boulware Deser, 1985]

BUT

Somewhat curiously:

the D-dimensional, non-rotating, Reisner-Nordström black holes are also of the Kerr-Schild type,

however, the known 5-D charged and rotating black hole solutions are not [Kunz et al, Beckenridge et al, R. Kallosh et al]

Also :

the Kerr-Schild ansatz, used to obtain the 5-dimensional Kerr (AdS) black hole solutions of Einstein's equations, does not solve the Einstein-Gauss-Bonnet field equations.

The Einstein Gauss-Bonnet tensor for Kerr-Schild spacetimes

$$E^\mu_\nu = T^\mu_\nu \qquad \text{with} \qquad E^\mu_\nu = \Lambda\delta^\mu_\nu + \kappa^{-1} G^\mu_\nu + \alpha H^\mu_\nu \,.$$

$$H^\mu_\nu \equiv 2R^{\mu\alpha}{}_{\beta\gamma} R^{\beta\gamma}{}_{\nu\alpha} - 4R^{\mu\alpha}{}_{\nu\beta} R^\beta_\alpha - 4R^\mu_\alpha R^\alpha_\nu + 2RR^\mu_\nu$$
$$-\tfrac{1}{2}\delta^\mu_\nu (R^{\alpha\beta}{}_{\gamma\delta} R^{\gamma\delta}{}_{\alpha\beta} - 4R^\alpha_\beta R^\beta_\alpha + R^2) \,.$$

When the metric is of the Kerr-Schild type the Ricci tensor R^μ_ν is linear in f

The Riemann tensors $R^\mu{}_{\nu\rho\sigma}$ and $R^{\mu\nu}{}_{\rho\sigma}$ turn out to be only quadratic in f

The contracted products $R^{\mu\alpha}{}_{\beta\gamma} R^{\beta\gamma}{}_{\nu\alpha}$ and $R^{\mu\alpha}{}_{\nu\beta} R^\beta_\alpha$ are also quadratic in f

Hence: the Gauss-Bonnet tensor H^μ_ν is only quadratic in f

at least for maximally symmetric backgrounds :

$$\overline{R}_{\mu\nu\rho\sigma} = -\tfrac{1}{\mathcal{L}^2}(\overline{g}_{\mu\rho}\overline{g}_{\nu\sigma} - \overline{g}_{\mu\sigma}\overline{g}_{\nu\rho}) \quad \text{where} \quad \kappa^{-1} - \tfrac{2\tilde{\alpha}}{\mathcal{L}^2} = \mp\sqrt{\kappa^{-2} - \tfrac{4\tilde{\alpha}}{l^2}}$$

More precisely:

$$E^{\mu}_{\nu} = \left(\kappa^{-1} - \frac{2\tilde{\alpha}}{\mathcal{L}^2}\right) \left[\frac{(D-1)}{\mathcal{L}^2} f h^{\mu} h_{\nu} + R^{\mu}_{(L)\nu} - \frac{1}{2}\delta^{\mu}_{\nu} R_{(L)}\right]$$

$$+2\alpha \left(\frac{K}{\mathcal{L}^2} f h^{\mu} h_{\nu} + R^{\mu\alpha}_{(L)\beta\gamma} R^{\beta\gamma}_{(L)\nu\alpha} - 2R^{\mu\alpha}_{(L)\nu\beta} R^{\beta}_{(L)\alpha} - 2R^{\mu}_{(L)\alpha} R^{\alpha}_{(L)\nu} + R_{(L)} R^{\mu}_{(L)\nu}\right)$$

$$-\frac{\alpha}{2}\delta^{\mu}_{\nu} \left(R^{\alpha\beta}_{(L)\gamma\delta} R^{\gamma\delta}_{(L)\alpha\beta} - 4R^{\alpha}_{(L)\beta} R^{\beta}_{(L)\alpha} + R^2_{(L)}\right)$$

with the following definitions

- $R^{\mu\nu}_{(L)\rho\sigma} = \overline{g}^{\nu\alpha}(\overline{D}_{\rho}\Delta^{\mu}_{\alpha\sigma} - \overline{D}_{\sigma}\Delta^{\mu}_{\alpha\rho})$, $R^{\mu}_{(L)\nu} = \overline{g}^{\mu\sigma}\overline{D}_{\rho}\Delta^{\rho}_{\nu\sigma}$
$R_{(L)} = \overline{D}_{\rho}[h^{\rho}\overline{D}_{\mu}(f h^{\mu})]$,

- $\Delta^{\mu}_{\nu\rho} = \frac{1}{2}[\overline{D}_{\nu}(f h^{\mu} h_{\rho}) + \overline{D}_{\rho}(f h^{\mu} h_{\nu}) - \overline{D}^{\mu}(f h_{\nu} h_{\rho})]$.,

- $K =$
$3(h^{\alpha}\partial_{\alpha} f)\overline{D}_{\beta}h^{\beta} + 2(D-1) f \overline{D}_{\alpha}(h^{\alpha}\overline{D}_{\beta}h^{\beta}) + (4D-7) f \overline{D}_{\alpha}h^{\beta}(\overline{D}_{\beta}h^{\alpha} - \overline{D}^{\alpha}h_{\beta})$.

Trace of the Einstein-Gauss-Bonnet tensor

5-D (anti-)de Sitter backgrounds in spheroidal coordinates:

$$d\bar{s}^2 = -\frac{(1+r^2/\mathcal{L}^2)\Delta_\theta}{\Xi_a\Xi_b}dt^2 + \frac{r^2\rho^2}{(1+r^2/\mathcal{L}^2)(r^2+a^2)(r^2+b^2)}dr^2 + \frac{\rho^2}{\Delta_\theta}d\theta^2$$

$$+\frac{r^2+a^2}{\Xi_a}\sin^2\theta\, d\phi^2 + \frac{r^2+b^2}{\Xi_b}\cos^2\theta\, d\psi^2$$

the null and geodesic vector:

$$h_\mu dx^\mu = \frac{\Delta_\theta}{\Xi_a\Xi_b}\, dt + \frac{r^2\rho^2}{(1+r^2/\mathcal{L}^2)(r^2+a^2)(r^2+b^2)}\, dr + \frac{a\sin^2\theta}{\Xi_a}d\phi + \frac{b\cos^2\theta}{\Xi_b}d\psi\,.$$

Kerr-Schild line element : $ds^2 = d\bar{s}^2 + f(r,\theta)h_\mu h_\nu dx^\mu dx^\nu$

A Remarkably simple form for the trace: $E = -\frac{(rQ_t)''}{2r\rho^2}$

$Q_t = (D-2)\kappa^{-1}Q_l + \frac{\tilde{\alpha}Q_q}{D-3}$ with $Q_l = \rho^2 f$ and $Q_q = 2(4r^2-\rho^2)\frac{f^2}{\rho^2}$

The Einstein-Gauss-Bonnet tensor
($a=b$, $5D$, Minkowski background)

Consider Kerr-Schild metrics $ds^2 = d\bar{s}^2 + f(r)h_\mu h_\nu dx^\mu dx^\nu$ where $d\bar{s}^2$ is the flat 5-D line element in spheroidal coordinates with equal rotation coefficients:

$$d\bar{s}^2 = -dt^2 + \frac{r^2}{r^2+a^2}dr^2 + (r^2+a^2)(d\theta^2 + \sin^2\theta d\phi^2 + \cos^2\theta d\psi^2)$$

The null and geodesic vector is $h_\mu = \left(1, \frac{r^2}{r^2+a^2}, 0, a\sin^2\theta, a\cos^2\theta\right)$

The trace of the EGB tensor simplifies into $E = -\frac{(rQ_t)''}{2r(r^2+a^2)}$

$$Q_t = (D-2)\kappa^{-1}Q_l + \frac{\tilde{\alpha}Q_q}{D-3}$$

$Q_l = f(r^2+a^2)$ and $Q_q = \frac{2(3r^2-a^2)f^2}{r^2+a^2}$

Careful examination then shows that all components of the EGB tensor can then be expressed in terms of E^r_r and E^ϕ_ψ as :

$$E^t_t = -\frac{a^2}{3(r^2+a^2)}\left(\frac{a^2+r^2}{r}E^{r\prime}_r + \frac{2E^\phi_\psi}{\cos^2\theta}\right) + E^r_r$$

$$E^t_\phi = -\frac{a\sin^2\theta}{3}\left(\frac{a^2+r^2}{r}E^{r\prime}_r + \frac{2E^\phi_\psi}{\cos^2\theta}\right)$$

$$E^t_\psi = -\frac{a\cos^2\theta}{3}\left(\frac{a^2+r^2}{r}E^{r\prime}_r + \frac{2E^\phi_\psi}{\cos^2\theta}\right)$$

$$E^\theta_\theta = \frac{1}{3}\left(\frac{a^2+r^2}{r}E^{r\prime}_r - \frac{E^\phi_\psi}{\cos^2\theta}\right) + E^r_r$$

$$E^\phi_\phi = \frac{1}{3}\left(\frac{a^2+r^2}{r}E^{r\prime}_r + (2-3\cos^2\theta)\frac{E^\phi_\psi}{\cos^2\theta}\right) + E^r_r$$

$$E^\psi_\psi = \frac{1}{3}\left(\frac{a^2+r^2}{r}E^{r\prime}_r - (1-3\cos^2\theta)\frac{E^\phi_\psi}{\cos^2\theta}\right) + E^r_r$$

As for E_r^r et E_ψ^ϕ they are expressed in terms of Q_t and Q_q as

$$E_r^r = \frac{1}{6r(r^2+a^2)^2}\left[-(3r^2+a^2)Q_t' + 4\tilde{\alpha}a^4\left(\frac{Q_q}{3r^2-a^2}\right)'\right]$$

and (an admitedly ugly expression)

$$\frac{E_\psi^\phi}{\cos^2\theta} = \frac{a^2[(a^2+5r^2)Q_t' - r(r^2+a^2)Q_t'']}{6r^3(r^2+a^2)^2}$$

$$+\frac{2\tilde{\alpha}a^2(27r^4+42r^2a^2+31a^4)Q_q}{(3r^2-a^2)^3(r^2+a^2)^2} - \frac{2\tilde{\alpha}a^2(18r^6+27r^4a^2+16r^2a^4-a^6)Q_q'}{3r^3(3r^2-a^2)^2(r^2+a^2)^2}$$

$$+\frac{\tilde{\alpha}a^2(3r^2+2a^2)Q_q''}{3r^2(3r^2-a^2)(r^2+a^2)}$$

(Of course, various checks were made...)

Recovering standard results ($a = 0$)

$E_t^t = E_r^r = -\frac{Q_t'}{2r^3} \quad , \quad E_\theta^\theta = E_\phi^\phi = E_\psi^\psi = -\frac{Q_t''}{6r^2}$ with

$Q_t = 3\kappa^{-1} Q_l + \frac{\tilde{\alpha} Q_q}{2}$ and $Q_l = r^2 f \quad , \quad Q_q = 6f^2$

Electromagnetic potential $A^\mu = (U(r), 0, 0, 0, 0)$.

A Kerr-Schild solution of the EGB equations of motion exists and is

$$U(r) = \frac{q}{r^2} \quad , \quad Q_t = \frac{2q^2}{r^2} + 6m$$

$$\Longrightarrow \quad f(r) = \frac{r^2}{2\kappa\tilde{\alpha}} \left(-1 + \sqrt{1 + \frac{8\kappa^2 \tilde{\alpha}}{3r^4} \left(3m + \frac{q^2}{r^2} \right)} \right)$$

Reisner-Gauss-Bonnet solution [Boulware-Deser], in Kerr-Schild form.

m is a constant of integration : the total mass [Deser-Tekin] [Padilla] [DKO]...

A "no-go" result

$$d\bar{s}^2 = -dt^2 + \frac{r^2}{r^2+a^2}dr^2 + (r^2+a^2)(d\theta^2 + \sin^2\theta d\phi^2 + \cos^2\theta d\psi^2)$$

$h_\mu = \left(1, \frac{r^2}{r^2+a^2}, 0, a\sin^2\theta, a\cos^2\theta\right)$ and $ds^2 = d\bar{s}^2 + f(r)h_\mu h_\nu dx^\mu dx^\nu$

$A_\mu = U(r)h_\mu$; Maxwell equations yield $U = \frac{q}{r^2+a^2}$

Einstein-Maxwell Gauss-Bonnet trace equation:

$$\frac{(rQ_t)''}{2r(r^2+a^2)} = \frac{2q^2(r^2-a^2)}{(r^2+a^2)^4} \implies Q_t = \frac{2c}{r} + 6m + \frac{q^2}{r^2+a^2} - \frac{q^2\,\mathrm{Arctan}\frac{r}{a}}{ar} + \frac{\pi q^2}{2ar}$$

$Q_t = (D-2)\kappa^{-1}Q_l + \frac{\tilde{\alpha}Q_q}{D-3}$ with $Q_l = f(r^2+a^2)$ and $Q_q = \frac{2(3r^2-a^2)f^2}{r^2+a^2}$
hence

$$f(r) = \frac{3(r^2+a^2)^2}{2\kappa\tilde{\alpha}(3r^2-a^2)}\left(-1 + \sqrt{1 + \frac{8\tilde{\alpha}\kappa^2(3r^2-a^2)}{9(r^2+a^2)^3}\left[3m + \frac{c}{r} + \frac{q^2}{2(r^2+a^2)} + \frac{q^2}{2ar}\left(\frac{\pi}{2} - \mathrm{Arctan}\frac{r}{a}\right)\right]}\right)$$

For all the other field equations to be satisfied we must have

$$E_r^r = \frac{2q^2}{(r^2+a^2)^3} \qquad , \qquad E_\psi^\phi = 0 \qquad (*)$$

Now, E_r^r and E_ψ^ϕ are known fonctions of $f(r)$.

It is an easy exercice to see that, with the function f obtained from the trace equation, equations (*) are NOT satisfied.

if $c \neq 0$ then $E_r^r \to \frac{c}{r^5}$ and $\frac{E_\psi^\phi}{\cos^2\theta} \to -\frac{7a^2c}{6r^7}$

if $c = 0$ then $E_r^r \to \frac{32a^4q^2}{45r^{10}}$ and $\frac{E_\psi^\phi}{\cos^2\theta} \to -\frac{16a^2q^2}{3r^8}$

if $c = q = 0$ then $E_r^r \to -\frac{32a^4\tilde{\alpha}\kappa^2m^2}{r^{12}}$ and $\frac{E_\psi^\phi}{\cos^2\theta} \to \frac{336a^2\tilde{\alpha}\kappa^2m^2}{r^{10}}$

Hence :There is no Kerr-Schild solution of the (5D) Einstein-Maxwell-Gauss-Bonnet field equations

SUMMARY AND OUTLOOK

- We studied Kerr-Schild metrics on maximally symmetric backgrounds
- We showed that the Einstein-Gauss-Bonnet tensor is quadratic in the Kerr-Schild function f.
- Specializing to 5-dimensional backgrounds in spheroidal coordinates we found a simple expression for the trace of the Einstein-Gauss-Bonnet tensor.
- Specializing further to a flat backgound and equal rotation coefficients we wrote the whole Einstein-Gauss-Bonnet tensor in closed form.
- We used those results to show in a transparent manner that the Einstein-Maxwell Gauss-Bonnet equations do not possess rotating Kerr-Schild solutions.
- The techniques developped may prove useful in the quest for Einstein-Gauss-Bonnet rotating black hole solutions and to elucidate under which conditions Kerr-Schild solutions can exist.

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