On second quantization on NC spaces with twisted symmetries

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Introduction

QFT on NC space(time)s: how and why?

Various possible routes:

- Path-integral field quantization Filk 96, Douglas, Schwarz, Oeckl 99, Seiberg-Witten 99, Alvarez-Gaume', Nekrasov, Szabo,..., Grosse-Wulkenhaar (renormalizable QFT's),...,
- Field quantization in operator approach (Canonical or á la Wightman? Standard o deformed Poincaré?...) Doplicher-Fredenhagen-Roberts 95, *et al* 95-06,...,Chaichian *et al* 04-07, Balachandran *et al* 05-07, Lizzi *et al* 06, Abe 06, Zahn 06, ...,G.F.-Wess 07, ,...., Aschieri *et al* 07 (quantizing $\star$-Poisson bracket),...
  In [G.F.-Wess 07] a surprising result: Wightman axioms with careful twisted Poincaré covariance yield QFT (free or interacting) with the same n-point functions and commutation relation relations. What's happening?!?
  (Already [G.F.-Schupp 96]: twisted symmetries compatible with Bose-Fermi statistics)
- Here **2nd Quantization**: from covariant QM of n identical bosons/fermions on a NC space(time) to QFT on the latter. Main motivations: importance of the particle interpretation; keeping Bose-Fermi statistics to avoid drastic consequences.

Various issues involved: consistency with QM axioms (unitarity, quantum statistics,...)? Deformed space(time) symmetry? Causality? Divergences & renormalizability?...

Bottom-up approach based on $\star$-products to guess a consistent NC framework:

- Start from G-covariant many-particle QM on a G-symmetric, *commutative* space(time) M.
E.g.: G=Galilei group and $M=\mathbb{R}\times\mathbb{R}^3$; G=Poincaré group and M=Minkowski spacetime.
- QFT can be obtained by 2nd quantization.
- We may introduce a moderate (very special) non-locality in the interactions using the $\star$-product induced by a twist $\mathcal{F}(\lambda)$ of $H\equiv U\mathbf{g}$ [$\mathbf{g}:=Lie(G)$, λ=deformation parameter].
- Express all ordinary products as formal expansions in λ of $\star$-ones upon inverting the definition

$$f\star g=fg+O(\lambda)\qquad\Rightarrow\qquad fg=f\star g+O_\star(\lambda).$$

So we *reformulate commutative notions* [wavefunctions, differential operators (Hamiltonian, etc), creation & annihilation operators,..., their transformations, 2nd quantization procedure itself] *purely in terms of their noncommutative analogs*.
- Forget original products to obtain a *closed framework for Second Quantization on a NC space*.

Same strategy as adopted by J. Wess and collaborators [03-06] e.g. to formulate noncommutative diffeomorphisms and related notions (metric, connections, tensors etc).

We stick to *NC space(time)s symmetric under* **triangular** *deformations (by twisting) of Lie groups, requiring full covariance of the framework* under such a "twisted symmetry group" (Hopf algebra). Maybe no or little new dynamics, but at least a "noncommutative way" to look at it; moreover, this can pave the way for more interesting (and complicated) deformations.

Plan

1. Introduction
2. Twisting $H = U\mathbf{g}$ to a noncocommutative Hopf algebra $\hat{H}$
3. Twisting modules and module (∗-)algebras [to be applied to $\mathcal{H}, \mathcal{O}, \mathcal{M} = C^{\infty}(M)$, $\mathcal{D}(\mathcal{M}), \mathcal{L}^2$, their tensor powers, their (anti)symmetric parts, $\mathcal{A}$, the field algebra Φ,...]
4. Symmetric QM with n bosons/fermions in abstract Hilbert space
5. Second quantization: from wavefunctions to quantum fields (non-relativistic)
6. Second quantization: from wavefunctions to quantum fields (relativistic)
7. Examples: QM and QFT on Moyal NC space(time)

Twisting $H = U\mathbf{g}$ to a noncocomm. Hopf algebra $\hat{H}$

Real deformation parameter λ. $\hat{H}$, $H[[\lambda]]$ have

1. same $*$-algebra (over $\mathbb{C}[[\lambda]]$) and counit ε
2. coproducts $\Delta, \hat{\Delta}$ related by

$$\Delta(g) \equiv \sum_I g^I_{(1)} \otimes g^I_{(2)} \quad \longrightarrow \quad \hat{\Delta}(g) = \mathcal{F}\Delta(g)\mathcal{F}^{-1} \equiv \sum_I g^I_{(\hat{1})} \otimes g^I_{(\hat{2})}$$

3. antipodes $S, \hat{S}$ s.t. $\hat{S}(g) = \alpha^{-1} S(g) \alpha$, with $\alpha = \sum_I S\left(\overline{\mathcal{F}}^{(1)}_I\right) \overline{\mathcal{F}}^{(2)}_I$, $\overline{\mathcal{F}} = \mathcal{F}^{-1}$.

where the *twist* [Drinfel'd 83] is for our purposes a *unitary* element $\mathcal{F} \in (H^s \otimes H^s)[[\lambda]]$, ($H^s \subseteq H$ Hopf $*$-subalgebra) fulfilling

$$\mathcal{F} = \mathbf{1} \otimes \mathbf{1} + O(\lambda), \qquad (\epsilon \otimes \mathrm{id})\mathcal{F} = (\mathrm{id} \otimes \epsilon)\mathcal{F} = \mathbf{1},$$

$$(\mathcal{F} \otimes \mathbf{1})[(\Delta \otimes \mathrm{id})(\mathcal{F})] = (\mathbf{1} \otimes \mathcal{F})[(\mathrm{id} \otimes \Delta)(\mathcal{F})] =: \mathcal{F}_3. \tag{1}$$

$\hat{H}$ has unitary triangular structure $\mathcal{R} = \mathcal{F}_{21}\mathcal{F}^{-1}$. $\hat{H}^s := H^s[[\lambda]]$ is a Hopf $*$-subalgebra of $\hat{H}$.

Twisting module ($*$-)algebras

Recall defs: a $\hat{*}$-algebra $\hat{\mathcal{A}}$ over $\mathbb{C}[[\lambda]]$ is a left $\hat{H}$-module $\hat{*}$-algebra if $\exists$ a $\mathbb{C}[[\lambda]]$-bilinear map $(g,\hat{a}) \in \hat{H} \times \hat{\mathcal{A}} \to g\hat{\triangleright}\hat{a} \in \hat{\mathcal{A}}$, called *left action*, such that (omitting product symbols)

$$\begin{aligned} (gg')\hat{\triangleright}\hat{a} &= g\hat{\triangleright}\,(g'\hat{\triangleright}\hat{a}), \qquad (g\hat{\triangleright}\hat{a})^{\hat{*}} = [\,\hat{S}(g)]^{\hat{*}}\hat{\triangleright}\hat{a}^{\hat{*}}, \\ g\hat{\triangleright}\,(\hat{a}\hat{b}) &= \textstyle\sum_I [g^I_{\hat{(1)}}\hat{\triangleright}\hat{a}][g^I_{\hat{(2)}}\hat{\triangleright}\hat{b}]\,. \end{aligned} \tag{2}$$

A $\hat{H}$-($\hat{*}$-) module $\hat{\mathcal{M}}$ is a linear space fulfilling only the first (two) relations.

Since $\hat{H} = H[[\lambda]]$ as $*$-algebras (and $\mathcal{F}$ is unitary), $\mathcal{M}[[\lambda]]$ is a $\hat{H}$-($\hat{*}$-)module under

$$g\hat{\triangleright}a = g \triangleright a; \qquad (\, a^{\hat{*}} := S(\alpha^{-1}) \triangleright a^*\,) \tag{3}$$

Given a $\mathcal{A}$, endow $\mathcal{M}[[\lambda]] := V(\mathcal{A})[[\lambda]] \equiv$vector space underlying $\mathcal{A}[[\lambda]]$ with a new product, the $\star$-*product*, defined by

$$a \star b := \sum_I \left(\overline{\mathcal{F}}_I^{(1)} \triangleright a\right) \left(\overline{\mathcal{F}}_I^{(2)} \triangleright b\right), \tag{4}$$

this becomes a H-module ($*$-)algebra $\mathcal{A}_\star$: associativity follows from (1), whereas $(2)_3$ from

$$g \triangleright (a\star b) = \left[g^I_{(1)}\overline{\mathcal{F}}^{(1')}_{I'} \triangleright a\right]\left[g^I_{(2)}\overline{\mathcal{F}}^{(2')}_{I'} \triangleright b\right] = \left[\overline{\mathcal{F}}^{(1')}_{I'} g^I_{\hat{(1)}} \triangleright a\right]\left[\overline{\mathcal{F}}^{(2')}_{I'} g^I_{\hat{(2)}} \triangleright b\right] = \left[g^I_{\hat{(1)}} \triangleright a\right] \star \left[g^I_{\hat{(2)}} \triangleright$$

Moreover, $(a\star b)^{\hat{*}} = b^{\hat{*}}\star a^{\hat{*}}$. We stress: works even if $\mathcal{A}$ not abelian!

The $\star$ is ineffective if a or b is H^s-invariant:

$$g \triangleright a = \epsilon(g)a \quad \text{or} \quad g \triangleright b = \epsilon(g)b \quad \forall g \in H^s \qquad \Rightarrow \qquad a \star b = ab. \tag{5}$$

Given H-($*$-)modules $\mathcal{M}, \mathcal{N}$, then also $\mathcal{M}\otimes\mathcal{N}$ is, under the action

$$g \triangleright (m \otimes n) = \sum_I \left(g^I_{(1)} \triangleright m\right) \otimes \left(g^I_{(2)} \triangleright n\right), \tag{6}$$

and $\mathcal{M}\otimes\mathcal{N}[[\lambda]]$ is a $\hat{H}$-($\hat{*}$-)module, under the action

$$g \hat{\triangleright} (m \otimes n) = \sum_I \left(g^I_{(\hat{1})} \hat{\triangleright} m\right) \otimes \left(g^I_{(\hat{2})} \hat{\triangleright} n\right) \in \mathcal{M}\otimes\mathcal{N}[[\lambda]]. \tag{7}$$

$\mathcal{F}\triangleright^{\otimes 2}$ is an intertwiner between them. Applying (6) to the "$\star$-tensor product" [Aschieri et al]

$$(m \otimes_\star n) := \overline{\mathcal{F}} \triangleright^{\otimes 2} (m \otimes n) = \sum_I (\overline{\mathcal{F}}^{(1)}_I \triangleright m) \otimes (\overline{\mathcal{F}}^{(2)}_I \triangleright n),$$

one finds $g \triangleright (m \otimes_\star n) = \sum_I \left(g^I_{(\hat{1})} \triangleright m\right) \otimes_\star \left(g^I_{(\hat{2})} \triangleright n\right)$, i.e. a realizaton of (7).

Moreover, if $\mathcal{M}^s \subset \mathcal{M}\otimes\mathcal{N}$ is a H-($*$-) submodule, then $\mathcal{F} \triangleright^{\otimes 2} \mathcal{M}^s$ is a $\hat{H}$-($\hat{*}$-)submodule.

Given H-module ($*$-)algebras $\mathcal{A}, \mathcal{B}$ the tensor ($*$-)algebra $\mathcal{A}\otimes\mathcal{B}$ $[(a \otimes b)(a' \otimes b') = aa' \otimes bb']$ also is a H-module ($*$-)algebra under $\triangleright$.

Introducing the $\star$-product (4) $\mathcal{A}\otimes\mathcal{B}$ is deformed into a $\hat{H}$-module ($*$-)algebra $(\mathcal{A}\otimes\mathcal{B})_\star$. One finds

$$(a \otimes_\star b) \star (c \otimes_\star d) = \sum_I a \star (\overline{\mathcal{R}}^{(1)}_I \triangleright c) \otimes_\star (\overline{\mathcal{R}}^{(1)}_I \triangleright b) \star d, \tag{8}$$

$\overline{\mathcal{R}} \equiv \mathcal{R}^{-1}$. $\otimes_\star$ is the associated *braided tensor product*, (*involutive*, as $\mathcal{R}\,\mathcal{R}_{21} = \mathbf{1} \otimes \mathbf{1}$). Note that $a \otimes_\star b = (a \otimes \mathbf{1}) \star (\mathbf{1} \otimes b)$, $\mathcal{A}_{1\star} \equiv (\mathcal{A} \otimes \mathbf{1})_\star \sim \mathcal{A}_\star$, So $\mathcal{B}_{2\star} \equiv (\mathbf{1} \otimes \mathcal{B})_\star \sim \mathcal{B}_\star$. So $(\mathcal{A}\otimes\mathcal{B})_\star$ encodes the $\star$ within $\mathcal{A}, \mathcal{B}$ and the $\otimes_\star$ between them.

H-module algebras defined by generators and relations. Given a $\mathcal{M}$, fix a (discrete) basis $\{a_i\}_{i\in\mathcal{I}}$ of $\mathcal{M}$. The *free* algebra $\mathcal{A}^f$ generated by $\{a_i\}_{i\in\mathcal{I}}$ is automatically a H-module ($*$-)algebra under

$$g \triangleright (a_{i_1} a_{i_2} ... a_{i_k}) = \sum_I \left(g^I_{(1)} \triangleright a_{i_1}\right) \left(g^I_{(2)} \triangleright a_{i_2}\right) ... \left(g^I_{(k)} \triangleright a_{i_k}\right).$$

By the previous procedure one deforms $\mathcal{A}^f$ into a $\hat{H}$-module ($*$-) algebra $\mathcal{A}^f_\star$. Now assume $\mathcal{A} = \mathcal{A}^f/\mathcal{I}$, where $\mathcal{I}$ is a H-invariant ($*$-)ideal generated by some set of polynomial relations [a]

$$f^J(a_1, a_2, ...) = 0, \qquad J \in \mathcal{J}. \tag{9}$$

We can define $\star$-polynomials $f^J_\star$ requiring $f^J_\star(a_1\star, a_2\star, ...) = f^J(a_1, a_2, ...)$ in $V(\mathcal{A}^f)[[\lambda]] = V(\mathcal{A}^f_\star)$. The $\star$-polynomial relations

$$f^J_\star(a_1\star, a_2\star, ...) = 0, \qquad J \in \mathcal{J} \tag{10}$$

generate a $\hat{H}$-invariant ($\hat{*}$-)ideal $\mathcal{I}_\star$, hence $\mathcal{A}_\star := \mathcal{A}^f_\star/\mathcal{I}_\star$ is a H-module ($*$-)algebra $\mathcal{A}_\star$, with generators a_i and relations (10). By construction the Poincaré-Birkhoff-Witt series of $\mathcal{A}$, $\mathcal{A}_\star$ coincide.

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[a] This covers most cases of interest: if e.g. $\exists\, a_0$ spanning a 1-dimensional submodule and $a_0 a_i - a_i = a_i a_0 - a_i = 0$ for all i, then $\mathcal{A}$ has unit, $a_0 \equiv \mathbf{1}$. If $a_i a_j - a_j a_i = 0$ for all i, j, then $\mathcal{A}$ is abelian. Imposing further polynomial relations one gets the algebra of functions on an algebraic manifold M. If instead $\mathcal{A}$ is the UEA of a Lie algebra (in particular of the vector fields over M) among the relations there are $a_i a_j - a_j a_i = c^k_{ij} a_k$. And so on.

Similarly, the Poincaré-Birkhoff-Witt series of $\mathcal{A}\otimes\mathcal{B}$, $(\mathcal{A}\otimes\mathcal{B})_\star$ coincide. Denoting by $\{a_i\}_{i\in\mathcal{I}}$, $\{b_i\}_{i\in\mathcal{I}'}$ sets of generators of $\mathcal{A}$, $\mathcal{B}$ (assumed unital), a set of generators of both $\mathcal{A}\otimes\mathcal{B}$ and $(\mathcal{A}\otimes\mathcal{B})_\star$ will consist of $\{a_{i1}, b_{i'2}\}_{i\in\mathcal{I},i'\in\mathcal{I}'}$, where for any $\alpha\in\mathcal{A}[[\lambda]]$, $\beta\in\mathcal{B}[[\lambda]]$ we set

$$\alpha_1 := \alpha\otimes\mathbf{1}, \qquad \beta_2 := \mathbf{1}\otimes\beta.$$

As generators of $\mathcal{A}\otimes\mathcal{B}$ [resp. $(\mathcal{A}\otimes\mathcal{B})_\star$] they will all separately fulfill (9) [resp. (10)] and the analogous relations for $\mathcal{B}$, together with

$$a_{i1}b_{i'2} = b_{i'2}a_{i1} \qquad \text{[resp. } a_{i1}\star b_{i'2} = \sum_I (\overline{\mathcal{R}}_I^{(1)}\hat{\triangleright} b_{i'2})\star(\overline{\mathcal{R}}_I^{(1)}\hat{\triangleright} a_{i1})\text{].} \tag{11}$$

The $\{a_{i1}\}_{i\in\mathcal{I}}$ will generate a H- (resp. $\hat{H}$-) module (*-)subalgebra, which we shall call $\mathcal{A}_1$ (resp. $\mathcal{A}_{1\star}$). As a H- (resp. $\hat{H}$-) module (*-)algebra, this will be isomorphic to $\mathcal{A}$ (resp. $\mathcal{A}_\star$). Similarly for $\mathcal{B}_2$ (resp. $\mathcal{B}_{2\star}$).

As the original product of $\mathcal{A}$ no more appears in these $\star$-relations, one can introduce $\hat{H}$-module ($*$-)algebras $\widehat{\mathcal{A}}$, $\hat{B}$, $\widehat{\mathcal{A}\otimes\mathcal{B}}$ resp. isomorphic to $\mathcal{A}_\star$, $\mathcal{B}_\star$, $(\mathcal{A}\otimes\mathcal{B})_\star$ just in terms of these generators and relations. Change of notation: $a_i, b_, \triangleright, \star \longrightarrow \hat{a}_i, \hat{b}_i, \hat{\triangleright},$ (omitting the symbol $\star$); e.g. $\widehat{\mathcal{A}}\equiv$ ($*$-)algebra generated by $\{\hat{a}_i\}$ fulfilling

$$f_\star^J(\hat{a}_1, \hat{a}_2, ...) = 0, \qquad J\in\mathcal{J}$$

(and $\hat{*}$-structure defined by $\hat{a}_i{}^{\hat{*}} = S(\alpha^{-1})\triangleright\widehat{a_i^*}$).

H itself is a left H-module $*$-algebra under the left adjoint action

$$g \hat{\triangleright} h = g^I_{(\hat{1})} h \; \hat{S}(g^I_{(\hat{2})}) \tag{12}$$

(no $\hat{}$). Applying the above procedure to $\mathcal{A} = H$ one gets [Aschieri et al.] a $\hat{H}$-module $*$-algebra $H_\star$, isomorphic to $\hat{H}$ under the noncocomm. adjoint action $\hat{\triangleright}$. More generally, if a H-module $*$-algebra $\mathcal{A}$ admits a $*$-algebra map $\sigma : H \longrightarrow \mathcal{A}$ s.t. $\triangleright$ can be expressed in the "adjoint-like" form

$$g \hat{\triangleright} \hat{a} = \hat{\sigma}\left(g^I_{(\hat{1})}\right) \hat{a} \hat{\sigma}\left[\hat{S}\left(g^I_{(\hat{2})}\right)\right] \tag{13}$$

(no $\hat{}$), then $\mathcal{A}[[\lambda]]$ becomes a $\hat{H}$-module $*$-algebra under $\hat{\triangleright}$ defined by (13) (with $\hat{}$'s and extended $\hat{\sigma} : \hat{H} = H[[\lambda]] \rightarrow \mathcal{A}[[\lambda]]$). The *deforming map* $D^\sigma_{\mathcal{F}} : a \in V(\mathcal{A})[[\lambda]] \rightarrow \check{a} \in V(\mathcal{A})[[\lambda]]$ defined by

$$\check{a} \equiv D^\sigma_{\mathcal{F}}(a) := \sigma\left(\mathcal{F}^{(1)}_I\right) a \; \sigma\left[S\left(\mathcal{F}^{(2)}_I \alpha\right)\right] = \left(\overline{\mathcal{F}}^{(1)}_I \triangleright a\right) \sigma\left(\overline{\mathcal{F}}^{(2)}_I\right) \tag{14}$$

intertwines between $\triangleright, \hat{\triangleright}$:

$$g \hat{\triangleright} [D^\sigma_{\mathcal{F}}(a)] = D^\sigma_{\mathcal{F}}\left(g \triangleright a\right). \tag{15}$$

Moreover $[D^\sigma_{\mathcal{F}}(a)]^* = D^\sigma_{\mathcal{F}}\left[a^{\hat{*}}\right]$, implying $\left(g \hat{\triangleright} \check{a}\right)^* = [\hat{S}(g)]^* \hat{\triangleright} (\check{a})^*$. So if $\mathcal{M} \subseteq V(\mathcal{A})$ is a H-$*$-submodule, $D^\sigma_{\mathcal{F}}(\mathcal{M})$ is a $\hat{H}$-$*$-submodule. Finally,

$$D^\sigma_{\mathcal{F}}(a \star b) = D^\sigma_{\mathcal{F}}(a) D^\sigma_{\mathcal{F}}(b). \tag{16}$$

So we can promote $D^\sigma_{\mathcal{F}}$ to a $\hat{H}$-module $*$-algebra isomorphism $D^\sigma_{\mathcal{F}} : \mathcal{A}_\star \rightarrow \check{\mathcal{A}} = \mathcal{A}[[\lambda]]$. If $\mathcal{A}^s \subset \mathcal{A}$ is a H-module $*$-subalgebra, $\check{\mathcal{A}}^s = D^\sigma_{\mathcal{F}}(\mathcal{A}^s_\star) \subset \mathcal{A}[[\lambda]]$ is a $\hat{H}$-module ($\hat{*}$-)subalgebra.

Clearly, for $\mathcal{A} = H$ one can use $\sigma = \mathrm{id}$ [Gurevich & Majid '94]. In [G.F. '96] a σ for general H-covariant Heisenberg or Clifford algebras was proposed, see below.

Clearly, if $*$-algebra maps $\sigma_{\mathcal{A}}: H \to \mathcal{A}$, $\sigma_{\mathcal{B}}: H \to \mathcal{B}$ exist,

$$\hat{\sigma}_{\mathcal{A}\otimes\mathcal{B}} := (\sigma_{\mathcal{A}} \otimes \sigma_{\mathcal{B}}) \circ \hat{\Delta} : \hat{H} \to (\mathcal{A}\otimes\mathcal{B})[[\lambda]]$$

is a $*$-algebra map. Replacing $\hat{\sigma}_{\mathcal{A}\otimes\mathcal{B}}$ in ($\hat{13}$) we make $(\mathcal{A}\otimes\mathcal{B})[[\lambda]]$ into a $\hat{H}$-module $*$-algebra. One can define a *deforming map*, i.e. a $\hat{H}$-$*$-module isomorphism $D_{\mathcal{F}}^{\sigma_{\mathcal{A}\otimes\mathcal{B}}} : (\mathcal{A}\otimes\mathcal{B})_\star \to (\mathcal{A}\otimes\mathcal{B})[[\lambda]]$ by

$$D_{\mathcal{F}}^{\sigma_{\mathcal{A}\otimes\mathcal{B}}}(a) := \mathcal{F}_\sigma \sum_I \left[\overline{\mathcal{F}}_I^{(1)} \triangleright a\right] \sigma^{(2)} \left(\overline{\mathcal{F}}_I^{(2)}\right) \mathcal{F}_\sigma^{-1} \tag{17}$$

[with $\mathcal{F}_\sigma := (\sigma_{\mathcal{A}} \otimes \sigma_{\mathcal{B}})(\mathcal{F})$]. Again we see that, if $\mathcal{M} \subset \mathcal{A}\otimes\mathcal{B}$ is a H-submodule, then $D_{\mathcal{F}}^{\sigma^n}(\mathcal{M}) \subset \mathcal{A}^{\otimes n}[[\lambda]]$ is a $\hat{H}$-submodule.

H-covariant QM with n bosons/fermions (abstract setting) and associated Fock space

Assume the 1-particle Hilbert space is a H-$*$-module: $\exists$ a dense subspace $\mathcal{H}$ and $*$-algebra map (embedding) $\rho : H \to \mathcal{O} \equiv$alg. of operators on $\mathcal{H}$, $g \triangleright u = \rho(g)u$ on $u \in \mathcal{H}$. The compatibility condition $g \triangleright (Ou) = \left(g^I_{(1)} \triangleright O\right) g^I_{(2)} \triangleright u$ induces on $\mathcal{O}$ a H-module $*$-algebra structure:

$$g \triangleright u = \rho(g)u, \qquad\qquad g \triangleright O = \rho\left(g^I_{(1)}\right) O\, \rho\Big[S\Big(g^I_{(2)}\Big)\Big]. \tag{18}$$

Replacing ρ in (18) by $\rho^{(n)} := \rho^{\otimes n} \circ \Delta^{(n)}$ transformation of n-particle states and observables. (Previous constructions with $\sigma = \rho, \rho^{(n)}$ apply!)

The completely (anti)symmetric part $\mathcal{H}^n_\pm$ of $\mathcal{H}^{\otimes n}$ is a H-$*$-submodule and describes the Hilbert space of n-boson $(+)$ or n-fermion $(-)$ states. The completely symmetric part $\mathcal{O}^n_+$ of $\mathcal{O}^{\otimes n}$ is a H- module $*$-subalgebra and its elements maps each of $\mathcal{H}^n_+, \mathcal{H}^n_-$ into itself. $\rho^{(n)}(H) \subset \mathcal{O}^n_+$ is usually a physically relevant module $*$-subalgebra: if e.g. $\mathcal{H}$ has a rotational symmetry $so(3)$, the components of the total angular momentum of the n-particle system belong to $\rho^{(n)}(Uso(3))$.

Let $\{e_i\}_{i\in\mathbb{N}}$ be an orthonormal basis of $\mathcal{H}$. For any $i_1, i_2, \ldots, i_n \in \mathbb{N}$ let

$$e^{\pm}_{i_1,i_2,\ldots,i_n} := N\, \mathcal{P}^{n\, j_1 j_2 \ldots j_n}_{\pm\, i_1 i_2 \ldots i_n} \left(e_{j_1} \otimes e_{j_2} \otimes \ldots \otimes e_{j_n}\right) \in \mathcal{H}^n_\pm$$

(N $\equiv$normalization factor). An orthonormal basis $\mathcal{B}^n_+$ (resp. $\mathcal{B}^n_-$) of $\mathcal{H}^n_+$ (resp. $\mathcal{H}^n_-$) is obtained

choosing $i_1 \le i_2 \le \ldots \le i_n$ (resp. $i_1 < i_2 < \ldots < i_n$).

Introduce occupation numbers n_j: each counts for how many h it occurs $i_h = j$ there; for n identical fermions it can be only $n_j = 0, 1$. The vectors of $\mathcal{B}^n_\pm$ are characterized by a sequence of occupation numbers fulfilling $n = \sum_{j\in\mathbb{N}} n_j$, so one can denote them as

$$|n_1, n_2, ...\rangle := e^{\pm}_{i_1,i_2,...,i_n}. \tag{19}$$

Let $|0\rangle$ $\equiv$vacuum state. **Fock space**: completion of

$$\mathcal{H}^\infty_\pm := \mathbb{C}|0\rangle \oplus \mathcal{H} \oplus \mathcal{H}^2_\pm \oplus ... \oplus \mathcal{H}^n_\pm \oplus ...$$

Define creation/annihilation op.'s as usual. They fulfill the canonical (anti)commutation relations

$$[a^i, a^j]_\pm = 0, \qquad [a^+_i, a^+_j]_\pm = 0, \qquad [a^i, a^+_j]_\pm = \delta^i_j. \tag{CCR}$$

Assuming $|0\rangle$ to be H-invariant, a^+_i, a^i must transform as $e_i = a^+_i|0\rangle$ and $\langle i| = \langle 0|a^i$ respectively:

$$g\hat{\triangleright}\hat{a}^+_i = \rho^j_i(g)\hat{a}^+_j \qquad g\hat{\triangleright}\hat{a}^i = \hat{\rho}^{\vee}{}^j_i(g)\hat{a}^j := \rho^i_j\left[\hat{S}(g)\right]\hat{a}^j. \tag{20}$$

(with no $\hat{}$). So $\{a^+_i\}$ and $\{a^i\}$ resp. span carrier spaces of the representations $\rho, \rho^\vee$ of H. As the CCR are H-invariant, $\{a^+_i, a^i\}$ generate a (Heisenberg or Clifford) H-module $*$-algebra $\mathcal{A}$.

Applying the previous deformation procedure one obtains a H-module $*$-algebra $\mathcal{A}$ with generators $\hat{a}_i^+, \hat{a}^i$ [G.F. '96] transforming as above (with $\hat{}$), fulfilling $\hat{a}_i^{+*} = \hat{a}^i$ and

$$\begin{aligned} \hat{a}^i\hat{a}^j &= \pm R^{ij}_{vu}\hat{a}^u\hat{a}^v, \\ \hat{a}_i^+\hat{a}_j^+ &= \pm R^{vu}_{ij}\hat{a}_u^+\hat{a}_v^+, \qquad (TCR) \\ \hat{a}^i\hat{a}_j^+ &= \delta^i_j\mathbf{1}_{\mathcal{A}} \pm R^{ui}_{jv}\hat{a}_u^+\hat{a}^v, \end{aligned}$$

where $R := (\rho\otimes\rho)(\mathcal{R}) = \mathbf{1}\otimes\mathbf{1} + O(\lambda)$. We could have presented $\widehat{\mathcal{A}} \sim \mathcal{A}_\star$ also in terms of generators a_i^+, a^i and $\star$-products. On the other hand, one can define a Lie $*$-algebra map $\sigma : \mathbf{g} \to \mathcal{A}$ by

$$\sigma(g) := (g \triangleright a_j^+)a^j = \rho^i_j(g)a_i^+a^j, \qquad g \in \mathbf{g}; \tag{21}$$

σ is extended as a $*$-algebra map $\sigma : H = U\mathbf{g}[[\lambda]] \to \mathcal{A}[[\lambda]]$ over $\mathbb{C}[[\lambda]]$ by setting $\sigma(\mathbf{1}_H) = \mathbf{1}_{\mathcal{A}}$. (It is a generalization of the Jordan-Schwinger realization of $\mathbf{g} = su(2)$.) So we can make $\mathcal{A}[[\lambda]]$ into a $\hat{H}$-module $*$-algebra. Under $\hat{\triangleright}$ a_i^+, a^i *do not* transform as $\hat{a}_i^+, \hat{a}^i$ in (20), but the elements [G.F. '96]

$$\check{a}_i^+ = D^\sigma_{\mathcal{F}}(a_i^+), \qquad \check{a}^i = D^\sigma_{\mathcal{F}}[\rho^i_j(\alpha^{-1})a^j] \tag{22}$$

do. Moreover, the latter fulfill the (TCR) and $\check{a}^i = \check{a}_i^{+*}$. Namely, $\check{a}_i^+, \check{a}^i$ provide a *realization of* $\hat{a}_i^+, \hat{a}^i$ *within* $\mathcal{A}[[\lambda]]$. Hence, $*$-representations of $\mathcal{A}[[\lambda]]$ are also $*$-representations of $\widehat{\mathcal{A}}$. This suggests that, at least near $0 = \lambda \in \mathbb{C}$, $*$-representations of $\mathcal{A}$, in particular the Fock one, are also $*$-representations of $\widehat{\mathcal{A}}$ (to be verified case by case). Let us compare a_i^+ with $\check{a}_i^+, \hat{a}_i^+$:

$$\check{a}_{i_1}^+ ... \check{a}_{i_n}^+|0\rangle = (F^n)^{-1}{}^{j_1,...,j_n}_{i_1,...,i_n}\; a_{j_1}^+ ... a_{j_n}^+|0\rangle = a_{i_1}^+ \star ... \star a_{i_n}^+|0\rangle =: \hat{a}_{i_1}^+ ... \hat{a}_{i_n}^+|0\rangle. \tag{23}$$

$\check{a}_i^+, \check{a}^i, \hat{a}_i^+, \hat{a}^i, a_i^+\star, a^i\star$ all act on the Fock space of bosons/fermions (no change of statistics!).

Differential and integral calculus over G-symmetric M

Let $\mathcal{M} = C^\infty(M)$, $\Xi \supset \mathbf{g}$ the algebra of vector fields over M, $\mathcal{A} = \mathcal{D} := U\Xi{\ltimes}\, \mathcal{M}$ (cf. [Aschieri et al]). (By construction $\mathcal{M}$ is a H-module $*$-subalgebra of $\mathcal{D}$). We apply the above $\star$-deformation procedure with some twist $\mathcal{F} \in (H \otimes H)[[\lambda]]$, $H = U\mathbf{g}$.

If M is a submanifold of some $\mathbb{R}^m$ characterized by a set of equations $f_J(x) = 0$ [with $x = (x^1, \ldots x^m)$] symmetric under $\mathbf{g}$, $\mathcal{M}$ is the abelian H-module algebra generated by $x^1, \ldots x^m$ fulfilling the relations $f_J(x) = 0$ in addition to $x^a x^b - x^b x^a = 0$, Ξ is the Lie subalgebra of vector fields $\xi = \sum_{h=1}^m \xi^h(x)\partial_{x^h}$ over $\mathbb{R}^m$ such that $[\xi, f_J(x)] = 0$, and the $\star$-deformed (or "hatted") objects can be described globally in terms of **generators** $\hat{x}^h, \hat{\partial}_{x^h}$ **and relations**.
One can globally define a Lie $*$-algebra map $\sigma : H[[\lambda]] \to U\Xi[[\lambda]]$ starting from

$$\sigma(g) := \sum_{h=1}^m (g \triangleright x^h)\partial_{x^h} \in \Xi, \qquad\qquad g \in \mathbf{g}.$$

and the corresponding deforming map $D_{\mathcal{F}}^\sigma$ for $\mathcal{D}$, and then for tensor powers of $\mathcal{D}$.

If M $\equiv$Riemannian, G $\equiv$its group of isometries, $d\nu$ $\equiv$invariant volume form, also $\int_X d\nu(x)$ is:

$$\int_X d\nu(x)(g \triangleright f) = \epsilon(g) \int_X d\nu f \qquad \Leftrightarrow \qquad \int_X d\nu f(g \triangleright h) = \int_X d\nu [S(g) \triangleright f]h.$$

This implies for the corresponding $\star$-product

$$\int_X d\nu(x) f(x) \star h(x) = \int_X d\nu(x) f(\mathbf{x})[\alpha \triangleright h(x)] = \int_X d\nu(x)[S(\alpha) \triangleright f(\mathbf{x})]h.$$

These eqns hold also for integration over n independent x-variables.

From wavefunctions to quantum fields (non-relativistic)

Let $H = U\mathbf{g}$ include the UEA of the Lie group G of isometries of a commutative spacetime $\mathbb{R}\times X$, with $X \equiv$a Riemannian manifold on which QM is well-defined; then $d\nu$, $\int_X d\nu$ are H-invariant (E.g. $X = \mathbb{R}^3$, $G\equiv$ Galilei group). **Fix an inertial reference frame.**

First: $n = 1$ nonrelativistic quantum particle on X (with spin zero or factored out):

1. $\exists$ H-equivariant, unitary transformation $\kappa : u \in \mathcal{H} \leftrightarrow \psi_u \in \mathcal{X} \subset C^\infty(X) \cap \mathcal{L}^2(X, d\nu)$, $g \triangleright \psi_u = \psi_{g \triangleright u}$,

$$\langle u|v\rangle = \int_X d\nu\, [\psi_u(\mathbf{x})]^* \psi_v(\mathbf{x}) = \int_X d\nu\, [\psi_u(\mathbf{x})]^{\hat{*}} \star \psi_v(\mathbf{x}). \tag{24}$$

2. $\kappa(Ou) = \tilde{\kappa}(O)\kappa(u)$ for any $u \in \mathcal{H}$ defines a H-equivariant map $\tilde{\kappa} : \mathcal{O} \leftrightarrow \mathcal{D}$. [for $X = \mathbb{R}^3$, $\mathcal{O}$ is generated by $\{q^a, p^a\}$, and $\tilde{\kappa}(q^a) = x^a \cdot$, $\tilde{\kappa}(p^a) = -i\hbar \frac{\partial}{\partial x^a}$].

The maps $\kappa, \tilde{\kappa}$ provide a *commutative, H-equivariant configuration space realization* of $\{\mathcal{H}, \mathcal{O}\}$ on $\mathcal{X}, \mathcal{D}$, **depending on the choice of the reference frame**.

This is immediately extended to n identical quantum particles on X:
1) $\kappa^{\otimes n} : \mathcal{H}^{\otimes n} \leftrightarrow \mathcal{X}^{\otimes n}$ and the restrictions to the completely (anti)symmetric subspaces $\kappa^{\otimes n} : \mathcal{H}^n_\pm \leftrightarrow (\mathcal{X}^{\otimes n})_\pm$ are H-equivariant unitary transfs, (24) holds with n-fold integration
2) $\tilde{\kappa} : \mathcal{O}^{\otimes n} \leftrightarrow \mathcal{D}^{\otimes n}$ and the restriction $\tilde{\kappa} : \mathcal{O}^{\otimes n}_+ \leftrightarrow \mathcal{D}^{\otimes n}_+$ are H-equivariant maps.

Applying the $\star$-deformation procedure with a twist leaving t central, one defines wavefunctions $\hat{\psi}_u \equiv \wedge^n(\psi_u)$ of noncommutative coordinates, $\hat{\psi}_u(\mathbf{x}_1\star, ..., \mathbf{x}_n\star, t) = \psi_u(\mathbf{x}_1, ..., \mathbf{x}_1, t)\star$, and "hatted" differential operators $\hat{D}(\mathbf{x}\star, \partial\star) := D(\mathbf{x}, \partial)\star$, and one finds $\hat{D} = \wedge^n D [\wedge^n]^{-1}$. We define a deformed $\hat{H}$-invariant “integration over $\hat{X}$” $\int_{\hat{X}} d\hat{\nu}(\hat{\mathbf{x}})$ such that

$$\int_{\hat{X}} d\hat{\nu}(\hat{\mathbf{x}})\hat{f}(\hat{\mathbf{x}}) = \int_X d\nu(\mathbf{x}) f(\mathbf{x}),$$

and similarly for n-fold integration. Then in “hat-notation” $(24)_2$ for n particles becomes

$$\langle u, v\rangle = \int_{\hat{X}} d\hat{\nu}(\hat{\mathbf{x}}_1)...\int_{\hat{X}} d\hat{\nu}(\hat{\mathbf{x}}_n)[\hat{\psi}_u(\hat{\mathbf{x}}_1, ..., \hat{\mathbf{x}}_n)]^{\hat{*}}\hat{\psi}_v(\hat{\mathbf{x}}_1, ..., \hat{\mathbf{x}}_n). \tag{25}$$

The map $\wedge^n : \psi_u \in \mathcal{X}^{\otimes n} \rightarrow \hat{\psi}_u \in (\mathcal{X}^{\otimes n})_\star$ is therefore unitary and $\hat{H}$-equivariant. $\wedge^n$ also maps the action of the symmetric group S_n from $\mathcal{X}^{\otimes n}$ to $(\mathcal{X}^{\otimes n})_\star$. A permutation $\tau \in S_n$ is represented on $\mathcal{X}^{\otimes n}$, $(\mathcal{X}^{\otimes n})_\star$ respectively by the permutation operator $\mathcal{P}_\tau$ and the “twisted permutation operator” $\mathcal{P}^F_\tau = \wedge^n \mathcal{P}_\tau [\wedge^n]^{-1}$ [c.f. G.F.-Schupp 96].

Let $\hat{\kappa}^n := \wedge^n \kappa^{\otimes n}$, $\widehat{\tilde{\kappa}}^n(\cdot) := \wedge^n[\tilde{\kappa}^{\otimes n}(\cdot)][\wedge^n]^{-1}$. Then the maps $\hat{\kappa}^n_\pm, \widehat{\tilde{\kappa}}^n_+$ define a *$\hat{H}$-equivariant, noncommutative configuration space realization* of $\{\mathcal{H}^{\otimes n}_\pm, \mathcal{O}^{\otimes n}_+\}$ on $(\mathcal{X}^{\otimes n}_\pm)_\star, (\mathcal{D}^{\otimes n}_+)_\star$, depending on the choice of the reference frame.

Let $\{e_i\}_{i\in\mathbb{N}}$ be orthonormal basis of $\mathcal{H}$, $\varphi_i=\kappa(e_i)$, a_i^+, a^i associated wavefunction, creation, annihilation operators. The (nonrelativistic) field operator and its hermitean conjugate

$$\varphi(\mathbf{x}) := \varphi_i(\mathbf{x})a^i, \qquad \varphi^*(\mathbf{x})=\varphi_i^*(\mathbf{x})a_i^+ \tag{26}$$

in the Schrödinger picture (sum over i: infinitely many terms) are operator-valued distributions, basis-independent (i.e. invariant under a unitary transf. U of $\{e_i\}_{i\in\mathbb{N}}$) and fulfilling the CC(A)R

$$[\varphi(\mathbf{x}),\varphi(\mathbf{y})]_\mp = \text{h.c.} = 0, \qquad [\varphi(\mathbf{x}),\varphi^*(\mathbf{y})]_\mp = \varphi_i(\mathbf{x})\varphi_i^*(\mathbf{y}) = |g|^{-\frac{1}{2}}\delta(\mathbf{x}-\mathbf{y}) \tag{27}$$

($\mp$ for bosons/fermions). The *field $*$-algebra* Φ is spanned e.g. by the normal ordered monomials

$$\varphi^*(\mathbf{x}_1)....\varphi^*(\mathbf{x}_m)\varphi(\mathbf{x}_{m+1})...\varphi(\mathbf{x}_n)$$

($\mathbf{x}_1, ..., \mathbf{x}_n$ are independent points). So $\Phi \subset \Phi^e := \left(\bigotimes_{i=1}^{\infty} V\right)\otimes\mathcal{A}$ (1st, 2nd,... V means space of distributions depending on $\mathbf{x}_1, \mathbf{x}_2$,...). CCR of $\mathcal{A}$ are the only nontrivial comm. rel. in Φ^e .
H-invariant $|0\rangle \Rightarrow a_i^+$ transform as $e_i = a_i^+|0\rangle, \varphi_i$, whereas a^i transform as $\langle i| = \langle 0|a^i, \varphi_i^*$:

$$g \triangleright a_i^+ = \rho_i^j(g)a_j^+ \qquad g \triangleright a^i = \rho^{\vee}{}_i^j(g)a^j := \rho_j^i\big(S(g)\big)a^j. \tag{28}$$

When $g \in \mathbf{g}$ this is a very special (infinitesimal) unitary transformation U of $\{e_i\}_{i\in\mathbb{N}}$. Under this transf. φ, φ^* **are scalars. So $\star$-products with $\forall\chi\in\Phi^e$ make no difference:**

$$g \triangleright \varphi(\mathbf{x}) = \epsilon(g)\varphi(\mathbf{x}), \qquad \varphi(\mathbf{x})\star\chi = \varphi(\mathbf{x})\chi, \qquad \chi\star\varphi(\mathbf{x}) = \chi\varphi(\mathbf{x}), \qquad \& \quad \text{h. c.} \tag{29}$$

These properties, $\epsilon(\alpha)=1$ and the definition $a'^{i} := a_i^{+\hat{*}} = S(\alpha^{-1})\triangleright a^i = \rho^i_j(\alpha^{-1})a^j$ imply

$$\varphi(\mathbf{x}) = \varphi_i(\mathbf{x})\star a'^{i}, \qquad\qquad \varphi^*(\mathbf{x}) = \varphi^{\hat{*}}(\mathbf{x}) = a_i^+\star\varphi_i^{\hat{*}}(\mathbf{x}), \tag{30}$$

$\varphi_i(\mathbf{x})\varphi_i^*(\mathbf{y}) = \varphi_i(\mathbf{x})\star\varphi_i^{\hat{*}}(\mathbf{y})$ and therefore that the CCR (27) can be rewritten in the form

$$[\varphi(\mathbf{x}) \stackrel{\star}{,} \varphi(\mathbf{y})]_\mp = h.c. = 0, \qquad\qquad [\varphi(\mathbf{x}) \stackrel{\star}{,} \varphi^{\hat{*}}(\mathbf{y})]_\mp = \varphi_i(\mathbf{x})\star\varphi_i^{\hat{*}}(\mathbf{y}) \tag{31}$$

(here and below $[A \stackrel{\star}{,} B]_\mp := A\star B \mp B\star A$). Also Φ^e is a H-module $*$-algebra.

The field fulfills the following properties: for any $u\in(\mathcal{H}^{\otimes n})_\pm$

$$\psi_u(\mathbf{x}_1,\ldots,\mathbf{x}_n) = \kappa^{\otimes n}(u)(\mathbf{x}_1,\ldots,\mathbf{x}_n) = \frac{1}{\sqrt{n!}}\langle 0|\varphi(\mathbf{x}_n)\star\ldots\star\varphi(\mathbf{x}_1)u, \tag{32}$$

$$u = \frac{1}{\sqrt{n!}}\int_X d\nu(\mathbf{x}_1)\ldots\int_X d\nu(\mathbf{x}_n)\psi_u(\mathbf{x}_1,\ldots,\mathbf{x}_n)\varphi^{\hat{*}}(\mathbf{x}_1)\star\ldots\star\varphi^{\hat{*}}(\mathbf{x}_n)|0\rangle, \tag{33}$$

which are very useful for computing the unitary transformation $\kappa^{\otimes n}\restriction_{\mathcal{H}^n_\pm}$ and its inverse, taking into account automatically the combinatorial aspects of (anti)symmetrization.

Equations of motion

Assume the n-particle wavefunctions $\psi_\star^{(n)}$ fulfill some $\star$-differential Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\psi_\star^{(1)} = \mathsf{H}_\star^{(1)}\psi_\star^{(1)}, \qquad \mathsf{H}_\star^{(1)} = \left[-\frac{\hbar^2}{2m}D^a\star D_a + V\right]\star, \qquad D_a = \partial_a + ieA_a$$
$$i\hbar\frac{\partial}{\partial t}\psi_\star^{(n)} = \mathsf{H}_\star^{(n)}\psi_\star^{(n)}, \qquad \mathsf{H}_\star^{(n)} = \sum_{h=1}^{n}\mathsf{H}_\star^{(1)}(\mathbf{x}_h,t) + \sum_{h<k} W(\rho_{hk})\star. \quad n\geq 2 \tag{34}$$

Commuting t; $\star$-local interaction with external background potential $V(\mathbf{x},t)$ and $U(1)$ gauge potential $\mathbf{A}(\mathbf{x},t)$. $\rho_{hk}\equiv$distance between the points $\mathbf{x}_h,\mathbf{x}_k$ ($\rho_{hk}=|\mathbf{x}_h-\mathbf{x}_k|$ if $X=\mathbb{R}^3$). $\mathsf{H}_\star^{(n)}\equiv$pseudo-differential operator! It is hermitean provided $\mathsf{H}^{(1)}$ is and $\alpha\triangleright\mathsf{H}^{(1)}=\mathsf{H}^{(1)}$. $\mathsf{H}_\star^{(n)}$ is completely symmetric, so preserves the (anti)symmetry of $\psi_\star^{(n)}$. The Fock space Hamiltonian

$$\mathsf{H}_\star(\varphi) = \int_X d\nu(\mathbf{x})\varphi^{\hat{*}}(\mathbf{x})\star\mathsf{H}_\star^{(1)}(\mathbf{x},t)\varphi(\mathbf{x}) + \int_X d\nu(\mathbf{x})\int_X d\nu(\mathbf{y})\varphi^{\hat{*}}(\mathbf{y})\star\varphi^{\hat{*}}(\mathbf{x})\star W(\rho_{\mathbf{xy}})\star\varphi(\mathbf{x})\star\varphi(\mathbf{y}$$

commutes with $\mathbf{n}:=a_i^+a^i=a_i^+\star a'^i$, and $\kappa^{\otimes n}\circ\mathsf{H}_\star\restriction_{\mathcal{H}_\pm^n}=\mathsf{H}_\star^{(n)}$ for $n\geq 2$.

The Heisenberg field operator $\varphi_\star^H(\mathbf{x},t) := e^{-\frac{i}{\hbar}\int_0^t dt\,\mathsf{H}_\star}\varphi(\mathbf{x})e^{-\frac{i}{\hbar}\int_0^t dt\,\mathsf{H}_\star}$ fulfills

$$[\varphi_\star^H(\mathbf{x},t)\stackrel{\star}{,}\varphi_\star^H(\mathbf{y},t)]_\mp = \text{h.c.} = 0, \qquad [\varphi_\star^H(\mathbf{x},t)\stackrel{\star}{,}\varphi_\star^{H\hat{*}}(\mathbf{y},t)]_\mp = \varphi_i(\mathbf{x})\star\varphi_i^{\hat{*}}(\mathbf{y}),$$
$$i\hbar\frac{\partial}{\partial t}\varphi_\star^H = [\mathsf{H}_\star\stackrel{\star}{,}\varphi_\star^H]. \tag{35}$$

If $W=0$ (34)$_3$ amounts to the "second quantization of (33)$_1$", $i\hbar\frac{\partial\varphi_\star^H}{\partial t}=\mathsf{H}_\star^{(1)}\varphi_\star^H$, a $\star$-local equation. If $\mathsf{H}_\star^{(1)}$ is t-independent, so is $\mathsf{H}_\star$, then $\mathsf{H}_\star(\varphi_\star^H)=\mathsf{H}_\star(\varphi)$, and (34) can be equivalently formulated directly in the Heisenberg picture as equations in the unknown $\varphi_\star^H(t)$.

By further replacing $\hat{V}(\mathbf{x}\star,t)=V(\mathbf{x},t)\star$, $\hat{\mathbf{A}}(\mathbf{x}\star,t)=\mathbf{A}(\mathbf{x},t)\star$, $\hat{\varphi}_i(\mathbf{x}\star)=\varphi_i(\mathbf{x})\star$ we can reformulate the previous eq.'s purely with $\star$-products: **2nd quantization on the NC spacetime $\hat{X}\times\mathbb{R}$ compatible with QM axioms and Bose/Fermi statistics**. In "hat" notation, within $\hat{\Phi}^e$, $\hat{\Phi}$,

$$
\begin{aligned}
&\hat{\varphi}(\hat{\mathbf{x}})=\hat{\varphi}_i(\hat{\mathbf{x}})\hat{a}'^i, && \varphi^{\hat{*}}(\hat{\mathbf{x}})=\hat{a}_i^+\hat{\varphi}_i^{\hat{*}}(\hat{\mathbf{x}})\\
&[\hat{\varphi}(\hat{\mathbf{x}}),\hat{\varphi}(\hat{\mathbf{y}})]_\mp=\text{h.c.}=0, && [\hat{\varphi}(\hat{\mathbf{x}}),\hat{\varphi}^{\hat{*}}(\hat{\mathbf{y}})]_\mp=\hat{\varphi}_i(\hat{\mathbf{x}})\hat{\varphi}_i^{\hat{*}}(\hat{\mathbf{y}}),\\
&i\hbar\frac{\partial}{\partial t}\hat{\psi}=\hat{\mathsf{H}}^{(n)}\hat{\psi}, && \hat{\mathsf{H}}^{(n)}=\sum_{h=1}^{n}\hat{\mathsf{H}}^{(1)}(\hat{\mathbf{x}}_h,t)+\sum_{h<k}\hat{W}(\hat{\rho}_{hk})\\
&\hat{\mathsf{H}}=\int d\hat{\nu}(\hat{\mathbf{x}})\hat{\varphi}^{\hat{*}}(\hat{\mathbf{x}})\hat{\mathsf{H}}^{(1)}(\hat{\mathbf{x}},t)\hat{\varphi}(\hat{\mathbf{x}})+\int d\hat{\nu}(\hat{\mathbf{x}})\int d\hat{\nu}(\hat{\mathbf{y}})\hat{\varphi}^{\hat{*}}(\hat{\mathbf{y}})\hat{\varphi}^{\hat{*}}(\hat{\mathbf{x}})W(\hat{\rho}_{\mathbf{xy}})\hat{\varphi}(\mathbf{x})\hat{\varphi}(\mathbf{y}),\\
&[\hat{\varphi}_H(\hat{\mathbf{x}},t),\hat{\varphi}_H(\hat{\mathbf{y}},t)]_\mp=\text{h.c.}=0, && [\hat{\varphi}_H(\hat{\mathbf{x}},t),\hat{\varphi}_H^{\hat{*}}(\hat{\mathbf{y}},t)]_\mp=\hat{\varphi}_i(\hat{\mathbf{x}})\hat{\varphi}_i^{\hat{*}}(\hat{\mathbf{y}}),\\
&i\hbar\frac{\partial}{\partial t}\hat{\varphi}_H=[\hat{\mathsf{H}},\hat{\varphi}_H].
\end{aligned}
\tag{36}
$$

There is an advantage if the $\hat{\mathbf{x}}$-dependence of $\hat{V}(\hat{\mathbf{x}},t)$, $\hat{\mathbf{A}}(\hat{\mathbf{x}},t)\hat{\varphi}_i(\hat{\mathbf{x}})$ is simpler than the $\mathbf{x}$-dependence of $V(\mathbf{x},t)$, $\mathbf{A}(\mathbf{x},t)$, $\varphi_i(\mathbf{x})$, as it happens if the latter fulfill $\star$-differential equations. **Now you can forget how you have got (36), and check its consistency beyond the level of formal λ-power series using only deformed "maths".**

Note in particular that the field commutation relations, both in the Schroedinger and in the Heisenberg picture, are of the type ”field (anti)commutator= a distribution”.

$$\hat{\psi}_u(\hat{\mathbf{x}}_1,...,\hat{\mathbf{x}}_n) := \hat{\kappa}^n_\pm(u)(\hat{\mathbf{x}}_1,...,\hat{\mathbf{x}}_n) = \frac{1}{\sqrt{n!}}\langle 0|\hat{\varphi}(\hat{\mathbf{x}}_n)...\hat{\varphi}(\hat{\mathbf{x}}_1)u,$$
$$u = \frac{1}{\sqrt{n!}}\int_{\hat{X}} d\hat{\nu}(\hat{\mathbf{x}}_1)...\int_{\hat{X}} d\hat{\nu}(\hat{\mathbf{x}}_n)\hat{\psi}_u(\hat{\mathbf{x}}_1,...,\hat{\mathbf{x}}_n)\hat{\varphi}^{\hat{*}}(\hat{\mathbf{x}}_1)...\hat{\varphi}^{\hat{*}}(\hat{\mathbf{x}}_n)|0\rangle. \tag{37}$$

for any $u\in(\mathcal{H}^{\otimes n})_\pm$; choosing $u = e^\pm_{i_1,...,i_n} \in \mathcal{B}^n_\pm$ one finds in particular

$$\psi_u(\mathbf{x}_1,...,\mathbf{x}_n) = N\varphi_{(j_1}(\mathbf{x}_1)...\varphi_{j_n]}(\mathbf{x}_n),$$
$$\hat{\psi}_u(\hat{\mathbf{x}}_1,...,\hat{\mathbf{x}}_n) = \mathsf{F}^{n\,j_1...j_n}_{\;\;i_1...i_n} N\hat{\varphi}_{(j_1}(\hat{\mathbf{x}}_1)...\hat{\varphi}_{j_n]}(\hat{\mathbf{x}}_n)$$

where $(...]$ means indices (anti)symmetrization, and $\mathsf{F}^n := (\tilde{\kappa}\circ\rho)^{\otimes n}\left(\mathcal{F}^n\right)$ (a unitary operator). The group S_n acts on $\hat{\psi}_u(\hat{\mathbf{x}}_1,...,\hat{\mathbf{x}}_n) \in$ the (braided) tensor product $\hat{\mathcal{X}}\,\underline{\otimes}\,...\,\underline{\otimes}\,\hat{\mathcal{X}}$ by "twisted permutations" $\mathcal{P}^F_\tau = F^n\mathcal{P}_\tau F^{n-1}$ [G.F. & Schupp ’95]. This is an alternative way to fulfill Bose/Fermi statistics.

Examples: QM and QFT on Moyal NC space(time)

Here $\mathbf{g} = \mathcal{G} \equiv$ Galilei Lie algebra in the non-relativistic case, $\mathbf{g} = \mathcal{P} \equiv$ Poincaré Lie algebra in the relativistic case. Simplest choice for $\mathcal{F}$:

$$\mathcal{F} \equiv \sum_I \mathcal{F}_I^{(1)} \otimes \mathcal{F}_I^{(2)} := \exp\left(\tfrac{i}{2}\lambda\theta^{\mu\nu} P_\mu \otimes P_\nu\right) \to \exp\left(\tfrac{i}{2}\theta^{\mu\nu} P_\mu \otimes P_\nu\right).$$

where $\theta^{\mu\nu}$ is a fixed real antisymmetric matrix. Setting $M_\omega = \omega^{\mu\nu} M_{\mu\nu}$ ($\omega^{\mu\nu} = -\omega^{\nu\mu}$),

$$\hat{\Delta}(P_\mu) = \Delta(P_\mu) = P_\mu \otimes \mathbf{1} + \mathbf{1} \otimes P_\mu = \Delta(P_\mu),$$
$$\hat{\Delta}(M_\omega) = M_\omega \otimes \mathbf{1} + \mathbf{1} \otimes M_\omega + P \cdot \otimes [\omega, \theta] P \neq \Delta(M_\omega).$$

Translations undeformed!

When $\mathbf{g} = \mathcal{G}$ put $\theta^{0a} = 0$, $t = x^0$, $P_0 = H_0 \equiv$ non-relativistic kinetic energy, $M^{bc} = \epsilon^{abc} L^a$, $M^{0a} = K^a$, and the mass m is an additional generator, central. Only nontrivial comm. rel.:

$$[K^a, P^b] = im\hbar\delta^{ab}, \qquad [K^a, H_0] = i\hbar P^a,$$
$$[L^a, L^b] = i\epsilon^{abc}\hbar L^c, \qquad [L^a, P^b] = i\epsilon^{abc}\hbar P^c, \qquad [L^a, K^b] = i\epsilon^{abc}\hbar K^c. \tag{38}$$

The above $\mathcal{F}$ gives $x_i^\mu \star x_j^\nu = x_i^\mu x_j^\nu + i\theta^{\mu\nu}/2 \qquad \Rightarrow \qquad [x_i^\mu \overset{\star}{,} x_j^\nu] = \mathbf{1} i\theta^{\mu\nu}$,

$$a(x_i) \star b(x_j) = \exp\left[\frac{i}{2}\partial_{x_i}\theta\partial_{x_j}\right] a(x_i) b(x_j), \tag{4'}$$

after which we must set $x_i = x_j$ if $i = j$.

Simplest (nonrelativistic) models where one can see the effects of the $\star$-locality of the interaction:

1. Charged particle in constant magnetic field $\mathbf{B}$. The simplest gauge choice is $A^i(x)=\epsilon^{ijk}B^jx^k/2$. One finds $\mathsf{H}_\star^{(1)}$, is still differential of second order, but more complicated. In terms of "hatted" objects it can be formulated and solved as in the undeformed case. Choose x^3-axis parallel to $q\mathbf{B}=qB\vec{k}$ with $qB>0$, this gives $\hat{D}^3=\partial^3$, $\hat{D}^a=\partial^a-i\frac{qB}{2\hbar c}\epsilon^{ab}\hat{x}^b$ for $a,b\in\{1,2\}$, with $\epsilon^{12}=1=-\epsilon^{21}$, $\epsilon^{aa}=0$. These fulfill $[\partial^3,\hat{D}^a]=0$, $[\hat{D}^1,\hat{D}^2]=i\frac{qB}{\hbar c}[1-\frac{qB\theta^{12}}{2\hbar c}]$. Defining

$$a:=\alpha[\hat{D}^1-i\hat{D}^2],\qquad a^*=\alpha[-\hat{D}^1-i\hat{D}^2]\qquad \alpha:=\sqrt{\frac{\hbar c}{qB}}\Big/\sqrt{2-\frac{qB\theta^{12}}{2\hbar c}} \tag{39}$$

(we assume $qB\theta^{12}<4\hbar c$) one obtains the commutation relation $[a,a^*]=1$, and

$$\begin{aligned}\mathsf{H}^{\hat{(1)}}&=\frac{-\hbar^2}{2m}\hat{D}^i\hat{D}^i=\frac{-\hbar^2}{2m}\left[(\partial^3)^2-\frac{1}{2\alpha^2}(aa^*+a^*a)\right]=\mathsf{H}^{\hat{(1)}}{}_{\parallel}+\mathsf{H}^{\hat{(1)}}{}_{\perp}\\ \mathsf{H}^{\hat{(1)}}{}_{\parallel}&:=\frac{(-i\hbar\partial^3)^2}{2m},\qquad \mathsf{H}^{\hat{(1)}}{}_{\perp}:=\hbar\omega\left(a^*a+\frac{1}{2}\right),\qquad \omega:=\frac{qB}{mc}\left(1-\frac{qB\theta^{12}}{4\hbar c}\right)\end{aligned} \tag{40}$$

$[\mathsf{H}^{\hat{(1)}}{}_{\parallel},\mathsf{H}^{\hat{(1)}}{}_{\perp}]=0$. $\mathsf{H}^{\hat{(1)}}{}_{\parallel}$ has continuous spectrum $[0,\infty[$; the generalized eigenfuntions are the eigenfuntions $e^{ik\hat{x}^3}$ of $p^3=-i\hbar\partial^3$ with eigenvalue $\hbar k$. The second is formally an harmonic oscillator Hamiltonian with ω modified by the presence of the noncommutativity θ^{12}.

2. Charged particle in a plane wave electromagnetic field.
$A^a(x) = \varepsilon^a(\mathrm{p})\exp[-ip\cdot x] \equiv \varepsilon^a(\mathrm{p})\exp[i(\mathrm{p}\cdot\mathbf{x} - |\mathrm{p}|t)]$, (the amplitude vector fulfilling $\varepsilon^a(\mathrm{p})p^a = 0$). To check **(??)** it is useful to note the properties

$$e^{i\mathrm{p}\cdot\mathbf{x}} \star f(\mathbf{x}) = e^{i\mathrm{p}\cdot\mathbf{x}} f(\mathbf{x}+\theta\mathrm{p}/2) \qquad \Rightarrow \qquad e^{i\mathrm{p}\cdot\mathbf{x}} \star e^{ia\mathrm{p}\cdot\mathbf{x}} = e^{i\mathrm{p}\cdot\mathbf{x}} e^{ia\mathbf{p}\cdot\mathbf{x}} \tag{41}$$

where $(\theta\mathrm{p})^a := \theta^{ab}\mathrm{p}^b$, as $\mathrm{p}\theta\mathrm{p} = 0$. The Schrödinger equation for $n=1$ particle becomes

$$i\hbar\partial_t\psi_\star^{(1)}(\mathbf{x},t) = \frac{-\hbar^2}{2m}\left[\Delta\psi_\star^{(1)}(\mathbf{x},t) + 2iee^{-ip\cdot x}\varepsilon^a\partial_a\psi_\star^{(1)}\left(\mathbf{x}+\frac{\theta\mathrm{p}}{2},t\right) - e^2e^{-2ip\cdot x}|\varepsilon|^2\psi_\star^{(1)}(\mathbf{x}+$$

the nonlocality induced by the $\star$-product is here particularly simple, in that it involves the wavefunction at points $\mathbf{x}, \mathbf{x}+\theta\mathrm{p}/2, \mathbf{x}+\theta\mathrm{p}$ related by the constant shift $\theta\mathrm{p}/2$.

Relativistic QFT

By analogous considerations one can construct a consistent (at least free) QFT on a NC Minkowski spacetime with twisted symmetry. For the Moyal NC one reobtains recent results of G.F., J. Wess 07, in particular

$$[\varphi_0(x) \stackrel{\star}{,} \varphi_0(y)] = i\Delta(x-y), \qquad i\Delta(\xi) := \int \frac{d\mu(p)}{(2\pi)^3}[e^{-ip\cdot\xi} - e^{-ip\cdot\xi}] \tag{42}$$

(Δ undeformed!) for free fields, implying **the c.c.r.** $[\varphi_0(x^0,\mathbf{x}) \stackrel{\star}{,} \dot{\varphi}_0(x^0,\mathbf{y})] = i\,\delta^3(\mathbf{x}-\mathbf{y})$. In terms of generalized basis (eigenvectors of P_μ) and creation & annihilation operators:

$$\begin{aligned}
a^+_{\mathbf{p}}\star a^+_{\mathbf{q}} &= e^{-ip\theta q}\, a^+_{\mathbf{q}}\star a^+_{\mathbf{p}}, & \qquad \hat{a}^+_{\mathbf{p}}\hat{a}^+_{\mathbf{q}} &= e^{iq\theta p}\, \hat{a}^+_{\mathbf{q}}\hat{a}^+_{\mathbf{p}}, \\
a^{\mathbf{p}}\star a^{\mathbf{q}} &= e^{-ip\theta q}\, a^{\mathbf{q}}\star a^{\mathbf{p}}, & \hat{a}^{\mathbf{p}}\hat{a}^{\mathbf{q}} &= e^{iq\theta p}\, \hat{a}^{\mathbf{q}}\hat{a}^{\mathbf{p}}, \\
& \qquad\Leftrightarrow \\
a^{\mathbf{p}}\star a^+_{\mathbf{q}} &= e^{ip\theta q}\, a^+_{\mathbf{q}}\star a^{\mathbf{p}} + 2p^0\delta^3(\mathbf{p}-\mathbf{q}) & \hat{a}^{\mathbf{p}}\hat{a}^+_{\mathbf{q}} &= e^{ip\theta q}\, \hat{a}^+_{\mathbf{q}}\hat{a}^{\mathbf{p}} + 2p^0\delta^3(\mathbf{p}-\mathbf{q}), \\
a^{\mathbf{p}}\star e^{iq\cdot x} &= e^{-ip\theta q}\, e^{iq\cdot x}\star a^{\mathbf{p}}, \quad \&\text{ h.c.}, & \hat{a}^{\mathbf{p}} e^{iq\cdot\hat{x}} &= e^{-ip\theta q}\, e^{iq\cdot\hat{x}}\hat{a}^{\mathbf{p}}, \quad \&\text{ h.c.};
\end{aligned} \tag{43}$$

$$\check{a}^+_{\mathbf{p}} \equiv D^{\sigma}_{\mathcal{F}}\left(a^+_{\mathbf{p}}\right) = a^+_{\mathbf{p}} e^{-\frac{i}{2}p\theta\sigma(P)}, \qquad \check{a}^{\mathbf{p}} \equiv D^{\sigma}_{\mathcal{F}}\left(a^{\mathbf{p}}\right) = a^{\mathbf{p}} e^{\frac{i}{2}p\theta\sigma(P)}$$

$$\hat{a}^+_{\mathbf{p}_1}...\hat{a}^+_{\mathbf{p}_n}|0\rangle = a^+_{\mathbf{p}_1}\star...\star a^+_{\mathbf{p}_n}|0\rangle = \check{a}^+_{\mathbf{p}_1}...\check{a}^+_{\mathbf{p}_n}|0\rangle = \exp\left[-\frac{i}{2}\sum_{\substack{j,k=1\\ j<k}}^{n} p_j\theta p_k\right] a^+_{\mathbf{p}_1}...a^+_{\mathbf{p}}$$

where $\sigma(P_\mu) = \int d\mu(p)\, p_\mu a^+_{\mathbf{p}} a^{\mathbf{p}}$. By (45) generalized states differ from their undeformed counterparts only by multiplication by a phase factor. As $\check{a}^+_{\mathbf{p}}\check{a}^{\mathbf{p}} = a^+_{\mathbf{p}} a^{\mathbf{p}}$, $\sigma(P_\mu) = \int d\mu(p)\, p_\mu \check{a}^+_{\mathbf{p}}\check{a}^{\mathbf{p}}$, the inverse of $D^{\sigma}_{\mathcal{F}}$ is readily obtained.

This means that the results of G.F., J. Wess 07 are consistent with Bose-Fermi statistics and a description of (at least) free n-particle states by t-dependent wavefuntions.