NC Emergent Gravity
&
Fermions

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hep-th/0805.1157

Outline

To begin with … an Introduction

Geometry & NC *U(N)* gauge theory

Scalars

Fermions

… then gravity emerges …

Geometry & UV/IR mixing

Conclusion

Introduction

¿ GR & QM ?

Gravity
$\Updownarrow$
Quantum fluctuations of space-time

Noncommutative space-time

$$[x_i, x_j] = i\,\theta_{ij}$$

Desire simple, intrinsic relation between NC & Gravity

NC gauge theory contains gravity

The Model

$$S_{YM} = -\mathrm{Tr}\big[Y^a, Y^b\big]\big[Y^{a'}, Y^{b'}\big] g_{aa'} g_{bb'}$$

$Y^a \in L(\mathcal{H})$ ... matrices/operators; $\quad a = 0, 1, 2, 3;$

$$\big[Y^a, Y^b\big] = i\,\theta^{ab}$$

- θ^{ab} not constant
- Y^a interpreted as quantization of coordinate functions y^a on Poisson manifold $\big(\mathcal{M}, \theta^{ab}(y)\big)$ with Poisson structure $\theta^{ab}(y)$
- This implies in the semi classical limit

$$[Y^a, \Psi] \sim i\,\theta^{ab}(y)\frac{\partial}{\partial y^b}\Psi$$

Effective Metric

$$\begin{aligned} S[\Phi] &= -\mathrm{Tr}\,[Y^a,\Phi]\left[Y^b,\Phi\right] g_{ab} \\ &\sim \int \mathrm{d}^4y\,\rho(y) G^{ab}(y)\partial_a\Phi(y)\partial_b\Phi(y) \end{aligned}$$

where

$$G^{mn}(y) = \theta^{ma}(y)\theta^{nb}(y)g_{ab}$$

- Effective metric determined by Poisson sturcture θ^{mn}
- Φ couples to effective metric
- $\rho(y) = |G_{ab}|^{1/4}$
- $\theta^{mn} \longrightarrow$ vielbein

Fermions

The Action

$$S = (2\pi)^2 \operatorname{Tr} \bar{\Psi}\gamma_a [Y^a, \Psi]$$
$$\sim \int \mathrm{d}^4 y \rho(y) \bar{\Psi} i \gamma_a \theta^{ab}(y) \partial_b \Psi$$

The Dirac operator

$$\not{D}\Psi = \gamma_a [Y^a, \Psi] \sim i\gamma_a \theta^{ab}(y) \partial_b \Psi$$

⋆ no spin connection appears ⋆

Compare to standard covariant derivative

$$\not{D}_{\text{comm}} \Psi = i\gamma^a e_a^\mu \left(\partial_\mu + \Sigma_{ab}\, \omega_\mu^{ab}\right) \Psi$$

Spin connection?

Yes, this is a strange feature

- Rotation of spin will be different in this model
- Holonomies will be different than in GR

Nevertheless ...

... trajectory of fermion will follow geodesics
... Einstein-Hilbert action will be induced

<u>This is a reasonable action for fermions</u>

... Opens possiblity for experimental signature?

Emergent Gravity

How will the Einstein-Hilbert action join our game?
No need of adding further terms to the action.

S_{EH} emerges automatically upon quantization.
One-loop effective action Γ_Ψ will correspond to EH-action S_{EH}.

$$e^{-\Gamma_\Psi} = \int d\Psi d\bar{\Psi} e^{-S[\Psi,\bar{\Psi}]}$$
$$\Gamma_\Psi = -\frac{1}{2}\mathrm{Tr}\log \not{D}^2$$

$\not{D}^2$ defines quadratic form

$$S_{\mathrm{square}} = \left(2\pi^2\right)\mathrm{Tr}\,\bar{\Psi}\not{D}^2\Psi = \int d^4y\rho(y)\bar{\Psi}\not{D}^2\Psi$$

Emergent Gravity

Unimodular metric

$$\widetilde{G}_{ab} := e^{\sigma} G_{ab} \qquad e^{\sigma} = (\det G_{ab})^{-1/4} \quad \rightarrow \quad \det \widetilde{G} = 1$$

$$S_{\text{square}} = \int d^4y \sqrt{\widetilde{G}}\, \bar{\Psi} \widetilde{\not{D}}^2 \Psi$$

$$\widetilde{\not{D}}^2 \Psi = -\left(\widetilde{G}^{ab} \partial_a \partial_b \Psi + e^{-\sigma} \gamma_a \gamma_b \theta^{ma} \left(\partial_m \theta^{db} \right) \partial_d \Psi \right)$$
$$:= -\left(\widetilde{G}^{ab} \partial_a \partial_b \Psi + \widetilde{a}^d \partial_d \Psi \right)$$

Seeley-de Witt coefficients for Fermions

$$\frac{1}{2}\mathrm{Tr}\left(\log \widetilde{\not{D}}^2 - \log \widetilde{\not{D}}_0^2\right) = -\frac{1}{2}\mathrm{Tr}\int_0^\infty \frac{d\alpha}{\alpha}\left(e^{-\alpha \widetilde{\not{D}}^2} - e^{-\alpha \widetilde{\not{D}}_0^2}\right) e^{-\frac{1}{2\alpha \widetilde{\Lambda}^2}}$$

Heat kernel expansion

$$\mathrm{Tr} e^{-\alpha \widetilde{\not{D}}^2} = \sum_{n\geq 0} \alpha^{\frac{n-4}{2}} \int_{\mathcal{M}} d^4y\, a_n\left(y, \widetilde{\not{D}}^2\right)$$

Seeley-de Witt coefficients

$$a_0(y) = \frac{1}{16\pi^2}\,\mathrm{tr}\,\mathbb{I}$$

$$a_2(y) = \frac{1}{16\pi^2}\,\mathrm{tr}\left(\frac{R[\widetilde{G}]}{6}\mathbb{I} + \mathcal{E}\right)$$

Effective Action for Fermions

$$\Gamma_{\Psi} = \frac{1}{16\,\pi^2} \int d^4y \left(2\,\mathrm{tr}(\mathbb{I})\,\widetilde{\Lambda}^4 + \mathrm{tr}\left(\frac{R[\widetilde{G}]}{6}\,\mathbb{I} + \mathcal{E} \right) \widetilde{\Lambda}^2 + O(\log \widetilde{\Lambda}) \right)$$

Commutative case:

$$\mathrm{tr}\mathcal{E}_{\mathrm{comm}} = -R$$

This model: no spin connection $\rightarrow$ $\mathcal{E}$ will be modified

Still a reasonable Einstein-Hilbert action?

yes

Induced EH-Action?

$$\mathcal{E} = -\widetilde{G}^{mn}\left(\partial_m \Omega_n + \Omega_m \Omega_n - \widetilde{\Gamma}^k_{mn}\Omega_k\right)$$

$$\Omega_m = \frac{1}{2}\widetilde{G}_{mn}\left(\widetilde{a}^n + \widetilde{\Gamma}^n\right) \qquad \widetilde{a}^n = e^{-\sigma}\gamma_a\gamma_b\theta^{ma}\left(\partial_m\theta^{nb}\right)$$

$$\boxed{\int d^4y\, \mathrm{tr}\mathcal{E} = -\int d^4y\, (2R[\widetilde{G}] - G^{mn}\,(\partial_m\sigma)\,(\partial_n\sigma))}$$

for on-shell geometries $\leftrightarrow$ fulfill eom
Obtain EH action with an unsual numerical constant
+ dilaton-like term

R & $\mathrm{tr}\mathcal{E}$

Use Jacobi identity

$$\partial_a \theta^{-1}_{bc} + \partial_c \theta^{-1}_{ab} + \partial_b \theta^{-1}_{ca} = 0$$

$$\begin{aligned} R[\widetilde{G}] = e^{-\sigma} \Big[& -\frac{1}{2} \theta^{mn} G^{pq} \partial_p \partial_q \theta^{-1}_{mn} - \frac{1}{2} G^{pq} \left(\partial_p \theta^{mn}\right) \left(\partial_q \theta^{-1}_{mn}\right) \\ & - \left(\partial_m \theta^{ma}\right) G^{nk} \left(\partial_k \theta^{-1}_{na}\right) + \frac{1}{2} G^{mn} \left(\partial_m \sigma\right) \left(\partial_n \sigma\right) \\ & + \frac{1}{2} \left(\partial^m \theta^{-1}_{na}\right) \left(\partial^n \theta^{-1}_{mb}\right) g^{ab} - \frac{1}{2} G^{mn} \left(\partial^q \theta^{-1}_{ma}\right) \left(\partial_q \theta^{-1}_{nb}\right) g^{ab} \\ & - \frac{1}{2} \left(\partial_m \theta^{na}\right) \left(\partial_n \theta^{mb}\right) g_{ab} \Big] \end{aligned}$$

$$\mathrm{tr}\mathcal{E} = e^{-\sigma} \left[G^{mn} \left(\partial^l \theta^{-1}_{ma}\right) \left(\partial_l \theta^{-1}_{nb}\right) g^{ab} - \left(\partial^m \theta^{-1}_{na}\right) \left(\partial^n \theta^{-1}_{mb}\right) g^{ab} \right]$$

R & tr$\mathcal{E}$

Partial integration reduces Ricci scalar to

$$\int d^4y R[\widetilde{G}]\widetilde{\Lambda}^2 = e^{-\sigma}\times$$

$$\Big[\frac{1}{2}\left(\partial^m\theta^{-1}_{na}\right)\left(\partial^n\theta^{-1}_{mb}\right)g^{ab} - \frac{1}{2}G^{mn}\left(\partial^p\theta^{-1}_{ma}\right)\left(\partial_p\theta^{-1}_{nb}\right)g^{ab}$$

$$-\frac{1}{2}\left(\partial_p\theta^{pa}\right)G^{qk}\left(\partial_k\theta^{-1}\right) + \frac{1}{2}G^{mn}\left(\partial_m\sigma\right)\left(\partial_n\sigma\right)\Big]\widetilde{\Lambda}^2$$

SUSY

Cancellation of bosonic and fermionic contributions

$$\Gamma_\Psi + 4\Gamma_\Phi = \frac{1}{16\pi^2}\int d^4y \,\mathrm{tr}\,\mathcal{E}\,\widetilde{\Lambda}^2$$

- Induced gravitational action is nontrivial in the case of e.g. $N = 1$ supersymmetry
- UV/IR mixing remains in supersymmetric models
- Full cancellation only for $N = 4$ supersymmetry

Relation with gauge theory ...

$$S = (2\pi)^2 \,\mathrm{Tr}\, \bar{\Psi}\gamma_a \left[Y^a, \Psi\right]$$

regarded as action for fermions on $\mathbb{R}^4_\theta$ coupled to $U(1)$ gauge field via

covariant coordiantes

$$Y^a = X^a - \bar{\theta}^{ab} A_b(x)$$

$A_b(x)$ hermitian matrices, smooth functions on $\mathbb{R}^4_\theta$

$$\left[X^a, X^b\right] = i\,\bar{\theta}^{ab}$$

$\bar{\theta}^{ab}$ constant $\qquad \bar{g}^{ab} = \bar{\theta}^{am}\bar{\theta}^{bn} g_{mn} \qquad \bar{\rho} = |\bar{g}_{ab}|^{1/4}$

UV/IR mixing

The “conventional” gauge theory point of view.

Write the action

$$S = (2\pi)^2 \mathrm{Tr}\, \bar{\Psi}\gamma_a \left[Y^a, \Psi\right]$$

as NC $U(1)$ gauge theory...

$$S = \int d^4x\, \bar{\Psi} i\gamma^a \left(\bar{\partial}_a \Psi + ig\left[A_a, \Psi\right]\right)$$

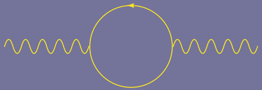
Consider again the one-loop effective action

$\longrightarrow$ Use fermionic Feynman rules

UV/IR mixing

$$\Gamma_\Psi = -\frac{1}{2}\mathrm{Tr}\log\Delta_0 - \frac{g^2}{2}\langle \int d^4x\, \bar\rho\bar{\bar\Psi}\,\gamma^a\,[A_a,\Psi] \int d^4y\, \bar\rho\bar{\bar\Psi}\,\gamma^b\,[A_b,\Psi]\rangle$$

... corresponds to the Feynman diagram



$$\Gamma_\Psi = -4\Gamma_\Phi - \int d^4x\, \bar\rho\, \bar g^{ac}\bar g^{bd}\, \bar F_{ab}\bar\partial^2\, \bar F_{cd}\frac{\Lambda^2}{2}$$

Rewrite geometric Γ_Ψ

We had

$$\Gamma_\Psi = \frac{1}{16\,\pi^2}\int d^4y\left(2\,\mathrm{tr}(\mathbb{I})\,\widetilde{\Lambda}^4 + \mathrm{tr}\left(\frac{R[\widetilde{G}]}{6}\,\mathbb{I} + \mathcal{E}\right)\widetilde{\Lambda}^2 + O(\log\widetilde{\Lambda})\right)$$
$$= -4\,\Gamma_\Phi + \frac{1}{16\pi^2}\int d^4y\,\mathrm{tr}\mathcal{E}\,\widetilde{\Lambda}^2$$

compare this to the result from gauge theory point of view

rewrite Γ_Ψ in x-coordiantes of Moyal-Weyl plane

UV/IR mixing ...
...an effect of gravity

Relation between *y* and *x* coordinates

$$y^a = x^a - \bar{\theta}^{ab}\bar{A}_b + O(\theta^2)$$

Relation between

$$i\,\theta^{ab}(y) = \left[Y^a, Y^b\right] = i\bar{\theta}^{ab} - i\bar{\theta}^{ac}\bar{\theta}^{bd}\bar{F}_{cd}$$
$$\theta^{-1}_{ab}(y) = \bar{\theta}^{-1}_{ab} - \bar{F}_{ab}$$

Obtain effective action in *x*-coordinates

$$\Gamma_\Psi = -4\Gamma_\Phi - \int d^4x\,\bar{\rho}\,\bar{g}^{ac}\bar{g}^{bd}\,\bar{F}_{ab}\bar{\partial}^2\bar{F}_{cd}\frac{\Lambda^2}{2}$$

<u>UV/IR mixing is understood as an effect of gravity.</u>

Conclusions

Framework of NC emergent gravity extends to fermions

Fermions couple to background geometry

Spin connection appears to be missing

Fermions will follow standard trajectories
with different spin rotation

Einstein-Hilbert action is induced

UV/IR mixing is an effect of gravity

Bosonic & fermionic contributions do not cancel

... only for $N = 4$ SUSY