Vacuum Configurations in the Frame Formalism

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Outline

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Frame formalism

$$[x^\mu, x^\nu] = i\kbar J^{\mu\nu}(x^\sigma).$$

Let μ be a typical 'large' source mass with 'Schwarzschild radius' $G_N\mu$.

Square $G_N\hbar$ of the Planck length and by $\kbar$.

Weak field: $\epsilon_{GF} = G_N\hbar\mu^2$ small

Almost commutative: $\epsilon_{NC} = \kbar\mu^2$ small

We assume

$$\epsilon \ll, \qquad \epsilon = \epsilon_{GF}/\epsilon_{GF}.$$

Ordinary gravity: limit $\epsilon \to 0$

Noncommutative gravity: $\epsilon \simeq 1$

Frame: the momenta p_α stand in duality to the position operators x^μ by the relation

$$[p_\alpha, x^\mu] = e^\mu_\alpha$$

The right-hand side of this identity defines the gravitational field. The left-hand side must obey Jacobi identities.

These identities yield relations between quantum mechanics in the given curved space-time and the noncommutative structure of the algebra.

The three aspects of reality then, the curvature of space-time, quantum mechanics and the noncommutative structure of space-time are intimately connected.

Poisson energy

We shall consider here the even more exotic possibility that the field equations of general relativity are encoded also in the structure of the algebra so that the relation between general relativity and quantum mechanics can be understood by the relation which each of these theories has with noncommutative geometry.

In spite of the rather lengthy formalism the basic idea is simple.

1) classical geometry with a moving frame $\tilde{\theta}^\alpha$

2) 'quantize' by $\tilde{\theta}^\alpha \mapsto \theta^\alpha$

3) 'quantize' by imposing the rule

$$\tilde{e}_\alpha \mapsto e_\alpha = \mathrm{ad}\, p_\alpha$$

4) chose the algebra so that

$$i\hbar[p_\alpha, J^{\mu\nu}] = [x^{[\mu}, [p_\alpha, x^{\nu]}]]$$

All together: $\tilde{g}_{\mu\nu} \mapsto J^{\mu\nu}$

We shall work exclusively in the 'quasi-classical' approximation,

One problem: to understand 'Poisson energy' (Fishy physics)

Another problem: to what extend is the Einstein tensor determined as integrability conditions for the underlying associative-algebra structure.

The value *in vacuo* of the commutative limit of this tensor we interpret as the Poisson energy

In the frame formalism the density of energy-momentum of the gravitational field is a vector-valued 3-form τ_S

The total energy-momentum is given as the integral over the sphere at infinity of the Sparling 2-form

$$\sigma_\alpha = -\tfrac{1}{2}\omega^*_{\alpha\beta}\theta^\beta.$$

In the noncommutative case there is a preferred frame.

Our assumption is that there is another 3-form

$$\tau_{PS} = \tau_{PS}(J)$$

which vanishes in the commutative limit and which is such that the sum

$$\tau = \tau_S + \tau_{PS}$$

is an exact 3-form.

We cannot give an explicit formula for τ_{PS}

Differential calculi

Assume that over $\mathcal{A}$ is a differential calculus which is such that the module of 1-forms is free and possesses a preferred frame θ^α which commutes,

$$[x^\mu, \theta^\alpha] = 0,$$

with the algebra.

We can write the differential

$$dx^\mu = e^\mu_\alpha \theta^\alpha, \qquad e^\mu_\alpha = e_\alpha x^\mu.$$

The differential calculus is defined as the largest one consistent with the module structure of the 1-forms so constructed.

The input of which we shall make the most use is the Leibniz rule

$$i\hbar e_\alpha J^{\mu\nu} = [e^\mu_\alpha, x^\nu] - [e^\nu_\alpha, x^\mu].$$

One can see here a differential equation for $J^{\mu\nu}$ in terms of e^μ_α.

If the matrix J is invertible then the classical Darboux theorem states that coordinates can be chosen so that the components of J are constants. We deduce in this case that

$$X_\alpha{}^{\mu\nu} = [e^\mu_\alpha, x^\nu]$$

is symmetric in the two indices. Under the change of coordinates, in fact also the momenta change; the quantities e^μ_α become constant and $X \to 0$.

Finally, we must insure that the differential is well defined. A necessary condition is that $d[x^\mu, \theta^\alpha] = 0$ from which it follows that the momenta p_α must satisfy the consistency condition

$$2p_\gamma p_\delta P^{\gamma\delta}{}_{\alpha\beta} - p_\gamma F^\gamma{}_{\alpha\beta} - K_{\alpha\beta} = 0.$$

The $P^{\gamma\delta}{}_{\alpha\beta}$ define the product π in the algebra of forms.

We write $P^{\alpha\beta}{}_{\gamma\delta}$ in the form

$$P^{\alpha\beta}{}_{\gamma\delta} = \tfrac{1}{2}\delta^{[\alpha}_\gamma \delta^{\beta]}_\delta + i\epsilon Q^{\alpha\beta}{}_{\gamma\delta}$$

of a standard projector plus a perturbation.

The compatibility condition

$$(P^{\alpha\beta}{}_{\zeta\eta})^* P^{\eta\zeta}{}_{\gamma\delta} = P^{\beta\alpha}{}_{\gamma\delta}$$

with the product is satisfied provided $Q^{\alpha\beta}{}_{\gamma\delta}$ is real.

It follows that

$$\begin{aligned} d[x^\mu, \theta^\alpha] &= [dx^\mu, \theta^\alpha] + [x^\mu, d\theta^\alpha] \\ &= e^\mu_\beta [\theta^\beta, \theta^\alpha] - \tfrac{1}{2}[x^\mu, C^\alpha{}_{\beta\gamma}]\theta^\beta\theta^\gamma. \end{aligned}$$

We find then that multiplication of 1-forms must satisfy

$$[\theta^\alpha, \theta^\beta] = \tfrac{1}{2}\theta^\beta_\mu [x^\mu, C^\alpha{}_{\gamma\delta}]\theta^\gamma\theta^\delta.$$

Consistency requires then that

$$\theta^{[\beta}_\mu [x^\mu, C^{\alpha]}{}_{\gamma\delta}] = 0.$$

Because of the condition (6) consistency also requires that

$$\theta^{(\alpha}_\mu [x^\mu, C^{\beta)}{}_{\gamma\delta}] = Q_-^{\alpha\beta}{}_{\gamma\delta}.$$

Consistency relations:

1) the Leibniz rule

2) the Jacobi identity

3) the two conditions on the differential

The Ricci rotation coefficients can be expressed as

$$C^{\alpha}{}_{\beta\gamma} = -4i\epsilon p_{\delta} Q_{-}^{\alpha\delta}{}_{\beta\gamma}$$

They must satisfy the gauge condition

$$e_{\alpha} C^{\alpha}{}_{\beta\gamma} = 0$$

We shall refer to all these conditions as the Jacobi conditions.

Metrics and connections

The metric is a map

$$g:\ \Omega^1(\mathcal{A}) \otimes \Omega^1(\mathcal{A}) \to \mathcal{A}.$$

Using the frame it is defined by

$$g(\theta^\alpha \otimes \theta^\beta) = g^{\alpha\beta},$$

and bilinearity of the metric implies that $g^{\alpha\beta}$ are complex numbers. To define the reality and symmetry of the metric as well as the bimodule structure of the linear connection one needs a 'flip',

$$\sigma(\theta^\alpha \otimes \theta^\beta) = S^{\alpha\beta}{}_{\gamma\delta}\theta^\gamma \otimes \theta^\delta,$$

which in the present notation is equivalent to a 4-index set of complex numbers $S^{\alpha\beta}{}_{\gamma\delta}$ which we can write as

$$S^{\alpha\beta}{}_{\gamma\delta} = \delta^\beta_\gamma \delta^\alpha_\delta + i\epsilon T^{\alpha\beta}{}_{\gamma\delta}.$$

In the present formalism the metric is ‘real’ if it satisfies the condition

$$\bar{g}^{\beta\alpha} = S^{\alpha\beta}{}_{\gamma\delta} g^{\gamma\delta}.$$

‘Symmetry’ of the metric can be defined either using the projection

$$P^{\alpha\beta}{}_{\gamma\delta} g^{\gamma\delta} = 0,$$

or the flip

$$S^{\alpha\beta}{}_{\gamma\delta} g^{\gamma\delta} = c g^{\alpha\beta}.$$

We choose the frame to be orthonormal in the commutative limit; we can write therefore

$$g^{\alpha\beta} = \eta^{\alpha\beta} - i\epsilon h^{\alpha\beta}.$$

In the linear approximation, the condition of the reality of the metric becomes

$$h^{\alpha\beta} + \bar{h}^{\alpha\beta} = T^{\beta\alpha}{}_{\gamma\delta} \eta^{\gamma\delta}.$$

We introduce also

$$g^{\mu\nu} = g(dx^\mu \otimes dx^\nu) = e^\mu_\alpha e^\nu_\beta g^{\alpha\beta}.$$

The covariant derivative is given by

$$D\xi = \sigma(\xi \otimes \theta) - \theta \otimes \xi.$$

In particular

$$\begin{aligned} D\theta^\alpha &= -\omega^\alpha{}_\gamma \otimes \theta^\gamma \\ &= -(S^{\alpha\beta}{}_{\gamma\delta} - \delta^\beta_\gamma \delta^\alpha_\delta) p_\beta \theta^\gamma \otimes \theta^\delta \\ &= -i\epsilon T^{\alpha\beta}{}_{\gamma\delta} p_\beta \theta^\gamma \otimes \theta^\delta, \end{aligned}$$

so the connection-form coefficients are linear in the momenta

$$\omega^\alpha{}_\gamma = \omega^\alpha{}_{\beta\gamma} \theta^\beta = i\epsilon p_\delta T^{\alpha\delta}{}_{\beta\gamma} \theta^\beta.$$

The $\omega^\alpha{}_\gamma$ measures the variation of the metric; the array $T^{\alpha\delta}{}_{\beta\gamma}$ is directly related to the anti-commutation rules for the 1-forms, and to the momenta p_δ. As $\bar{k} \to 0$ the right-hand side remains finite and

$$\omega^\alpha{}_\gamma \to \tilde{\omega}^\alpha{}_\gamma.$$

The connection is torsion-free if the

components satisfy the constraint

$$\omega^{\alpha}{}_{\eta\delta} P^{\eta\delta}{}_{\beta\gamma} = \tfrac{1}{2} C^{\alpha}{}_{\beta\gamma}.$$

The connection is metric if

$$\omega^{\alpha}{}_{\beta\gamma} g^{\gamma\delta} + \omega^{\delta}{}_{\gamma\eta} S^{\alpha\gamma}{}_{\beta\zeta} g^{\zeta\eta} = 0,$$

or linearized,

$$T^{(\alpha\gamma}{}_{\delta}{}^{\beta)} = 0.$$

The quasi-classical approximation

Direct connection between the rotation coefficients and the commutators $J^{\mu\nu}$:

$$[p_\alpha, p_\beta] = -2i\epsilon Q_{-}^{\gamma\delta}{}_{\alpha\beta} p_\gamma p_\delta - K_{\alpha\beta}.$$

The Jacobi anomaly is given by

$$\begin{aligned}
A^{\delta} &= \epsilon^{\alpha\beta\gamma\delta} [[p_\alpha, p_\beta], p_\gamma] \\
&= -2i\epsilon\epsilon^{\alpha\beta\gamma\delta} Q_{-}^{\eta\zeta}{}_{\alpha\beta} [p_\eta p_\zeta, p_\gamma] \\
&= -2i\epsilon\epsilon^{\alpha\beta\gamma\delta} Q_{-}^{\eta\zeta}{}_{\alpha\beta} (p_\eta [p_\zeta, p_\gamma] + [p_\eta, p_\gamma] p_\zeta) \\
&= -2i\epsilon\epsilon^{\alpha\beta\gamma\delta} Q_{-}^{\eta\zeta}{}_{\alpha\beta} (p_\eta [p_\zeta, p_\gamma] + [p_\zeta, p_\gamma] p_\eta)
\end{aligned}$$

$$= (2i\epsilon)^2 \epsilon^{\alpha\beta\gamma\delta} (Q_-^{\eta\zeta}{}_{\alpha\beta} Q_-^{ij}{}_{\zeta\gamma} p_\eta p_i p_j$$

$$+ Q_-^{\eta\zeta}{}_{\alpha\beta} Q_-^{ij}{}_{\zeta\gamma} p_i p_j p_\eta)$$

$$- 2(2i\epsilon)^2 \epsilon^{\alpha\beta\gamma\delta} Q_-^{\eta\zeta}{}_{\alpha\beta} K_{\zeta\gamma} p_\eta$$

If we introduce the left dual

$$Q_-^{\eta\zeta*\gamma\delta} = \frac{1}{2} \epsilon^{\alpha\beta\gamma\delta} Q_-^{\eta\zeta}{}_{\alpha\beta}$$

we can write the anomaly as

$$A^\delta = 2(2i\epsilon)^2 Q_-^{\eta\zeta*\gamma\delta} (Q_-^{ij}{}_{\zeta\gamma} p_\eta p_i p_j + Q_-^{ij}{}_{\zeta\gamma} p_i p_j p_\eta - K_{\zeta\gamma} p_\eta)$$

It must be set to zero case by case.

One can express the commutator of an arbitrary function f with x^λ as a derivative:

$$[x^\lambda, f] = i\hbar J^{\lambda\sigma} \partial_\sigma f (1 + o(\epsilon)).$$

Then the Leibniz rule and the Jacobi identity can be written in leading order as

$$e_\alpha J^{\mu\nu} = \partial_\sigma e_\alpha^{[\mu} J^{\sigma\nu]}$$

$$\epsilon_{\kappa\lambda\mu\nu} J^{\gamma\lambda} e_\gamma J^{\mu\nu} = 0. -$$

Written in terms of the frame components, these two Jacobi equations become

$$e_\gamma J^{\alpha\beta} - C^{[\alpha}{}_{\gamma\delta} J^{\beta]\delta} = 0,$$

$$\epsilon_{\alpha\beta\gamma\delta} J^{\alpha\epsilon} e_\epsilon J^{\beta\gamma} = 0.$$

In terms of the dual quantities

$$J^*_{\alpha\beta} = \tfrac{1}{2}\epsilon_{\alpha\beta\gamma\delta} J^{\gamma\delta}$$

we have

$$e_\alpha J^*_{\beta\gamma} + C^\delta{}_{\alpha[\beta} J^*_{\gamma]\delta} + C^\delta{}_{\alpha\delta} J^*_{\beta\gamma} = 0.$$

Equation (0.1) can be written also in terms of the $J^*_{\alpha\beta}$ as

$$C^\alpha{}_{[\alpha\gamma} J^*_{\beta]\delta} J^{\delta\gamma} = 0.$$

Similar to Equation (0.1) one can derive the identity

$$e_\alpha J^{-1}_{\beta\gamma} = J^{-1}_{\alpha\delta} C^\delta{}_{\beta\gamma} 2577$$

for the derivative of the inverse if it exists. The two are consistent because of the cocycle condition.

One can solve this for the rotation coefficients. One obtains

$$C^\alpha{}_{\beta\gamma} = J^{\alpha\eta} e_\eta J^{-1}_{\beta\gamma}.$$

This can be rewritten as

$$C^\alpha{}_{\beta\gamma} = J^{\alpha\delta} e_\delta J^{\zeta\eta} J^{-1}_{\zeta\beta} J^{-1}_{\eta\gamma}$$

and also, using (0.1) as

$$C^\alpha{}_{\beta\gamma} = J^{\alpha\delta}(C^\zeta{}_{\epsilon\delta} J^{\epsilon\eta} - C^\eta{}_{\epsilon\delta} J^{\epsilon\zeta}) J^{-1}_{\zeta\beta} J^{-1}_{\eta\gamma}.$$

It follows that in the quasi-classical approximation, the linear connection and the curvature can be directly expressed in terms of the commutation relations. In particular if the latter are constants then the curvature vanishes.

From (**??**) it follows that

$$dJ^{-1} = 0.$$

We conclude then that $J^{-1} = dg$ for some 1-form g. We write this as

$$J^{-1}_{\alpha\beta} = \frac{1}{2}[p_{[\alpha}, g_{\beta]}.$$

We are now in position to discuss the 'ground-state configurations'

$$g_\alpha = i\hbar(1+h)p_\alpha.$$

For these

$$J^{-1}_{\alpha\beta} = \frac{1}{2}i\hbar(1+h)[p_{[\alpha}, p_{\beta]} + \frac{1}{2}i\hbar[p_{[\alpha}, h], \partial_{\beta]}.$$

As usual one can introduce the Dirac operator $\theta = -p_\alpha\theta^\alpha$. It satisfies

$$\theta^2 = \tfrac{1}{2}[p_\alpha, p_\beta]\theta^\alpha\theta^\beta$$

From this follows

$$d\theta + \theta^2 = -\tfrac{1}{2}K_{\alpha\beta}\theta^\alpha\theta^\beta$$

If we consider J^{-1} as a Maxwell field strength then there is a source given by

$$e^\alpha J^{-1}_{\alpha\beta} = J^{-1\,\alpha\gamma}C_{\alpha\beta\gamma}.$$

It follows that the commutator must necessarily satisfy the constraint

$$e_\alpha \left(J^{\alpha\eta} e_\eta J^{-1}_{\beta\gamma} \right) = 0.$$

This can also be written as

$$\left(e_\alpha J^{\alpha\zeta} + J^{\alpha\eta} C^\zeta{}_{\alpha\eta} \right) e_\zeta J^{-1}_{\beta\gamma} = 0.$$

We have assumed that the noncommutativity is small and we have derived some relations to first-order in the parameter ϵ. We shall now make an analogous assumption concerning the gravitational field; we shall assume that ϵ_{GF} is also small and that we consider only the equations to first-order in it as well. With these two assumptions the equations become relatively easy to solve.

If we equate the Expression (19) for the rotation coefficients with that in terms of the components of the frame we find after a few simple applications of the Leibniz rule that

$$(\delta J^{-1})_{\alpha\beta\gamma} = e^\mu_{[\beta} e_{\gamma]} J^{-1}_{\alpha\mu.}$$

The cocyle condition (19) is equivalent to the condition

$$e^{\mu}_{[\beta}e_{\gamma]}J^{-1}_{\alpha\mu.}=0.$$

An interesting particular solution is given by constants:

$$J^{-1}_{\alpha\mu.}=J^{-1}_{0\alpha\mu.}.$$

It follows then that

$$J^{\mu\nu}=J^{\mu\alpha}_{0}e^{\nu}_{\alpha},\qquad J^{(\mu\alpha}_{0}e^{\nu)}_{\alpha}=0.$$

One verifies that

$$C^{\alpha}{}_{\alpha\gamma}=J^{\alpha\eta}e_{\eta}J^{-1}_{\alpha\gamma}=e_{\eta}J^{\alpha\eta}J^{-1}_{\alpha\gamma}$$

and so the left-hand side vanishes if and only if

$$e_{\beta}J^{\alpha\beta}=0.$$

Finally we note that using (11) we have the very strong relation

$$J^{\alpha\eta}e_{\eta}J^{-1}_{\beta\gamma}+4i\epsilon p_{\delta}Q^{\alpha\delta}_{-}{}_{\beta\gamma}=0$$

This can be written as

$$e_{\eta}J^{-1}_{\beta\gamma}+4i\epsilon J^{-1}_{\eta\alpha}p_{\delta}Q^{\alpha\delta}_{-}{}_{\beta\gamma}=0$$

To lowest order we can assume that J^{-1} and p commute, this because

$$[p_\delta, J^{-1}_{\eta\alpha}] = e_\delta J^{-1}_{\beta\gamma}$$

and, referring back to the equation we see that this quantity is of first order.

From the definitions it follows that

$$dJ^{-1}_{\beta\gamma} - 4i\epsilon dg_\alpha p_\delta Q_-^{\alpha\delta}{}_{\beta\gamma} = 0$$

If perchance $dp = 0$ then we can solve

$$J^{-1}_{\beta\gamma} = J^{-1}_{0\beta\gamma} - 4i\epsilon g_\alpha p_\delta Q_-^{\alpha\delta}{}_{\beta\gamma}$$

Ground-state examples

By the expression 'ground state' we mean a convenient solution about which we can apply the perturbation procedure and analyse the structure of the solutions in a neighborhood. In the examples this coincides with what one would normally call a ground state.

We set as usual

$$[x^\mu, x^\nu] = i\bar{k} J^{\mu\nu}.$$

The commutator will be of course restricted by Jacobi identities. To calculate its frame components we shall need the transformations θ^α_μ defined by $\theta^\alpha = \theta^\alpha_\mu d\xi^\mu$ with

$$\theta^\alpha_\mu = \left(\begin{array}{cc} \bar{\theta}^0_0 & \bar{\theta}^0_m \\ \\ \bar{\theta}^a_0 & \bar{\theta}^a{}_m \end{array} \right).$$

The frame components of the commutator are given by

$$J^{\alpha\beta} = J^{\mu\nu}\theta^\alpha_\mu\theta^\beta_\nu$$

Written as an equality between matrices this becomes

$$J^{\alpha\beta} = J^{\mu\nu}\theta^\alpha_\mu\theta^\beta_\nu = (\theta J \theta^T)^{\alpha\beta}$$

One finds that if one set $J^{ab} = \epsilon^{abc}\Xi_c$

$$J^{0a} = \bar{\theta}^a_0\bar{\theta}^0_j\bar{\theta}^0_0\bar{\theta}^a{}_j J^{0j} + \bar{\theta}^a{}_i\bar{\theta}^0_j J^{ij}$$

$$J^{ab} = \bar{\theta}^{[a}\bar{\theta}^{b]}{}_j J^{0j} + \bar{\theta}^a{}_j J^{ij}\bar{\theta}_j{}^b$$

$$\Xi_c = \epsilon_{abc}\bar\theta^a\bar\theta^b{}_i J^{0i} + \epsilon_{abc}\bar\theta^a{}_i\epsilon^{ijk}\Xi_k\bar\theta_j{}^b$$

We shall need in each case the inverse of this matrix.

Regular lattice structures

The simplest case is with $J^{\mu\nu} = J_0^{\mu\nu}$ a matrix of constants. This case wa treated in Section so we shall only present a variant.There is a special case of particular interest, that in which the mixed components $J_0^{\mu\alpha}$ are constant and the matrix they form is invertible. We write then

$$x^\mu = J_0^{\mu\alpha} D_\alpha, \qquad D_\alpha = p_\alpha + \mathcal{A}_\alpha.$$

The interest in this decomposition resides in the properties of the 1-forms $\mathcal{A} = \mathcal{A}_\alpha\theta^\alpha$ and $\theta = -p_\alpha\theta^\alpha$ considered as gauge potentials. Let $\mathcal{U} \subset \mathcal{A}$ be the group of unitary elements of the algebra and define for arbitrary $\mathcal{A}$ and $g \in \mathcal{U}$

$$\mathcal{A}' = g^{-1}\mathcal{A}g + g^{-1}dg.$$

Since

$$dg = e_\alpha g \theta^\alpha = -[\theta, g]$$

in the particular case with $\mathcal{A} = \theta$ we have $\theta' = \theta$. We conclude that, being the difference between two gauge potentials, the generators x^μ transform as adjoint representations of $\mathcal{U}$:

$$x'^\mu = g^{-1} x^\mu g.$$

Plane symmetric ground state

The Kasner frame

There is a natural distinguished vector $\xi^a = (0, 0, z)$. We set as usual

$$[x^\mu, x^\nu] = i\bar{k} J^{\mu\nu}.$$

The generic form for the commutator is

$$J^{0i} = f\xi^i, \qquad J^{ij} = z\epsilon^{ijk}\xi_k, \qquad [t, r] = i\bar{k} z f$$

with $\xi^2 = z^2$. From symmetry arguments the commutator must be of the form

$$J^{0a} = f\xi^a, \qquad J^{ab} = g\epsilon^{abc}\xi_c.$$

The inverse is of the form

$$J^{-1}_{0a} = -r^{-2} f^{-1} \xi_a, \qquad J^{-1}_{ab} = -r^{-2} g^{-1} \epsilon_{abc} \xi^c$$

We try for the momenta the Ansatz $p_0 = \rho$, $p_a = \sigma \xi_a$.

To calculate the frame components of the commutator we shall need the transformations θ^α_μ defined by $\theta^\alpha = \theta^\alpha_\mu d\xi^\mu$ with

$$\theta^\alpha_\mu = \begin{pmatrix} \bar{B} & \bar{B}\xi_m \\ \\ \bar{C}\xi^a & \bar{F}\delta^a_m + \bar{D}\xi^a \xi_m + \bar{E}\epsilon^a{}_{mn}\xi^n \end{pmatrix}.$$

To define a concrete Ansatz for the components on the left-hand side of this equation we have introduced on the right-hand side the two sets $\xi^i = \xi_i = (0, 0, z)$ and $\xi^a = \xi_a = (0, 0, z)$.

We shall need also the frame components $e_\alpha = e^\mu_\alpha \partial_\mu$ with

$$e^\mu_\alpha = \begin{pmatrix} A & B\xi_a \\ \\ C\xi^m & D\xi^m \xi_a + E\epsilon^m{}_{ab}\xi^b \end{pmatrix}.$$

The condition that e^μ_α be the inverse of θ^α_μ implies that

$$\bar{A} = D\Delta^{-1} \qquad \bar{B} = -Br^{-2}\Delta^{-1} \quad \bar{C} = -Cr^{-2}\Delta^{-1}$$

$$\bar{D} = Ar^{-4}\Delta^{-1} \quad \bar{E} = -r^{-2}E^{-1} \qquad \bar{F} = 0$$

The simplest anisotropic homogeneous solution to Einstein equations is the Kasner metric:

$$ds^2 = -dt^2 + t^{2q_1}(dx^1)^2 + t^{2q_2}(dx^2)^2 + t^{2q_3}(dx^3)^2.$$

The vacuum equations with vanishing cosmological constant impose the constraints on the parameters q_i

$$q_1 + q_2 + q_3 = 1, \qquad q_1^2 + q_2^2 + q_3^2 = 1.$$

The metric (23) is a member of a 1-parameter family of solutions.

The moving frame is given by

$$\theta^0 = dt, \qquad \theta^i = (t^Q)^i_j dx^j,$$

where Q is a symmetric 3×3 matrix. It can be simply written also in the coordinates

$y^i = (t^Q)^i_j x^j$ as

$$\theta^0 = dt, \qquad \theta^i = dy^i - Q^i_j t^{-1} y^j dt,$$

The Ricci rotation coefficients for the Kasner frame are given by the non-vanishing value

$$\theta^a_{0\,b0} = Q^a_b t^{-1},$$

and the nonvanishing components of the Ricci curvature tensor are

$$R^0{}_0 = -\mathrm{Tr}\,(Q{-}Q^2)\,t^{-2}, \qquad R^a{}_b = -(1{-}\mathrm{Tr}\,Q)\,Q^a_b\,t^{-2}.$$

We impose the commutation relations

$$[x, y] = i\kslash J^{12}, \qquad [t, z] = i\kslash J^{03}(\tau) = i\kslash J(t),$$

with $\tau = \tau(t)$. The Jacobi identities are satisfied if

$$J^{12} = c$$

with c a constant which we shall set equal to one. The algebra is the tensor product

$$\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2.$$

of a factor generated by (x, y) and a factor generated by (t, z). The tensor product structure, we shall see, is respected by the differential calculus; the classical limit is just the metric product of two manifolds. The algebra just defined is too restrictive to describe a general element of the Kasner family. It can be explicitly and easily solved however and it is a convenient ground state. From the definitions follow the commutation relations

$$[p_0, t] = 1, \qquad [p_b, x^a] = (\tau^Q)^a_b.$$

If the momenta are expressed in terms of the position generators the Leibniz rules are satisfied automatically.

The first factor is generated by the elements (x, y). We set

$$i\bar{k} p_1 = -y, \qquad i\bar{k} p_2 = x.$$

Then we have

$$\theta^1 = dx, \qquad \theta^2 = dy.$$

and we have completely described the geometry of the first factor. For the second factor we suppose $p_3 = p_3(t)$ so that the only nontrivial commutation relation is

$$[p_3, z] = \dot{p}_3 i\hbar J.$$

We have further

$$\theta^3(e_3) = \tau^{-q}[p_3, z] = 1, \qquad q = q_3$$

from which we conclude that

$$[p_3, z] = \tau^q.$$

We write this as a differential equation

$$\dot{p}_3 i\hbar J = \tau^q$$

for $p_3(t)$. We must force the algebra of momenta to be quadratic. This means that

$$[p_0, p_3] = -2i\epsilon Q^{33}{}_{03}(p_3)^2 + F^3{}_{03}p_3 + K_{03}$$

for some set of coefficients. We know however that by definition

$$[p_0, p_3] = \dot{p}_3$$

We find then a second differential equation

$$\dot{p}_3 + 2i\epsilon Q^{33}{}_{03}(p_3)^2 - F^3{}_{03}p_3 - K_{03} = 0$$

for $p_3(t)$. From (23) we find then an expression for J as integrability condition.

By the definition of the $C^\alpha{}_{\beta\gamma}$ we have

$$[p_0, p_3] = q\tau p_3 + ic_3$$

for some real c_3. We can choose

$$i\hbar p_3 = F(\tau)$$

which yields

$$\dot{\tau}F' = q\tau F - \hbar c_3.$$

From (11) we find that

$$q\tau = -4\mu F Q^{33}{}_{03}.$$

That is, F is linear in τ. With $F = \alpha\tau$, $q = -4\mu\alpha Q^{33}{}_{03}$ we find

$$\dot{\tau} = q\tau^2 + c, \qquad c = -\hbar c_3/\alpha$$

We find therefore the following expression for J

$$i\hbar J = \frac{i\hbar\tau^q}{\alpha(q\tau^2 + c)}.$$

We define

$$i\hbar p_0 = z.$$

The frame is given by

$$\theta^0 = -i\hbar J^{-1} dt, \qquad \theta^3 = \tau^{-q} dz.$$

Therefore

$$[p_0, t] = -J.$$

We have thus completely described the geometry of the second factor. There are two free quantities, the constants q and c. The solution is a very particular one. One can find more general solutions by adding a perturbation. From this point of view the most interesting Kasner solution is highly non-perturbative.

The non-vanishing rotation coefficients are given by

$$C^3{}_{30} = -\tau^{-q} J \partial_t \tau^q = -qJ\tau^{-1}\dot{\tau}.$$

If we use this value in the defining equation we find

$$[p_0, p_3] - C^3{}_{03} p_3 = -qJ\tau^{-1}(\dot{\tau} - q^{-1}\tau\dot{p}_3)$$

We check that there is a ground state. The frame components of the commutator are given by

$$J^{\alpha\beta} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \tau^{-q} \\ 0 & 0 & -\tau^{-q} & 0 \end{pmatrix}$$

We see that

$$i\hbar[p_\alpha, p_\beta] = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -J^{-1}\tau^{q} \\ 0 & 0 & J^{-1}\tau^{q} & 0 \end{pmatrix} = -J^{-1}_{\alpha\beta}.$$

Therefore the configuration is a ground state if $J \to 1$.

$$\hat{\theta}_\alpha = J^{-1}_{\alpha\beta}\theta^\beta = dp_\alpha.$$

The 2-form

$$\frac{1}{2} J^{-1}_{\alpha\beta} \theta^\alpha \theta^\beta = dxdy - J^{-1} dzdt$$

is closed.