Star-twisted gauge theories

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Summary

- Introduction: twisted vs. *-gauge transformations
- Half-way twisting: start-twisted gauge theories
- Outlook

Twisted vs. *-gauge transformations

(Quantum) Field Theory gets interesting when the space is noncommutative

$$[x^\mu, x^\nu] = i\theta^{\mu\nu}$$

In particular, noncommutativity deforms the gauge theory action:

$$S = -\frac{1}{2g^2}\int d^4x \,\mathrm{tr}\Big[F_{\mu\nu}\star F^{\mu\nu}\Big]$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]_\star$$

Twisted vs. *-gauge transformations

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In particular, noncommutativity deforms the gauge theory action:

$$S = -\frac{1}{2g^2}\int d^4x\,\mathrm{tr}\Big[F_{\mu\nu}\star F^{\mu\nu}\Big]$$

where

$$f(x)\star g(x) \equiv f(x)\exp\left[\frac{i}{2}\theta^{\mu\nu}\overleftarrow{\partial_\mu}\,\overrightarrow{\partial_\nu}\right]g(x)$$

This also results in a deformation of gauge symmetry (*-star gauge symmetry)

$$\delta_\varepsilon A_\mu = \partial_\mu \varepsilon + i[\varepsilon, A_\mu]_\star$$

Actually, because of the *-commutator,

$$[\varepsilon, A_\mu]_\star = [\varepsilon^a, A_\nu^b]_\star \{T^a, T^b\} + \{\varepsilon^a, A_\nu^b\}_\star [T^a, T^b]$$

there are restrictions on the possible gauge groups

- The gauge group is $U(N)$
- A_μ takes values in the universal enveloping algebra of a group G

(Jurco, Schralm, Schupp & Wess '00)

In addition, the NCYM action is also invariant under twisted gauge transformations

(Vassilevich '06; Aschieri, Dimitrijevic, Meyer, Schraml & Wess '06)

$$\delta_\varepsilon A_\mu = \partial_\mu \varepsilon + i[\varepsilon, A_\mu]$$

while the action on products uses a deformed Leibniz rule

$$\delta_\varepsilon(\Phi_1 \star \Phi_2) = (\delta_\varepsilon \Phi_1) \star \Phi_2 + \Phi_1 \star (\delta_\varepsilon \Phi_2)$$

$$+ \sum_{n=1}^{\infty} \frac{(-i/2)^n}{n!} \theta^{\alpha_1\beta_1} \ldots \theta^{\alpha_n\beta_n} \Big\{ [\partial_{\alpha_1}, [\ldots [\partial_{\alpha_n}, \delta_\varepsilon] \ldots]] \Phi_1 \star \partial_{\beta_1} \ldots \partial_{\beta_n} \Phi_2$$

$$+ \partial_{\alpha_1} \ldots \partial_{\alpha_n} \Phi_1 \star [\partial_{\beta_1}, [\ldots [\partial_{\beta_n}, \delta_\varepsilon] \ldots]] \Phi_2 \Big\}$$

In more precise terms, introducing the twist operator

$$\mathcal{F} = e^{-\frac{i}{2}\theta^{\mu\nu}\partial_\mu \otimes \partial_\nu} \longrightarrow f \star g = \mu\Big[\mathcal{F}^{-1} f \otimes g\Big]$$

the deformed Leibniz rule corresponds to a twist of the coproduct

$$\Delta(\delta_\varepsilon) = \delta_\varepsilon \otimes \mathbf{1} + \mathbf{1} \otimes \delta_\varepsilon \longrightarrow \Delta(\delta_\varepsilon)_{\mathcal{F}} = \mathcal{F}\Big(\delta_\varepsilon \otimes \mathbf{1} + \mathbf{1} \otimes \delta_\varepsilon\Big)\mathcal{F}^{-1}$$

For field theory purposes, this can be seen as a gauge transformation of the *-product

(Álvarez-Gaumé, Meyer & M.A.V.-M. '06)

$$\delta_\varepsilon(\Phi_1 \star \Phi_2) = (\delta_\varepsilon \Phi_1) \star \Phi_2 + \Phi_1 \star (\delta_\varepsilon \Phi_2) + \Phi_1(\delta_\varepsilon \star)\Phi_2$$

Now, the equations of motion constraint the possible gauge groups

$$\partial_\mu F^{\mu\nu} - i[A_\mu, F^{\mu\nu}]_\star = 0 \qquad\qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]_\star$$

- ☐ The gauge group is $U(N)$
- ☐ A_μ takes values in the universal enveloping algebra of a gauge group G

*-gauge symmetry has been argued to play a custodial role with respect to twisted-gauge symmetry

(Álvarez-Gaumé, Meyer & M.A.V.-M. '06)
(Giller, Gonera, Kosinski & Maslanka '07)

Summarizing (with a change of notation)

$$S = -\frac{1}{2g^2}\int d^4x \,\mathrm{tr}\Big[F_{\mu\nu} \star_\theta F^{\mu\nu}\Big] \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]_\theta$$

- *-gauge transformations:

$$\delta^\theta_\varepsilon A_\mu = \partial_\mu \varepsilon + i[\varepsilon, A_\mu]_\theta$$

$$\Delta(\delta_\varepsilon)_0 = \delta^\theta_\varepsilon \otimes \mathbf{1} + \mathbf{1} \otimes \delta^\theta_\varepsilon$$

- Twisted-gauge transformations:

$$\delta^0_\varepsilon A_\mu = \partial_\mu \varepsilon + i[\varepsilon, A_\mu]_0$$

$$\Delta(\delta_\varepsilon)_\theta = \mathcal{F}_\theta\Big(\delta^0_\varepsilon \otimes \mathbf{1} + \mathbf{1} \otimes \delta^0_\varepsilon\Big)\mathcal{F}^{-1}_\theta$$

Let us look at an “intermediate” case:

$$S = -\frac{1}{2g^2}\int d^4x\,\mathrm{tr}\Big[F_{\mu\nu}\star_\theta F^{\mu\nu}\Big] \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]_\theta$$

- *-twisted gauge transformations:

$$\delta_\varepsilon^{\theta'} A_\mu = \partial_\mu\varepsilon + i[\varepsilon, A_\mu]_{\theta'}$$

Is there any way to implement these transformations as an invariance of the action?

Let us look at an "intermediate" case:

$$S = -\frac{1}{2g^2}\int d^4x\, \mathrm{tr}\Big[F_{\mu\nu} \star_\theta F^{\mu\nu}\Big] \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]_\theta$$

$\uparrow$ $\qquad\qquad$ $\uparrow$

- <u>*-twisted gauge transformations</u>:

$$\delta_\varepsilon^{\theta'} A_\mu = \partial_\mu \varepsilon + i[\varepsilon, A_\mu]_{\theta'}$$

$\uparrow$

Is there any way to implement these transformations as an invariance of the action?

We look now for a deformation of the Leibniz rule, such that $F_{\mu\nu}$ transforms covariantly with respect to θ'-*-gauge transformations

$$\delta_\varepsilon^{\theta'} F_{\mu\nu} = [i\varepsilon, \partial_\mu A_\nu - \partial_\nu A_\mu]_{\theta'} + i[\partial_\mu \varepsilon, A_\nu]_{\theta'} - i[\partial_\nu \varepsilon, A_\mu]_{\theta'} - i\delta_\varepsilon^{\theta'}[A_\mu, A_\nu]_\theta$$

Imposing now that $\delta_\varepsilon^{\theta'} F_{\mu\nu} = [i\varepsilon, F_{\mu\nu}]_{\theta'}$ and using

$$f \star_\theta g = \sum_{n=0}^{\infty} \frac{(-i/2)^n}{n!} ({\theta'}^{\mu_1\nu_1} - \theta^{\mu_1\nu_1}) \ldots ({\theta'}^{\mu_n\nu_n} - \theta^{\mu_n\nu_n}) (\partial_{\mu_1} \ldots \partial_{\mu_n} f) \star_{\theta'} (\partial_{\nu_1} \ldots \partial_{\nu_n} g)$$

we find

$$\begin{aligned} \delta_\varepsilon^{\theta'} \left(A_\mu \star_\theta A_\nu \right) \;=\; & \sum_{n=0}^{\infty} \frac{(-i/2)^n}{n!} (\theta^{\alpha_1\beta_1} - {\theta'}^{\alpha_1\beta_1})(\theta^{\alpha_2\beta_2} - {\theta'}^{\alpha_2\beta_2}) \ldots (\theta^{\alpha_n\beta_n} - {\theta'}^{\alpha_n\beta_n}) \\ & \times \;\Big\{ \Big([\partial_{\alpha_1}, [\partial_{\alpha_2}, \ldots [\partial_{\alpha_n}, \delta_\varepsilon^{\theta'}] \ldots]] A_\mu \Big) \star_\theta \Big(\partial_{\beta_1} \partial_{\beta_2} \ldots \partial_{\beta_n} A_\nu \Big) \\ & \quad + \Big(\partial_{\alpha_1} \partial_{\alpha_2} \ldots \partial_{\alpha_n} A_\mu \Big) \star_\theta \Big([\partial_{\beta_1}, [\partial_{\beta_2}, \ldots [\partial_{\beta_n}, \delta_\varepsilon^{\theta'}] \ldots]] A_\nu \Big) \Big\} \end{aligned}$$

Using now the same Leibniz rule for the product of two field strength tensors

$$\begin{aligned}\delta_\varepsilon^{\theta'}\left(F_{\mu\nu}\star_\theta F^{\mu\nu}\right) &= \sum_{n=0}^{\infty}\frac{(-i/2)^n}{n!}(\theta^{\alpha_1\beta_1}-\theta'^{\alpha_1\beta_1})(\theta^{\alpha_2\beta_2}-\theta'^{\alpha_2\beta_2})\ldots(\theta^{\alpha_n\beta_n}-\theta'^{\alpha_n\beta_n})\\ &\times \left\{\left([\partial_{\alpha_1},[\partial_{\alpha_2},\ldots[\partial_{\alpha_n},\delta_\varepsilon^{\theta'}]\ldots]]F_{\mu\nu}\right)\star_\theta\left(\partial_{\beta_1}\partial_{\beta_2}\ldots\partial_{\beta_n}F^{\mu\nu}\right)\right.\\ &\quad\left.+\left(\partial_{\alpha_1}\partial_{\alpha_2}\ldots\partial_{\alpha_n}F_{\mu\nu}\right)\star_\theta\left([\partial_{\beta_1},[\partial_{\beta_2},\ldots[\partial_{\beta_n},\delta_\varepsilon^{\theta'}]\ldots]]F^{\mu\nu}\right)\right\}\end{aligned}$$

We find that

$$\delta_\varepsilon^{\theta'}\left(F_{\mu\nu}\star_\theta F^{\mu\nu}\right)=[i\varepsilon,F_{\mu\nu}\star_\theta F^{\mu\nu}]_{\theta'}$$

This guarantees the invariance of the pure Yang-Mills action under *-twisted gauge transformations:

$$\delta_\varepsilon^{\theta'}\left[-\frac{1}{2g^2}\int d^4x\,\mathrm{tr}\left(F_{\mu\nu}\star_\theta F^{\mu\nu}\right)\right]=0$$

Matter fields

We consider matter transforming under *-twisted gauge transformations as

$$\begin{aligned} \delta_\varepsilon^{\theta'}\psi &= i\varepsilon \star_{\theta'} \psi && \text{fundamental,} \\ \delta_\varepsilon^{\theta'}\psi &= -i\psi \star_{\theta'} \varepsilon && \text{antifundamental,} \\ \delta_\varepsilon^{\theta'}\psi &= [i\varepsilon, \psi]_{\theta'} && \text{adjoint.} \end{aligned}$$

One can defined the corresponding covariant derivatives

$$\begin{aligned} \nabla_\mu\psi &= \partial_\mu\psi - iA_\mu \star_\theta \psi && \text{fundamental,} \\ \nabla_\mu\psi &= \partial_\mu\psi + i\psi \star_\theta A_\mu && \text{antifundamental,} \\ \nabla_\mu\psi &= \partial_\mu\psi - i[A_\mu, \psi]_\theta && \text{adjoint.} \end{aligned}$$

Using now again the Leibniz rule

$$\delta_\varepsilon^{\theta'}\Big(\Phi_1 \star_\theta \Phi_2\Big) = \sum_{n=0}^{\infty} \frac{(-i/2)^n}{n!}(\theta^{\alpha_1\beta_1} - \theta'^{\alpha_1\beta_1})(\theta^{\alpha_2\beta_2} - \theta'^{\alpha_2\beta_2})\ldots(\theta^{\alpha_n\beta_n} - \theta'^{\alpha_n\beta_n})$$

$$\times \Big\{\Big([\partial_{\alpha_1},[\partial_{\alpha_2},\ldots[\partial_{\alpha_n},\delta_\varepsilon^{\theta'}]\ldots]]\Phi_1 \star_\theta \Big(\partial_{\beta_1}\partial_{\beta_2}\ldots\partial_{\beta_n}\Phi_2\Big)$$

$$+ \Big(\partial_{\alpha_1}\partial_{\alpha_2}\ldots\partial_{\alpha_n}\Phi_1\Big) \star_\theta \Big([\partial_{\beta_1},[\partial_{\beta_2},\ldots[\partial_{\beta_n},\delta_\varepsilon^{\theta'}]\ldots]]\Phi_2\Big)\Big\}$$

it can be proved that the derivatives are indeed covariant

$$\delta_\varepsilon^{\theta'}(\nabla_\mu\psi) = i\varepsilon \star_{\theta'} \nabla_\mu\psi \qquad \text{fundamental}$$

$$\delta_\varepsilon^{\theta'}(\nabla_\mu\psi) = -i(\nabla_\mu\psi) \star_{\theta'} \varepsilon \qquad \text{antifundamental}$$

$$\delta_\varepsilon^{\theta'}(\nabla_\mu\psi) = [i\varepsilon, \nabla_\mu\psi]_{\theta'} \qquad \text{adjoint}$$

With this we can construct invariant actions

We can make a bit more mathematical construction by defining the differential operator

$$X_f^\theta \equiv \sum_{n=0}^{\infty} \frac{(-i/2)^n}{n!} \theta^{\mu_1\nu_1} \dots \theta^{\mu_n\nu_n} \partial_{\mu_1} \dots \partial_{\mu_n} f \partial_{\nu_1} \dots \partial_{\nu_n}$$

with a left θ'-action given by

$$X_f^\theta \triangleright_{\theta'} g \equiv \sum_{n=0}^{\infty} \frac{(-i/2)^n}{n!} \theta^{\mu_1\nu_1} \dots \theta^{\mu_n\nu_n} (\partial_{\mu_1} \dots \partial_{\mu_n} f) \star_{\theta'} (\partial_{\nu_1} \dots \partial_{\nu_n} g) = f \star_{\theta'-\theta} g$$

Analogously, we can define

$$g \triangleleft_{\theta'} X_f^\theta = \sum_{n=0}^{\infty} \frac{(-i/2)^n}{n!} \theta^{\mu_1\nu_1} \dots \theta^{\mu_n\nu_n} (\partial_{\mu_1} \dots \partial_{\mu_n} g) \star_{\theta'} (\partial_{\nu_1} \dots \partial_{\nu_n} f) = g \star_{\theta'-\theta} f$$

and

$$\mathrm{Adj}(X_f^\theta) \triangleright_{\theta'} g = \sum_{n=0}^{\infty} \frac{(-i/2)^n}{n!} \theta^{\mu_1\nu_1} \dots \theta^{\mu_n\nu_n} \Big[\partial_{\mu_1} \dots \partial_{\mu_n} f, \partial_{\nu_1} \dots \partial_{\nu_n} g\Big]_{\theta'} = [f, g]_{\theta'-\theta}$$

The transformation field in different representations can be written then as

$$\begin{aligned}
\delta_\varepsilon^{\theta'}\Phi &= iX^\theta_{\varepsilon^a T^a} \triangleright_{\theta+\theta'} \Phi && \text{fundamental,}\\
\delta_\varepsilon^{\theta'}\Phi &= -i\Phi \triangleleft_{\theta+\theta'} X^\theta_{\varepsilon^a T^a} && \text{antifundamental,}\\
\delta_\varepsilon^{\theta'}\Phi &= i\mathrm{Adj}(X^\theta_{\varepsilon^a T^a}) \triangleright_{\theta+\theta'} \Phi && \text{adjoint}
\end{aligned}$$

Let's take Φ_1 and Φ_2 such that $\Phi_1 \star_\theta \Phi_2$ also transforms in one of these representations

Then, if $\theta^{\mu\nu} = \theta'^{\mu\nu}$ (*-gauge transformations)

$$\delta_\varepsilon^\theta(\Phi_1 \star_\theta \Phi_2) = \mu\Big[\mathcal{F}_\theta^{-1}\Delta(\delta_\varepsilon^\theta)\Phi_1 \otimes \Phi_2\Big], \qquad \Delta(\delta_\varepsilon^\theta) = \delta_\varepsilon^\theta \otimes \mathbf{1} + \mathbf{1} \otimes \delta_\varepsilon^\theta$$

Taking $\theta^{\mu\nu} \neq \theta'^{\mu\nu}$ (*-twisted gauge tranformations)

$$\delta_\varepsilon^{\theta'}(\Phi_1 \star_\theta \Phi_2) = \mu\Big[\mathcal{F}_\theta^{-1}\Delta(\delta_\varepsilon^{\theta'})_{\theta-\theta'}(\Phi_1 \otimes \Phi_2)\Big]$$

where

$$\Delta(\delta_\varepsilon^{\theta'})_{\theta-\theta'} = \mathcal{F}_{\theta-\theta'}\Big(\delta_\varepsilon^{\theta'} \otimes \mathbf{1} + \mathbf{1} \otimes \delta_\varepsilon^{\theta'}\Big)\mathcal{F}_{\theta-\theta'}^{-1}$$

This continuously interpolates between:

- ☐ $\theta^{\mu\nu} = \theta'^{\mu\nu}$ (*-gauge transformations)
- ☐ $\theta'^{\mu\nu} = 0$ (twisted-gauge transformations)

Concluding remarks

- We have shown that the NCYM action admits a continuous family of twisted invariances interpolating between *-gauge symmetry and twisted gauge symmetry

for example:

$$\theta'^{\mu\nu} = \lambda\theta^{\mu\nu} \qquad\qquad 0 \le \lambda \le 1$$

- An interesting case arises when

$$\theta^{\mu\nu} = 0 \qquad \text{but} \qquad \theta'^{\mu\nu} \neq 0$$

We find that ordinary (commutative) Yang-Mills theories admit a continous family of twisted invariances

Do these invariances play any dynamical role?

- In all cases the twist can be interpreted as a transformation of the *-product

$$\delta_\varepsilon^{\theta'}(\Phi_1 \star_\theta \Phi_2) = (\delta_\varepsilon^{\theta'}\Phi_1) \star_\theta \Phi_2 + \Phi_1 \star_\theta (\delta_\varepsilon^{\theta'}\Phi_2) + \Phi_1(\delta_\varepsilon^{\theta'}\star_\theta)\Phi_2$$

where

$$\Phi_1(\delta_\varepsilon^{\theta'}\star_\theta)\Phi_2 =$$

$$\sum_{n=1}^{\infty}\frac{(-i/2)^n}{n!}(\theta^{\alpha_1\beta_1}-\theta'^{\alpha_1\beta_1})\dots(\theta^{\alpha_n\beta_n}-\theta'^{\alpha_n\beta_n})\Big\{[\partial_{\alpha_1},[\dots[\partial_{\alpha_n},\delta_\varepsilon^{\theta'}]\dots]]\Phi_1\star_\theta\partial_{\beta_1}\dots\partial_{\beta_n}\Phi_2$$

$$+\partial_{\alpha_1}\dots\partial_{\alpha_n}\Phi_1\star_\theta[\partial_{\alpha_1},[\dots[\partial_{\alpha_n},\delta_\varepsilon^{\theta'}]\dots]]\Phi_2\Big\}$$

*-gauge symmetry might still be playing a custodial role

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Thanks