Geometry of the GW model

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Outline

General motivation

Approaches to NC gravity are different and depend much on the primary aspect, or the starting point which one takes, like e.g.

- field theory
- symmetries
- geometry

Many issues, from motivational to technical, are shared or present at various levels in different approaches.

General motivation

Originally the reason to introduce noncommutativity was to deal with divergences. In various aspects, e.g.

- NC gravity regularizes singularities in classical solutions
- NC field theory regularizes divergences in QFT
- gravity regularizes divergences of QFT

Personal view: the fact that finite-dimensional representations i.e. matrices have interpretation as geometry (fuzy sphere, matrix models) is a nontrivial new possibility to approach renormalization.

Completely opposite views, e.g. Caroll's: interpretational consistency of cosmology with quantum mechanics needs infinite-dimensional space of states.

General motivation

One of the common goals is to find NC versions of spaces like Schwarzschild, de Sitter or FRW. That is, to relate metric and curvature of the space to its noncommutativity in a unique and consistent way.

In the intersection of all approaches is the flat noncommutative Minkowski space, characterized by constant noncommutativity of coordinates and infinite-dimensionality of the Hilbert space of states.

Real motivation

Our real motivation is understand the properties of the Grosse-Wulkenhaar model, in particular its renormalizability

$$S = \int \frac{1}{2}\, \partial_\mu \varphi \partial^\mu \varphi + \frac{\mu^2}{2}\, \varphi^2 + \frac{\Omega^2}{2}\, \tilde{x}^\mu \varphi\, \tilde{x}_\mu \varphi + \frac{\lambda}{4!}\, \varphi^4$$

The model is defined on the flat noncommutative space. Fields on this space can be represented either as ordinary functions with the Moyal- Weyl multiplication or as infinite-dimensional matrices.

Translational symmetry broken, LS duality

Real motivation

In one of the first renormalizability proofs the estimates were done by truncation of infinite-dimensional to $n \times n$ matrices and taking the limit $n \to \infty$. The geometry of these matrix spaces is certainly different from the flat geometry of the Moyal-Weyl space.

Is it possible, if one defines GW model as a limit of finite-matrix approximations, to interpret the oscillator term geometrically?

This would also be one of the realizations of the idea that gravity (curvature) can regularize quantum field theory.

Relations

NC gravity in the noncommutative frame formalism is defined through a generalization of geometry i.e. differential geometry.

There are two ingredients, in general, which define a NC space: the commutation relations of coordinates (algebra)

$$[x^\mu, x^\nu] = i\hbar J^{\mu\nu},$$

and the commutation relations of coordinates and differentials (geometry)

$$[x^\mu, dx^\nu] = i\hbar \chi^{\mu\nu}.$$

Compatibility

If we assume that the Leibniz rule holds, there is a simple consistency relation between the geometry and the algebra

$$[dx^\mu, x^\nu] + [x^\mu, dx^\nu] = i\bar{k}\, dJ^{\mu\nu}.$$

For example, the NC Minkowski space is flat,

$$[x^\mu, x^\nu] = \mathrm{const}$$

if differential is defined such that

$$[x^\mu, dx^\nu] = 0$$

which is consistent with the commutation relation of coordinates.

Differential geometry

Differentials dx^μ are 1-forms. One can generalize the main (practically, all) notions of differential geometry, e.g. metric, connection, curvature and torsion to a noncommutative setting. For example, the inverse metric g is a function

$$g(dx^\mu \otimes dx^\nu) = g^{\mu\nu}(x),$$

the connection ω is a 1-form while the torsion Θ and the curvature Ω are 2-forms. Usually the linearity of these mappings is assumed.

Definition

In general, the nontrivial input is the definition of the differential, as it is not unique. Here it is given in terms of the noncommutative moving frame θ^α as

$$df = (e_\alpha f)\,\theta^\alpha .$$

e_α are the derivations dual to the 1-forms θ^α: $\theta^\alpha(e_\beta) = \delta^\alpha_\beta$
In addition, the frame forms satisfy

$$[x^\mu, \theta^\alpha] = 0.$$

It means means that we can consistently require

$$g(\theta^\alpha \otimes \theta^\beta) = g^{\alpha\beta} = \mathrm{const}.$$

Properties

As in general relativity, the physical input is the frame.

By the choice of the frame we define symmetries of the noncommutative space.

The formalism has general coordinate invariance but the local gauge invariance is broken.

In contrast to the commutative case, the algebraic structure and its compatibility with the geometric structure are additional constraints (Jacobi, Leibniz).

The status of the NC gravity action is unclear (at present).

Momenta

Noncommutative geometry has important property that certain spaces can be represented by finite-dimensional matrices.

In the matrix case, the choice of the frame is equivalent to the choice of the momenta p_α, as every every derivation can be represented as a commutator

$$e_\alpha f = [p_\alpha, f].$$

However the choice of the momenta is not completely arbitrary. Imposing the Leibniz rule and $d^2 = 0$ constrains the momenta

$$[p_\alpha, p_\beta] = \frac{1}{i\bar{k}} K_{\alpha\beta} + F^\gamma{}_{\alpha\beta} p_\gamma - 2i\bar{k} Q^{\gamma\delta}{}_{\alpha\beta} p_\gamma p_\delta .$$

They satisfy a quadratic algebra.

Connection

It can be shown that the rotation coefficients

$$d\theta^\alpha = -\frac{1}{2} C^\alpha{}_{\beta\gamma}\theta^\beta\theta^\gamma$$

are linear in the momenta

$$C^\gamma{}_{\alpha\beta} = F^\gamma{}_{\alpha\beta} - 4i\bar{k}\, Q^{\gamma\delta}{}_{\alpha\beta}\, p_\delta .$$

In the usual approximation the connection $\omega^\alpha{}_\beta = \omega^\alpha{}_{\gamma\beta}\theta^\gamma$ which is metric compatible and torsion free is given by

$$\omega_{\alpha\beta\gamma} = \frac{1}{2}(C_{\alpha\beta\gamma} - C_{\beta\gamma\alpha} + C_{\gamma\alpha\beta}).$$

Riemann curvature

The curvature is defined by

$$\Omega^\alpha{}_\beta = d\omega^\alpha{}_\beta + \omega^\alpha{}_\gamma\omega^\gamma{}_\beta = \frac{1}{2}R^\alpha{}_{\beta\rho\sigma}\theta^\rho\theta^\sigma .$$

Expressing it in terms of the momenta we obtain

$$R^\alpha{}_{\beta\rho\sigma}\theta^\rho\theta^\sigma = 2\Big(T^{\alpha\gamma}{}_{\sigma\beta}K_{\rho\gamma} - \frac{1}{4}F^\alpha{}_{\delta\beta}F^\delta{}_{\rho\sigma} + \frac{1}{4}F^\alpha{}_{\rho\gamma}F^\gamma{}_{\sigma\beta}$$

$$+\frac{1}{2}i\bar{k}p_\epsilon(2F^\epsilon{}_{\rho\gamma}T^{\alpha\gamma}{}_{\sigma\beta} + 2F^\alpha{}_{\gamma\beta}Q^{\gamma\epsilon}{}_{\rho\sigma} - F^\gamma{}_{\rho\sigma}T^{\alpha\epsilon}{}_{\gamma\beta} + F^\alpha{}_{\rho\gamma}T^{\gamma\epsilon}{}_{\sigma\beta} + F^\gamma{}_{\sigma\beta}T^{\alpha\epsilon}{}_{\rho\gamma})$$

$$+(i\bar{k})^2 p_\epsilon p_\eta(-2T^{\alpha\gamma}{}_{\sigma\beta}Q^{\epsilon\eta}{}_{\rho\gamma} + 2T^{\alpha\epsilon}{}_{\gamma\beta}Q^{\gamma\eta}{}_{\rho\sigma} + T^{\alpha\epsilon}{}_{\rho\gamma}T^{\gamma\eta}{}_{\sigma\beta})\Big)\theta^\rho\theta^\sigma$$

with

$$T_{\alpha\beta\gamma\delta} = 2(-Q_{\alpha\beta\gamma\delta} + Q_{\beta\gamma\delta\alpha} + Q_{\beta\delta\gamma\alpha})$$

Curvature scalar

For the curvature scalar we get

$$
\begin{aligned}
R = {} & 8K_{\alpha\gamma} Q^{\gamma\beta}{}_{\beta}{}^{\alpha} + \frac{1}{4} F^{\alpha\beta\gamma} F_{\alpha\beta\gamma} \\
& + i\hbar p_\epsilon \big(8F^{\epsilon}{}_{\alpha\gamma} Q^{\gamma\beta}{}_{\beta}{}^{\alpha} - 2F_{\alpha\gamma\beta} Q^{\epsilon\alpha\gamma\beta} \big) \\
& + (i\hbar)^2 p_\epsilon p_\eta \big(-16 Q^{\gamma\beta}{}_{\beta}{}^{\alpha} Q^{\epsilon\eta\alpha\gamma} - 16 Q^{\gamma\beta}{}_{\beta}{}^{\epsilon} Q_{\gamma\alpha}{}^{\alpha\eta} \\
& \qquad - 4 Q^{\epsilon\alpha\gamma\beta} Q^{\eta}{}_{\alpha\gamma\beta} - 8 Q^{\epsilon\alpha\gamma\beta} Q^{\eta}{}_{\gamma\alpha\beta} \big)
\end{aligned}
$$

Heisenberg algebra

In the Fock basis coordinates x and y are represented as

$$x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 & . & . & . & . \\ 1 & 0 & \sqrt{2} & . & . & . & . \\ 0 & \sqrt{2} & 0 & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & 0 & \sqrt{n-1} & . \\ . & . & . & . & \sqrt{n-1} & 0 & . \\ . & . & . & . & . & . & . \end{pmatrix}$$

$$y = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 & . & . & . & . \\ 1 & 0 & -\sqrt{2} & . & . & . & . \\ 0 & \sqrt{2} & 0 & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & 0 & -\sqrt{n-1} & . \\ . & . & . & . & \sqrt{n-1} & 0 & . \\ . & . & . & . & . & . & . \end{pmatrix}$$

Truncation

Truncation to the first n rows and n columns gives

$$x_n = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 & . & . & . \\ 1 & 0 & \sqrt{2} & . & . & . \\ 0 & \sqrt{2} & 0 & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & 0 & \sqrt{n-1} \\ . & . & . & . & \sqrt{n-1} & 0 \end{pmatrix}$$

$$y_n = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 & . & . & . \\ 1 & 0 & -\sqrt{2} & . & . & . \\ 0 & \sqrt{2} & 0 & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & 0 & -\sqrt{n-1} \\ . & . & . & . & \sqrt{n-1} & 0 \end{pmatrix}$$

Truncated Heisenberg algebra

This truncation changes the initial algebra

$$[x, y] = i$$

to

$$[x_n, y_n] = i(1 - nP_n)$$

with

$$P_n = \begin{pmatrix} 0 & 0 & 0 & . & . & . \\ 0 & 0 & 0 & . & . & . \\ 0 & 0 & 0 & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & 0 & 0 \\ . & . & . & . & 0 & 1 \end{pmatrix}$$

In the following, we simplify notation by omitting n.

Matrices and relations

The truncated Heisenberg algebra can be considered as a three-dimensional space generated by hermitian coordinates x, y and $z = nP$. In the limit $n \to \infty$, that is, $P \to 0$ (or $z \to 0$), it reduces to a two-dimensional space.

Matrices x_n and y_n approximate the real axis: their spectrum consists of zeroes of Hermité polynomials H_n. For growing n, more and more points are included in the spectrum.

As we deal with finite matrices, we have additional relations

$$a^n = 0, \qquad Pa = 0, \qquad a^{n-1}(1-P) = 0,$$

a is the lowering operator, $a = \frac{1}{\sqrt{2}}(x + iy)$.

Position algebra

The complete set of relations which defines this algebra is

$$[x, y] = i(1 - z)$$
$$[x, z] = i(yz + zy)$$
$$[y, z] = -i(xz + zx)$$

We shall find geometry of this space.

Momentum algebra

To obtain the geometry, we need to identify the momenta which satisfy a quadratic algebra. The momenta we introduce as

$$p_1 = iy, \quad p_2 = -ix, \quad p_3 = i(z - \frac{1}{2}).$$

Note that that p_1 and p_2 coincide with those for the Heisenberg. The momentum algebra is

$$[p_1, p_2] = \frac{1}{2i} + p_3$$
$$[p_2, p_3] = p_1 - i(p_1 p_3 + p_3 p_1)$$
$$[p_3, p_1] = p_2 - i(p_2 p_3 + p_3 p_2)$$

Weak limit

The (weak) limit in which the dimension of matrices tends to infinity can be defined as $z \to 0$ or $ip_3 \to \frac{1}{2}$. One can easily check that in this limit the two-dimensional subalgebra consistently decouples and becomes Heisenberg.

It is however not so with the geometry. Although we have $e_3 \to 0$ and $dz \to 0$, $dx \to \theta^1$, $dy \to \theta^2$, the space of 1-forms is 3-dimensional and the connection does not vanish

$$dx = (1-z)\theta^1 + (yz+zy)\theta^3$$

$$dy = (1-z)\theta^2 + (xz+zx)\theta^3$$

$$dz = (xz+zx)\theta^1 + (yz+zy)\theta^2$$

Connection

To calculate the connection we first identify the nonvanishing structure elements

$$K_{12} = \frac{1}{2}, \qquad F^1{}_{23} = 1,$$
$$Q^{13}{}_{23} = \frac{1}{2}, \qquad Q^{23}{}_{31} = \frac{1}{2}$$

(+ the symmetries). We obtain

$$\omega_{12} = -\omega_{21} = (-\frac{1}{2} + 2ip_3)\theta^3$$
$$\omega_{13} = -\omega_{31} = \frac{1}{2}\theta^2 + 2ip_2\theta^3$$
$$\omega_{23} = -\omega_{32} = -\frac{1}{2}\theta^1 - 2ip_1\theta^3$$

Scalar curvature

We can easily calculate the scalar curvature

$$R = \frac{11}{2} - 4(z - \frac{1}{2}) - 8(x^2 + y^2).$$

It does not vanish in the limit $z \to 0$ but tends to the value

$$R = \frac{15}{2} - 8(x^2 + y^2).$$

Thus we see that, defined as a limit of finite-dimensional matrix algebras, the Heisenberg algebra is not a flat space: it has residual non-constant curvature.

Conclusion

Now we can show that the GW action

$$S = \int \frac{1}{2}(1-\frac{\Omega^2}{2})\partial_\mu\varphi\,\partial^\mu\varphi + \frac{\mu^2}{2}\varphi^2 + \frac{\Omega^2}{2}\tilde{x}^\mu\tilde{x}_\mu\varphi\varphi + \frac{\lambda}{4!}\varphi^4$$

and the action for the scalar field nonminimally coupled to the curvature

$$S' = \int \sqrt{g}\left(\frac{1}{2}\,\partial_\mu\varphi\,\partial^\mu\varphi + \frac{m^2}{2}\varphi^2 - \frac{\xi}{2}R\varphi^2 + \frac{\Lambda}{4!}\varphi^4\right)$$

are the same in two (and more) dimensions, if S' is defined on a space which is a limit of finite-dimensional matrix spaces.

Conclusion

For $z \longrightarrow 0$ the two-dimensional subspace is flat, $\sqrt{g} = 1$, and $e_\alpha = \delta^\mu_\alpha \, \partial_\mu$ for $\mu = 1, 2$ while $e_3 = 0$.

Therefore the two actions are the same up to an overall rescaling

$$S = \kappa S'$$

for

$$\kappa = 1 - \frac{\Omega^2}{2}, \qquad \mu^2 = \kappa m^2 - \frac{\Omega^2 a}{b}, \qquad \lambda = \kappa \Lambda, \qquad \xi = \frac{\Omega^2}{\kappa b},$$

and with values a, b given by the curvature, $a = \frac{15}{2}$, $b = 8$.

Conclusion

The constant part of the curvature renormalizes the mass of the scalar field, while the space dependent part gives the harmonic oscillator potential.

The coupling constant ξ is not a priori fixed but can be related to Ω. If we identify the two actions at the self-duality point, $\Omega = 1$, we obtain $\xi = \frac{1}{4}$.

Conclusion

The harmonic oscillator potential in the GW action is in fact the curvature scalar of an appropriately defined noncommutative space.

Finite-matrix representations, which exist only for noncommutative spaces, are important as they can give valuable hints about renormalizability.

Noncommutative geometry i.e. noncommutative gravity has an important role.