Quantum Geometry of the M5-brane in a Constant C-Field from Multiple Membranes

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base on
arXiv:0901.1847 [hep-th]
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Bayrischzell Workshop, May 15-18, 2009

Outline

Quantum geometry

- An interesting example of quantum geometry is that the worldvolume of a D-brane become noncommutative when a constant NS B-field is turned on: Connes, Douglas, Schwarz (1997); Douglas, Hull (1997); Chu, Ho (1998); Schomerus (1999); Seiberg, Witten (1999)

Noncommutative geometry: $[X^i, X^j] = i\theta^{ij}$.

- The B-field modifies the BC of the open string to a mixed one. One can show that the ordinary canonical quantization is not compatitble with the mixed BC.
  Quantize the system canonically by taking into account of the boundary condition as a constraint, one obtains the NCG for the endpoint of the open string, i.e. the D-brane. Moreover one obtains automatically the open string metric. Chu, Ho (1998)

Q1: Geometry for M5 brane with constant 3-form C-field?

- M2-brane can end on M5-brane. One may thus try to quantize an open M2-brane with boundary condition modified by C-field and use it to deduce the kind of quantum geometry on the M5-brane.
- However, due to the nonlinearlity of the problem, only an approximate analysis can be carried out. Also the results is expressed in terms of commutator of the boundary string coordinates and is complicated.
  Bergshoeff, Berman, Schaar, Sundell (2000); Kawamoto, Sakakura (2000).

Q1. Different way to learn about the geometry?

Q2: Where is NCG in the fuzzel funnel description of D3-brane?

- Now a D3-brane can also be understood as a particular configuration of excited scalars of a large number of coincident D1-branes.

  Constable, Myers, Tafjord (1999)

- In this description, the D3-brane is described as a fuzzy funnel solution of the Nahm equation of the system of D1-branes.

The information of the NCG and the open string meric should somehow be encoded in this description of the D3-brane.

Q2. How to extract this?

Main Results

Main result 1

By considering a system of F1-strings ending on this D1-branes system, we show that the Nahm equation can be derived as the boundary condition of this system of F1-strings.
This provides us with a new way to extract the NCG and the open string metric of a D3-brane!

Main result 2

Applying this method to M5-brane, we are able to derive the form of the quantum geometry on a M5-brane when a C-field is turned on

$$\text{3-bracket geometry:} \quad [X^i, X^j, X^k] = i\theta^{ijk}.$$

Outline

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D1-branes as BIon spike solution of D3-brane

- Consider a system of N D1-strings ending on a D3-brane

$$\begin{array}{c|cccccccccc} D3 & 0 & 1 & 2 & 3 & & & & & & \\ D1's & 0 & & & & & & & & & 9 \end{array}.$$

  The intersecting brane system can be described in two different ways in terms of either the D3-brane theory or the D1-strings theory.

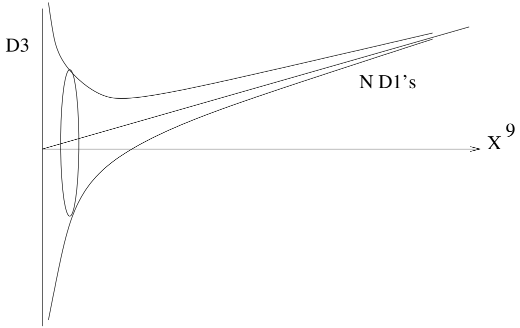
- From the D3-brane point of view, the Born-Infeld theory admits a static BPS solution

$$X^9(x^i) = \frac{N}{\sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}}, \qquad B_i = -\frac{N}{2r^3}\vec{r}$$

  which corresponds to placing N units of $U(1)$ magnetic charge on the D3-brane. This describes a spike of N D1-strings protruding out of the D3-brane. Callan-Maldacena (1997)

- At a fixed distance $X^9 = \sigma$ from the D3-brane, the cross-section of the D3-brane is a 2-sphere with radius

$$r(\sigma) = \frac{N}{\sigma}.$$



For B-field with $B_{12} \neq 0$, the spherical symmetrical is broken.

BIon spike solution

$$\frac{x_1^2}{r_1(\sigma)^2} + \frac{x_2^2}{r_2(\sigma)^2} + \frac{(x_3 - \sigma \tan\alpha)^2}{r_3(\sigma)^2} = 1, \quad \tan\alpha = 2\pi\alpha' B.$$

where

$$r_1(\sigma) = r_2(\sigma) = \frac{N}{\sigma}\cos\alpha, \quad r_3(\sigma) = \frac{N}{\sigma}.$$

This describes N D1-strings tilted away from the normal of the D3-brane by an angle α, and cross-section becomes an ellipsoid.

D3-brane as fuzzy funnel solution of D1-branes

- From the dual point of view, the D3-brane can be constructed as a particular excited configuration of the N D1-branes with three scalar fields Φ^i excited. The configuration satisfies:

Nahm equation

$$\partial_\sigma X^i = i\frac{1}{2}\epsilon_{ijk}[X^j, X^k], \quad i = 1, 2, 3.$$

- The solution $\Phi = 0$ corresponds to an infinitely long bundle of coincident D1-branes. Another solution is the fuzzy funnel

Fuzzy funnel solution

$$X^i(\sigma) = f(\sigma)\alpha^i, \qquad f = \frac{1}{2\sigma}, \quad [\alpha^i, \alpha^j] = 2\epsilon^{ijk}\alpha^k,$$

and $\alpha^i = N \times N$ representation of an $SU(2)$ subgroup of $SU(N)$.

Constable, Myers, Tafjord (1999)

- At a fixed point σ on the D1-branes, the geometry is that of a fuzzy sphere with radius $R^2 = \frac{1}{N}\mathrm{tr}(X^i)^2$,

$$R(\sigma) = \frac{\sqrt{N^2-1}}{|\sigma|} \approx \frac{N}{|\sigma|} \quad \text{for large } N.$$

- This solution describes a bunch of D1-strings expand into a D3-brane.

- For $B_{12} \neq 0$, the Nahm equation is modified to

B-field modified Nahm equation

$$\partial_\sigma \phi^i = i(\frac{1}{2}\epsilon_{ijk}[\phi^j, \phi^k] + \delta_i^3 iB),$$

where the rescaled fields ϕ^i are defined by

$$\phi^1 := \sqrt{1+B^2}X^1, \quad \phi^2 := \sqrt{1+B^2}X^2, \quad \phi^3 := X^3.$$

Karczmarez, Callan (2001)

- The modified Nahm equation has the solution

$$\phi^i = f(\sigma)\alpha^i - \delta_3^i \, B\, \sigma,$$

In terms of X^i, one see immediately that this solution reproduces precisely the features (ellipsoid's radii and tilting) of the BIon spike.

Remarks

Lesson I

BIon spikes description (D3-brane picture)

$\Updownarrow$

modified Nahm eqn (D1-branes description)

Outline

IIB matrix string theory as probe

- Since one can identify the endpoint of the open F1-string with the D3-brane worldvolume, so one should be able to understand the Nahm equation as a boundary condition of open F1-string.
- In order to identify the worldvolume coordinates of the D1-branes with the F1-string, we need to consider a matrix theory of F1-strings based on $U(N)$ matrices.

$\Longrightarrow$ Use IIB matrix string theory as a probe to our intersecting brane system.

A crash course on matrix string theory

- Matrix IIA strings in the lightcone gauge is given by a two-dimensional $U(N)$ SYM theory

$$S = \frac{1}{2\pi\alpha'} \int \mathrm{tr}\Big(\frac{1}{2}(D_\mu X^I)^2 + \pi^2 g_s^2 l_s^4 F_{\mu\nu}^2 - \frac{1}{4\pi^2 g_s^2 l_s^4}[X^I, X^J]^2 + \text{fermions}\Big).$$

  Eigenvalues of the X^I gives spacetime coordinates of the strings.

- A T-duality relates the theory of IIA strings with the theory of IIB strings, where the same action now describes IIB strings. Except the spinors are now of the same chirality and the action describes IIB strings stretched in the 9-th direction.

Nahm equation as BC of F1-strings

Back to the intersecting brane system of N D1 and a D3. Let us use a matrix IIB string as a probe to the system and derive the Nahm eqn as BC of the matrix string theory.

derivation ($B = 0$)

- Interaction with the B-field is given by the generalized coupling

$$S_B := \frac{1}{4\pi\alpha'} \int B_{\mu\nu} \, \mathrm{tr} D_a X^\mu D_b X^\nu \epsilon^{ab}.$$

  Also there is a coupling to the $U(1)$ gauge field on the D3-brane

$$\int d\tau \, \mathrm{tr} \left[A_\mu(x^\nu(\tau)) \frac{dX^\mu(\tau)}{d\tau} \right].$$

- One find the BC of the matrix F1-string:

$$\partial_\sigma X^i + B^i{}_j \partial_\tau X^j = F^{0i} = \frac{1}{2} \frac{\epsilon_{ijk}}{\sqrt{-G}} \tilde{F}_{jk}.$$

- One can show that $\tilde{F}$ should be identified with the boundary values of the scalar fields X^i of the F1-strings as

$$\tilde{F}^{ij} = i[X^i, X^j], \quad i, j = 1, 2, 3.$$

- When $B = 0$, it is $G_{\mu\nu} = \eta_{\mu\nu}$, and the BC reads

$$\partial_\sigma X^i = \frac{i}{2}\epsilon^{ijk}[X^j, X^k], \quad i, j = 1, 2, 3,$$

  Here X^i are the matrix coordinates of the F1-strings and can be identified with the coordinates of the D1-branes. Thus we arrive at the Nahm eqn.

NCG from Nahm equation

- When $B \neq 0$, $SO(1,3) \to SO(1,1) \times SO(2)$, and allows a metric of the form:

$$G_{\mu\nu} = \begin{pmatrix} -g_0 & 0 & 0 & 0 \\ 0 & g_1 & 0 & 0 \\ 0 & 0 & g_1 & 0 \\ 0 & 0 & 0 & g_0 \end{pmatrix}.$$

This implies that the BC reads

$$\partial_\sigma X^i + B^i{}_j \partial_\tau X^j = \frac{i\epsilon_{ijk}}{2} \frac{g_j g_k}{g_0 g_1} [X^j, X^k].$$

- Solve with an ansatz

$$X^i(\tau,\sigma) = X_0^i(\tau,\sigma)\mathbf{1} + Y^i(\sigma),$$

then

$$\partial_\sigma X_0^i + B^i{}_j \partial_\tau X_0^j = 0,$$
$$\partial_\sigma Y^i = \frac{i\epsilon_{ijk}}{2} \frac{g_j g_k}{g_0 g_1} \left([X_0^j, X_0^k]\mathbf{1} + [Y^j, Y^k] \right).$$

- Note that the function X_0^i obeys precisely the ordinary mixed BC for an open string in B-field. Results of open string quantization gives

$$[X_0^i(\tau,\sigma), X_0^j(\tau,\sigma')] = \begin{cases} i\theta^{ij}, & \text{for } \sigma = \sigma' = \sigma_0 \\ 0, & \text{otherwise} \end{cases}$$

where $\theta^{ij} = \left(B(1-B^2)^{-1}\right)^{ij}$.

- The eqn for Y reads

$$\begin{aligned} \partial_\sigma Y^i &= i\epsilon^{ij}[Y^j, Y^3], \quad i,j = 1,2, \\ \partial_\sigma Y^3 &= i\frac{g_1}{g_0}([Y^1, Y^2] + i\theta). \end{aligned}$$

These eqn reproduces the Nahm eqn if $g_1/g_0 = 1 + B^2$. This is precisely satisfied by the open string metric on the NCSYM:

$$g_0 = 1, \quad g_1 = 1 + B^2.$$

Remarks

Lesson II (new)

B-field modified Nahm equation $\Longleftrightarrow$ NCG θ^{ij} and open string metric G_{ij} of D3-brane

This understanding of the Nahm equation as boundary conditions for the F1-strings provides us a precise understanding of how the B-field modification in the Nahm eqn can be identified with the noncommutative parameter and the open string metric.

Combining lessons I and II:

Lessons I +II

BIon spikes description (D3-brane picture)

$\Updownarrow$

B-field modified Nahm equation (N D1-branes picture)

$\Updownarrow$

NCG θ^{ij} and open string metric G_{ij} of D3-brane

This means if we did not know about anything about NCG and open string metric, we could have derived them from the knowledge of the modified Nahm equation, or from the BIon spike!

Outline

M5-M2 branes intersection

- We want to repeat the above ideas for M5-branes and learn about the quantum geomety on M5-brane in the presence of a constant C-field.
- Consider a system of N M2-branes ending on a M5-brane

$$\begin{array}{c|cccccccccc} M5 & 0 & 1 & 2 & 3 & 4 & 5 & & & & \\ M2's & 0 & 1 & & & & & & & 9 & \end{array} .$$

  It is known that for $C = 0$, the intersecting brane system can again be described both from the M5-brane and the M2-branes point of view.
- What was known can be summarize in the following:

Known and unknown results

BIon spikes description of a bunch of M2-branes (M5-brane picture)

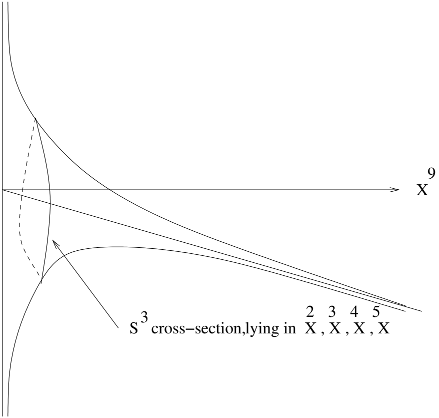
$\Updownarrow$

Basu-Harvey equation with $C = 0$, not known for $C \neq 0$ (N M2-branes picture)

$\Updownarrow$

Quantum Geometry of M5-brane

M5-brane spikes



- This solution describes a bundle of N M2-branes coming out of the M5-brane. The σ_2-dependent shift in the x_2 tells us that the M2-branes wedge is tilted away from the normal to the M5-brane with an angle α.

Basu-Harvey equations (2004)

- The Basu-Harvey eqn was proposed as an analogy to the Nahm equation to describe a M5 brane out of a bunch of membranes. It takes the form

$$\partial_2 X^i + \frac{i}{3!}\epsilon_{ijkl}[X^j, X^k, X^l] = 0, \quad i = 2,3,4,5.$$

  Here $(\tau, \sigma_1, \sigma_2)$ are the worldvolume coordinates of the membranes, $X^i(\sigma_2)$ describes the transverse fluctuations to the M2-branes.
- X^i is taken to be valued in an algebra $\mathcal{A}_4$ with generators T^i, $i = 1, \cdots, 4$ and the 3-bracket satisfies the $SO(4)$-invariant relation

$$[T^i, T^j, T^k] = i\epsilon_{ijkl}T^l.$$

- The Basu-Harvey equation is solved by

$$X^i(\sigma_2) = f(\sigma_2)T^i, \qquad f(\sigma_2) = \frac{1}{\sqrt{2\sigma_2}}.$$

  This define a fuzzy S^3 with radius $r^2(\sigma_2) := \sum (X^i)^2$ given by

$$r^2(\sigma_2) \sim \frac{1}{\sigma_2}.$$

This is precisely what one obtains for the M5 spike soluion.

- Just as the Nahm equation can be understood as a BPS equation of the D1-branes BI theory, the Basu-Harvey equation can also be understood as a BPS equation in the BL theory for multiple membranes. Bagger-Lambert (2006, 2007).
- A general theory of multiple membranes is based on a 3-algebra whose 3-bracket satisfies the fundamental identity

$$[[A,B,C],D,E] = [[A,D,E],B,C]+[A,[B,D,E],C]+[A,B,[C,D,E]].$$

- The action of the BL theory is

$$S = \int d^3\sigma \frac{1}{2}\langle D_a X^I, D_a X^I\rangle - \frac{1}{12}\langle [X^I,X^J,X^K],[X^I,X^J,X^K]\rangle + \cdots$$

BH eqn as BC of multiple membrane theory

As before, one can show that the BH eqn can be understood as the BC of the multiple membrane theory.
For $C \neq 0$, an analysis generalizing that for the D3-branes gives

$$[X^\mu, X^\nu, X^\lambda] = i\Theta^{\mu\nu\lambda},$$

where

$$\Theta^{\mu\nu\lambda} = \begin{cases} \epsilon^{\mu\nu\lambda}\frac{C'}{(1-C'^2)^2} & \mu,\nu,\lambda = 0,1,2, \\ \epsilon^{\mu\nu\lambda}\frac{C}{(1+C^2)^2} & \mu,\nu,\lambda = 3,4,5, \\ 0 & \text{otherwise}, \end{cases}$$

and $C := C_{345}$, $C' := C_{012}$.

Outline

Summary

- We have provided a new understanding of the D3-D1 branes system (or M2-M5 branes) in terms of BC of multiple open strings (or multiple open membranes).
- New type of quantum geometry from string theory:

Examples of quantum geometry from string

Moyal type: $[X^i, X^j] = i\theta^{ij}$.
Nambu type: $[X^i, X^j, X^k] = i\theta^{ijk}$.

More examples?

Further works

- Nature of the 3-bracket geometry:

  $$[X^1, X^2, X^3] := X^1X^2X^3+X^2X^3X^1+X^3X^1X^2-X^1X^3X^2-X^2X^1X^3-X^3X^2X^1 = c\mathbf{1}$$

  There exists respresentation like harmonic oscillator:

  $$X^1|\omega\rangle = (\omega+1+\rho)|\omega+1\rangle, \quad X^2|\omega\rangle = (\omega+\rho)|\omega+\rho\rangle, \quad X^3|\omega\rangle = \omega|\omega+\rho^2\rangle,$$

  where $\rho^3 = 1, \omega \in \mathbb{Z}[\rho]$. Takhtajan (1993)
  Exists $*$-product? Binary?
  Quantum dynamics in this quantum space? Localization of states?
  New physical effects? In progress.
- Deriving the 3-bracket geometry from quantization?
- "Non-Abelian" tensor multiplet on multiple M5-branes?

- Can dimensionally reduce the M2-M5 system on the X^1 direction. The system becomes a tilted F1-string ending on a D4-brane with RR 3-form potential $C_{(3)} = -C\, dX^3 dX^4 dX^5$ and a NS 2-form potential $B_{(2)} = C'\, dX^0 dX^2$.
  Our results imply that

$$[X^3, X^4, X^5] = i\,\frac{C}{K(1+C^2)^2}.$$

- This is intriguing. It is known that RR-backgrounds can give rise to nontrivial quantum geometry in the form of nonanticommutativity

$$\{\theta^\alpha, \theta^\beta\} = C^{\alpha\beta}$$

  Ooguri, Vafa (2003); Berkovits, Seiberg (2003)

  Our analysis implies that RR 3-form potential on D4-brane give rises to another new kind of quantum geometry for D-branes.
  Interesting to understand the string origin of 3-bracket geometry.