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Monopoles and instantons on quantum projective spaces

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Part 1:

A spectral triple on the quantum projective space $\mathbb{C}\mathrm{P}^2_q$
holomorphic calculus and spectral geometry

F. D'Andrea, L. Dabrowski, G. L.
The Noncommutative geometry of the quantum projective plane
Rev. Math. Phys. 20 (2008) 979–1006

Recently generalized to $\mathbb{C}\mathrm{P}^N_q$

F. D'Andrea, L. Dabrowski
Dirac operators on quantum projective spaces
arXive:0901.4735 [math.QA]

Part 2:

Monopoles connections on $\mathbb{C}\mathrm{P}^2_q$

F. D'Andrea, G. L.
Anti-selfdual Connections on the Quantum Projective Plane: Monopoles
arXiv:0903.3551v1 [math.QA]

F. D'Andrea, G. L.
Bounded and unbounded Fredholm modules for quantum projective spaces
arXiv:0903.3553v1 [math.QA]

A spectral triple for the quantum projective space $\mathbb{C}\mathsf{P}^2_q$

$$\mathcal{A}(\mathbb{C}\mathsf{P}^2_q),\, \mathcal{H},\, D$$

a q-deformation of the classical principal fibrations:

$$\mathsf{U}(2) \hookrightarrow \mathsf{SU}(3) \to \mathbb{C}\mathsf{P}^2 \qquad\qquad \mathsf{U}(1) \hookrightarrow S^5 \to \mathbb{C}\mathsf{P}^2$$

and q-deformation of associated bundles

With $0 < q < 1$, the function algebra

$$\mathcal{O} := \mathcal{A}(SU_q(3))$$

has generators u^i_j and relations

$$R^{ij}_{kl}(q) u^k_m u^l_n = u^k_m u^l_n R^{ij}_{kl}(q) \,, \qquad \sum_{p \in S_3} (-q)^{||p||} u^1_{p(1)} u^2_{p(2)} u^3_{p(3)} = 1$$

$$(u^i_j)^* = (-q)^{j-i} (u^{k_1}_{l_1} u^{k_2}_{l_2} - q u^{k_1}_{l_2} u^{k_2}_{l_1})$$

$R^{ij}_{kl}(q)$ is the R-matrix of the $SU_q(n)$ series

Coproduct, counit and antipode:

$$\Delta(u^i_j) = \sum_k u^i_k \otimes u^k_j \ ,$$

$$\epsilon(u^i_j) = \delta^i_j \ ,$$

$$S(u^i_j) = (u^j_i)^*$$

The symmetry algebra

Symmetries are via the $*$-Hopf algebra $\quad \mathcal{U} := U_q(su(3))$

generated by $K_i, K_i^{-1}, E_i, F_i,\ i = 1, 2$ with $K_i = K_i^*, F_i = E_i^*$, and relations (a_{ij} = Cartan matrix)

$$[K_i, K_j] = 0\ , \qquad K_i E_j K_i^{-1} = q^{a_{ij}/2} E_j\ , \qquad [E_i, F_j] = \delta_{ij} \frac{K_i^2 - K_i^{-2}}{q - q^{-1}}$$

$$E_i E_j^2 - (q + q^{-1}) E_i E_j E_i + E_j E_i^2 = 0 \qquad \forall\, i \neq j\ .$$

Coproduct, counit and antipode are given by (with $i = 1, 2$)

$$\Delta(K_i) = K_i \otimes K_i \ , \qquad \Delta(E_i) = E_i \otimes K_i + K_i^{-1} \otimes E_i \ ,$$

$$\epsilon(K_i) = 1 \ , \qquad \epsilon(E_i) = 0 \ ,$$

$$S(K_i) = K_i^{-1} \ , \qquad S(E_i) = -qE_i$$

Via a non-degenerate dual pairing $\langle\, , \,\rangle : \mathcal{U} \times \mathcal{O} \to \mathbb{C}$

define commuting left and right $\mathcal{U}$-actions on $\mathcal{O}$:

$$h \triangleright a = a_{(1)} \left\langle h, a_{(2)} \right\rangle \ , \quad a \triangleleft h = \left\langle h, a_{(1)} \right\rangle a_{(2)} \ ,$$

forall $h \in U_q(su(3)),\ a \in \mathcal{A}(SU_q(3))$

notation $\qquad \Delta(a) = a_{(1)} \otimes a_{(2)}$

The deformation of the quadratic Casimir of $U(su(3))$;
a central element in $\mathcal{U}$ given by

$$\mathcal{C}_q := (q-q^{-1})^{-2}\Big((H+H^{-1})\Big\{(qK_1K_2)^2+(qK_1K_2)^{-2}\Big\}+H^2+H^{-2}-6\Big)$$

$$+(qHK_2^2+q^{-1}H^{-1}K_2^{-2})F_1E_1+(qH^{-1}K_1^2+q^{-1}HK_1^{-2})F_2E_2$$

$$+\,qH[F_2,F_1]_q[E_1,E_2]_q+qH^{-1}[F_1,F_2]_q[E_2,E_1]_q$$

with $H := (K_1K_2^{-1})^{2/3}$ $\qquad [a,b]_q := ab-q^{-1}ba$

The restriction of $\mathcal{C}_q$ to the irrep (n_1,n_2) is

$$\mathcal{C}_q\Big|_{(n_1,n_2)} = [\tfrac{1}{3}(n_1-n_2)]^2+[\tfrac{1}{3}(2n_1+n_2)+1]^2+[\tfrac{1}{3}(n_1+2n_2)+1]^2$$

Some relevant subalgebras:

$U_q(su(2))$
the Hopf $*$-subalgebra of $U_q(su(3))$ generated by $\{K_1, K_1^{-1}, E_1, F_1\}$

$U_q(u(2))$
the Hopf $*$-subalgebra generated by $U_q(su(2))$ and $K_1K_2^2$, $(K_1K_2^2)^{-1}$

$K_1K_2^2$ commutes with all elements of $U_q(su(2))$

a class $\{\sigma_{\ell,N}\}$ of irreducible representation of $U_q(u(2))$
coming from a spin ℓ representation of $U_q(su(2))$, $\ell \in \frac{1}{2}\mathbb{N}$, and a representation of charge N on $K_1K_2^2$,

$$\sigma_{\ell,N}(K_1K_2^2) = q^N, \qquad N \in \frac{1}{2}\mathbb{Z}$$

a constraint on the labels: $\quad \ell + N \in \mathbb{Z}$

The quantum complex projective plane $\mathbb{C}\mathrm{P}^2_q$

The $*$-algebra $\mathcal{A}(SU_q(3))$ is an $U_q(su(3))$-bimodule for the canonical actions $\triangleright$ and $\triangleleft$

Call $\mathcal{A} := \mathcal{A}(\mathbb{C}\mathrm{P}^2_q)$ the fixed point subalgebra for the right action of the $*$-Hopf subalgebra $U_q(u(2)) \subset U_q(su(3))$:

$$\begin{aligned}\mathcal{A}(\mathbb{C}\mathrm{P}^2_q) &= \mathcal{A}(SU_q(3))^{U_q(u(2))} \\ &= \left\{ a \in \mathcal{A}(SU_q(3)) \,\middle|\, a \triangleleft h = \epsilon(h) a \,,\ \forall\, h \in U_q(u(2)) \right\}\end{aligned}$$

$\mathcal{A}$ is a left $U_q(su(3))$-module algebra;

$\mathcal{A}$ generated by elements $p_{ij} := (u_i^3)^* u_j^3$, of a projection

$$p^2 = p = p^*$$

this projection is a line bundle over $\mathbb{C}\mathrm{P}_q^2$ more later on

there are commutation rules (here sgn(0) := 0)

$$\begin{aligned}
p_{ii}p_{jk} &= q^{\mathrm{sgn}(i-j)+\mathrm{sgn}(k-i)} p_{jk}p_{ii} && i,j,k \text{ distinct },\\
p_{ii}p_{ij} &= q^{\mathrm{sgn}(j-i)+1} p_{ij}p_{ii} - (1-q^2)\textstyle\sum_{k<i} q^{6-2k} p_{kk}p_{ij} && i \neq j \ ,\\
p_{ij}p_{ik} &= q^{\mathrm{sgn}(k-j)} p_{ik}p_{ij} && i \notin \{j,k\} \ ,\\
p_{ij}p_{jk} &= q^{\mathrm{sgn}(i-j)+\mathrm{sgn}(k-j)+1} p_{jk}p_{ij} - (1-q^2)\textstyle\sum_{l<j} p_{il}p_{lk} && i,j,k \text{ distinct },\\
p_{ij}p_{ji} &= (1-q^2)\left(\textstyle\sum_{l<i} p_{jl}p_{lj} - \sum_{l<j} p_{il}p_{li}\right) && i \neq j \ ,
\end{aligned}$$

and 'projective plane' conditions

$$\sum_k p_{jk} p_{kl} = p_{jl} \,, \qquad\qquad \mathsf{Tr}_q(p) := q^4 p_{11} + q^2 p_{22} + p_{33} = 1$$

also: $*$-structure $(p_{ij})^* = p_{ji}$

There is a quantum sphere S^5_q

Call $\mathcal{B} := \mathcal{A}(S^5_q)$ the fixed point subalgebra for the right action of the $*$-Hopf subalgebra $U_q(su(2)) \subset U_q(su(3))$:

$$\begin{aligned} \mathcal{A}(S^5_q) &= \mathcal{A}(SU_q(3))^{U_q(u(2))} \\ &= \Big\{ a \in \mathcal{A}(SU_q(3)) \,\Big|\, a \triangleleft h = \epsilon(h) a \,,\ \forall\, h \in U_q(su(2)) \Big\} \end{aligned}$$

$\mathcal{B}$ is a left $\mathcal{U}$-module algebra, and is generated by $z_i := u^3_i$

Also: $$\mathcal{A}(\mathbb{C}\mathrm{P}^2_q) \simeq \Big\{ a \in \mathcal{A}(S^5_q) \,\Big|\, a \triangleleft K_1 K_2^2 = a \Big\}$$

and its generators are written as $p_{ij} := z_i^* z_j$

Relations for the sphere generators:

$$z_i z_j = q z_j z_i \quad \forall\ i < j\ , \qquad\qquad z_i^* z_j = q z_j z_i^* \quad \forall\ i \neq j\ ,$$

$$[z_1^*, z_1] = 0\ , \qquad [z_2^*, z_2] = (1 - q^2) z_1 z_1^*\ ,$$

$$[z_3^*, z_3] = (1 - q^2)(z_1 z_1^* + z_2 z_2^*)\ ,$$

$$z_1 z_1^* + z_2 z_2^* + z_3 z_3^* = 1\ .$$

The classical spin^c-structure

$\mathbb{C}\mathrm{P}^2$ does not admit a spin structure; only spin^c structures

it is a Kähler manifold $\Rightarrow$

the bundle of antiholomorphic forms $\Omega^{0,\bullet}$ with a natural $\mathbb{Z}_2$-grading is a canonical spin^c-bundle and a spin^c-Dirac operator is given by the Dolbeault-Dirac operator

$$D = \partial + \bar{\partial}$$

tensoring with line bundles one gets other spin^c-bundles

Associated bundles: line bundles in particular

$\sigma : U_q(u(2)) \to \mathrm{End}(\mathbb{C}^n)$ an n-dimensional $*$-representation

$\mathcal{A}(\mathbb{C}\mathrm{P}^2_q)$-bimodule of equivariant elements associated to σ:

$$\mathfrak{M}(\sigma) := \Big\{ v \in \mathcal{A}(SU_q(3))^n \,\Big|\, \sigma(S(h_{(1)})) \cdot (v \triangleleft h_{(2)}) = \epsilon(h) v \,, \ \forall\, h \in U_q(u(2)) \Big\}$$

These are finitely generated projective right-$\mathcal{A}(\mathbb{C}\mathrm{P}^2_q)$ modules

$\sigma = \sigma_{\ell,N}$, with $\ell \in \frac{1}{2}\mathbb{N}$, $N \in \frac{1}{2}\mathbb{Z}$, and condition $\ell + N \in \mathbb{Z}$

denote $\Sigma_{\ell,N} := \mathfrak{M}(\sigma_{\ell,N})$; in particular $\Sigma_{0,0} = \mathcal{A}$

Antiholomorphic forms again

$$\Omega^{0,0} := \mathcal{A} = \Sigma_{0,0}\ , \qquad \Omega^{0,1} := \Sigma_{\frac{1}{2},\frac{3}{2}}\ , \qquad \Omega^{0,2} := \Sigma_{0,3}\ ,$$

The full differential calculus $\qquad \Omega^{\bullet,\bullet} = \oplus \Omega^{i,j}$

$$\begin{array}{ccccccccc}
 & & & & \Omega^{0,0} & & & & \\
 & & & \swarrow & & \searrow & & & \\
 & & \Omega^{0,1} & & & & \Omega^{1,0} & & \\
 & \swarrow & & \searrow & & \swarrow & & \searrow & \\
\Omega^{0,2} & & & & \Omega^{1,1} & & & & \Omega^{2,0} \\
 & \searrow & & \swarrow & & \searrow & & \swarrow & \\
 & & \Omega^{1,2} & & & & \Omega^{2,1} & & \\
 & & & \searrow & & \swarrow & & & \\
 & & & & \Omega^{2,2} & & & &
\end{array}$$

$\Omega^{i,j} \simeq \Sigma_{\ell,N}$, with suitable values of (ℓ, N)

The double complex: $\partial : \Omega^{p,q} \to \Omega^{p+1,q}$ and $\bar{\partial} : \Omega^{p,q} \to \Omega^{p,q+1}$

denote $\partial_{i,j} := \partial|_{\Omega^{i,j}}$ and $\bar{\partial}_{i,j} := \bar{\partial}|_{\Omega^{i,j}}$ $\qquad$ and $\qquad \mathcal{R}_h a := a \triangleleft h$

$$\partial_{0,0} := \binom{\mathcal{R}_{E_2}}{\mathcal{R}_Y}, \qquad \bar{\partial}_{0,0} := \binom{\mathcal{R}_X}{\mathcal{R}_{F_2}},$$

$$\partial_{1,0} := -\begin{pmatrix} \mathcal{R}_{X^*} & \mathcal{R}_{E_2} \end{pmatrix}, \qquad \bar{\partial}_{0,1} := \begin{pmatrix} \mathcal{R}_{F_2} & \mathcal{R}_{Y^*} \end{pmatrix},$$

$$\partial_{0,1} := -\begin{pmatrix} -\mathcal{R}_Y & \mathcal{R}_{E_2} \\ \mathcal{R}_{E_2} & 0 \\ \mathcal{R}_Y & \mathcal{R}_{E_2} \\ 0 & \mathcal{R}_Y \end{pmatrix}, \qquad \bar{\partial}_{1,0} := \begin{pmatrix} \mathcal{R}_{F_2} & -\mathcal{R}_X \\ \mathcal{R}_X & 0 \\ \mathcal{R}_{F_2} & \mathcal{R}_X \\ 0 & \mathcal{R}_{F_2} \end{pmatrix},$$

$$\partial_{1,1} := \begin{pmatrix} -\mathcal{R}_{E_2} & \mathcal{R}_{X^*} & \mathcal{R}_{E_2} & 0 \\ \mathcal{R}_{X^*} & 0 & \mathcal{R}_{X^*} & \mathcal{R}_{E_2} \end{pmatrix}, \quad \partial_{0,2} := \binom{\mathcal{R}_{E_2}}{\mathcal{R}_Y},$$

$$\bar{\partial}_{1,1} := \begin{pmatrix} \mathcal{R}_{Y^*} & \mathcal{R}_{F_2} & \mathcal{R}_{Y^*} & 0 \\ -\mathcal{R}_{F_2} & 0 & \mathcal{R}_{F_2} & \mathcal{R}_{Y^*} \end{pmatrix}, \quad \bar{\partial}_{2,0} := \binom{\mathcal{R}_X}{\mathcal{R}_{F_2}},$$

$$\partial_{1,2} := -\begin{pmatrix} \mathcal{R}_{X^*} & \mathcal{R}_{E_2} \end{pmatrix}, \qquad \bar{\partial}_{2,1} := \begin{pmatrix} \mathcal{R}_{F_2} & \mathcal{R}_{Y^*} \end{pmatrix}$$

One finds: $\partial^2 = \bar\partial^2 = \partial\bar\partial + \bar\partial\partial = 0; \quad d = \partial + \bar\partial; \quad d^2 = 0$

An inner product on $\Omega^{\bullet,\bullet}$: $\langle \omega, \eta \rangle := \sum_{p,q} \varphi\Big(\omega^{\dagger}_{p,q}\, \eta_{p,q}\Big)$

with φ the Haar functional of $\mathcal{A}(SU_q(3))$

$\Omega^{2,2}$ is a rank one free module with basis a central element $\texttt{vol}$

use it to define an integral

$$⨍ \omega := \langle \texttt{vol}, \omega \rangle = \varphi(\omega_{2,2}) \,, \qquad \omega \in \Omega^{\bullet,\bullet}$$

since ∂ and $\bar\partial$ are constructed with the right action of elements in $\ker \epsilon$ and the Haar functional is invariant ($\varphi(a \triangleleft x) = \epsilon(x)\varphi(a)$), the integral is closed

$$⨍ \bar\partial\omega = ⨍ \partial\omega = 0$$

A spectral triple over $\mathbb{C}\mathrm{P}_q^2$ from antiholomorphic forms

$$\Omega^{(0,0)} \stackrel{\bar\partial}{\longrightarrow} \Omega^{(0,1)} \stackrel{\bar\partial}{\longrightarrow} \Omega^{(0,2)} \longrightarrow 0 \ .$$

$\mathcal{H}_+$ the completion of $\Omega^{(0,0)} \oplus \Omega^{(0,2)}$, $\mathcal{H}_-$ of $\Omega^{(0,1)}$,

$\mathcal{H} := \mathcal{H}_+ \oplus \mathcal{H}_-$

Dolbeault-Dirac operator:

$$D\omega := (\bar\partial^\dagger v,\ \bar\partial a + \bar\partial^\dagger b,\ \bar\partial v) \ , \qquad \omega = (a, v, b) \in \Omega^{(0,\bullet)}$$

$\bar\partial^\dagger$ the Hermitian conjugate of $\bar\partial$

$[D, f] \in \mathcal{B}(\mathcal{H})$ for all $f \in \mathcal{A}$

A self-adjoint extension of D defined once it is diagonalized
Compactness of $(D+i)^{-1}$ from asymptotic behaviour of $\mathrm{Sp}(D)$

$\ker D = \mathbb{C}$ are the constant 0-forms;
non-zero eigenvalues of D are $(n \geq 1)$

$$\pm\sqrt{\tfrac{2}{[2]}[n][n+2]} \qquad \text{with multiplicity } (n+1)^3\,,$$

$$\pm\sqrt{[n+1][n+2]} \qquad \text{with multiplicity } \tfrac{1}{2}n(2n+3)(n+3)$$

The spectrum of D is a q-deformation of the spectrum of the Dolbeault-Dirac operator of $\mathbb{C}\mathrm{P}^2$
$\mathrm{Sp}(D)$ grows exponentially: a 0^+-dimensional spectral triple

The spectrum of the Dirac operator from

$D^2\,\omega = [2]^{-1}\omega \triangleleft (\mathcal{C}_q - 2)$ $\qquad$ $\mathcal{C}_q$ is the quadratic Casimir

The Hodge star operator

the linear operator $*_H : \Omega^{i,j} \to \Omega^{2-j,2-i}$

$$\fint \omega^* \wedge_q \omega' = \left\langle *_H\, \omega, \omega' \right\rangle$$

Since $\Omega^{2,2} \simeq \mathcal{A}$ and the Haar state is faithful, equivalently

$$\omega^* \wedge_q \omega' = \left\langle *_H\, \omega, \omega' \right\rangle \mathtt{vol}$$

The calculus defined so that $*_H^2 \omega = (-1)^{\mathsf{dg}(\omega)} \omega$

$$\mathsf{d}^\dagger = *_H \mathsf{d} *_H \ .$$

$\mathfrak{e}(\omega)$ the left 'exterior product' : $\mathfrak{e}(\omega)\omega' := \omega \wedge_q \omega'$

$\mathfrak{i}(\omega) = \mathfrak{e}(\omega)^\dagger$ the 'contraction' by ω

$$*_H \omega = \mathfrak{i}(\omega^*)\mathtt{vol}$$

Line bundles

$\Sigma_{0,N}$ are line bundles of 'degree' N

As right $\mathcal{A}$-modules:

$$\Sigma_{0,N} \simeq P_N \mathcal{A}^{r_N}, \qquad P_N = \Psi_N \Psi_N^\dagger$$

Ψ_N is the column vector with components $\psi^N_{j,k,l}$:

$$(\psi^N_{j,k,l})^* := \sqrt{[j,k,l]!}\, z_1^j z_2^k z_3^l \,, \qquad \text{if } N \geq 0 \,; j+k+l = N \,,$$
$$(\psi^N_{j,k,l})^* := q^{-N+j-l} \sqrt{[j,k,l]!}\, (z_1^j z_2^k z_3^l)^* \,, \quad \text{if } N < 0 \,; j+k+l = -N$$

$$\Psi_N^\dagger \Psi_N = 1 \qquad \Rightarrow \qquad (P_N)^2 = P_N$$

The size is $r_N := \frac{1}{2}(N+1)(N+2)$; think of Ψ_N as a column vector of size r_N, and of $\Psi_N^\dagger$ as a row vector of the same size.

$[j,k,l]!$ are the q-trinomial coefficients:

$$[j,k,l]! = q^{-(jk+kl+lj)}\frac{[j+k+l]!}{[j]![k]![l]!}$$

q-factorial is

$$[n]! := [n][n-1]\dots[2][1], \quad n > 0; \qquad [0]! := 1$$

The Grassmannn connection:

$$\nabla : \Sigma_{0,N} \otimes_{\mathcal{A}} \Omega \to \Sigma_{0,N} \otimes_{\mathcal{A}} \Omega \qquad \nabla := P_N \circ \mathsf{d}$$

has curvature which is constant:

$$\nabla_N^2 = q^{N-1}[N]\nabla_1^2 \in \Omega^{(1,1)}$$

and anti-self-dual:

$$* \nabla_N^2 = -\nabla_N^2$$

the bundles are of rank 1

‘first Chern number ’ N

‘first Chern number’ $\frac{1}{2}N(N+1)$

Gauged Laplacian operator

$$\Box_N = \nabla_N^\dagger \nabla_N$$

Related to the Casimir $\mathcal{C}_q$

$$\Box_N = q^{-\frac{3}{2}} \frac{q^{\frac{3}{2}} + q^{-\frac{3}{2}}}{q^{\frac{N}{3}} + q^{-\frac{N}{3}}} \Big(\mathcal{C}_q - [\tfrac{1}{3}N]_q^2 - [\tfrac{1}{3}N+1]_q^2 - [\tfrac{2}{3}N+1]_q^2 \Big) + [2]_q [N]_q \ ,$$

The spectrum $\{\lambda_{n,N}\}_{n\in\mathbb{N}}$ of $\Box_N$

$$\begin{aligned} \lambda_{n,N} &= (1+q^{-3})[n]_q[n+N+2]_q + [2]_q[N]_q \qquad && \text{if } N \geq 0 \ , \\ \lambda_{n,N} &= (1+q^{-3})[n+2]_q[n-N]_q + [2]_q[N]_q \qquad && \text{if } N \leq 0 \ . \end{aligned}$$

with $n \in \mathbb{N}$.

not invariant under the exchange $N \leftrightarrow -N$,
not even when sending $q \leftrightarrow q^{-1}$

K-theory and K-homology

the C^*-algebra

$$0 \to \mathcal{K} \to C(\mathbb{C}\mathsf{P}^2_q) \to C(\mathbb{C}\mathsf{P}^1_q) \to 0$$

$$C(\mathbb{C}\mathsf{P}^1_q) \simeq C(S^2_q) = \mathcal{K} \oplus \mathbb{C}1$$

$$K_0(C(\mathbb{C}\mathsf{P}^2_q)) = \mathbb{Z}^3, \qquad K_1(C(\mathbb{C}\mathsf{P}^2_q)) = 0$$

$$K^0(C(\mathbb{C}\mathsf{P}^2_q)) = \mathbb{Z}^3, \qquad K^1(C(\mathbb{C}\mathsf{P}^2_q)) = 0$$

There are interesting twisted cocycles;
in particular a twisted volume form

$K_0(\mathcal{A})$: the a. group of finitely generated projective $\mathcal{A}$-module

$K^0(\mathcal{A})$: the a. group of even Fredholm modules

their pairing is via Chern characters; these are cyclic (co)-cycle,

a cyclic $2n$-cocycle: $\tau_n : \mathcal{A}^{2n+1} \to \mathbb{C}$ which is cyclic:

$$\tau_n(a_0, a_1, \ldots, a_{2n}) = \tau_n(a_{2n}, a_0, \ldots, a_{n-1}) ,$$

and Hochschild boundary closed: $b\,\tau_n = 0$

$$b\,\tau_n(a_0, \ldots, a_{2n+1}) := \sum_{j=0}^{2n} (-1)^j \tau_n(a_0, \ldots, a_j a_{j+1}, \ldots, a_{2n+1}) - \tau_n(a_{2n+1} a_0, a_1, \ldots, a_{2n})$$

Even cyclic cocycles $\quad \mathrm{ch}_n^{(\pi,\mathcal{H},F)}$, $2n \geq k$,
canonically associated to a $k+1$-summable Fredholm module:

a triple $(\pi, \mathcal{H}, F)$:

$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ a $\mathbb{Z}_2$-graded Hilbert space with grading γ

a graded representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H}_+) \oplus \mathcal{B}(\mathcal{H}_-)$

an odd operator F such that

$$[F, a_0][F, a_1] \ldots [F, a_k] \qquad \text{is traceclass}$$

then

$$\mathrm{ch}_n^{(\pi,\mathcal{H},F)}(a_0, \ldots, a_{2n+1}) := \tfrac{1}{2}(-1)^n \mathrm{Tr}_{\mathcal{H}}(\gamma F [F, a_0][F, a_1] \ldots [F, a_{2n}])$$

The coupling with idempotents:

$$\langle\, ,\, \rangle : K^0(\mathcal{A}) \times K_0(\mathcal{A}) \to \mathbb{Z}\ ,$$

$$\langle [(\pi, \mathcal{H}, F)], [e] \rangle = \tfrac{1}{2}(-1)^n \mathrm{Tr}_{\mathcal{H}\otimes\mathbb{C}^m}(\gamma F [F, e]^{2n+1})$$

it is an integer ; the index of a Fredholm operator

Classical Invariants

Back to line bundles over $\mathbb{C}\mathrm{P}^2_q$

The rank

The algebra $\mathcal{A} = \mathcal{A}(\mathbb{C}\mathrm{P}^2_q)$ with generators p_{ij} has a character (1-dimensional representation):

$$\tau_0 : \mathcal{A} \to \mathbb{C}, \qquad \tau_0(p_{ij}) := \delta_{i3}\delta_{j3}$$

then

$$\langle [\tau_0] | [P_N] \rangle := \tau_0(\mathrm{ch}_0(P_N)) = \tau_0(\mathrm{Tr}_{\mathbb{C}^{r_N}} P_N)$$

$$= 1$$

The monopole charge via a Fredholm module on $\mathcal{A}(\mathbb{C}\mathrm{P}^2_q)$

$\ell^2(\mathbb{N})$, with orthonormal basis $|n\rangle$;
the Hilbert space is $\mathcal{H}_1 := \ell^2(\mathbb{N}) \otimes \mathbb{C}^2$, and

$$F := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \qquad \gamma := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} ,$$

representation $\pi := \pi_+ \oplus \pi_-$ and

$$\pi_\pm(p_{11}) = \pi_\pm(p_{12}) = \pi_\pm(p_{13}) ,$$
$$\pi_-(p_{22}) = \pi_-(p_{23}) = 0$$
$$\pi_+(p_{22})|n\rangle = q^{2n}|n\rangle , \quad \pi_+(p_{23})|n\rangle = q^{n+1}\sqrt{1 - q^{2(n+1)}}|n+1\rangle ,$$

$\pi_+(p_{ij}) - \pi_-(p_{ij})$ is trace class for all i, j

the Fredholm module is 1-summable; a cyclic 0-cocycle:

$$\tau_1 : \mathcal{A}(\mathbb{C}\mathsf{P}^2_q) \to \mathbb{C}, \qquad \tau_1(a) := \tfrac{1}{2}\mathsf{Tr}_{\mathcal{H}}(\gamma F[F, a])$$

then

$$\begin{aligned}\langle[\tau_1]|[P_N]\rangle &:= \tau_1(\mathsf{ch}_0(P_N)) = \tfrac{1}{2}\,\mathsf{Tr}_{\mathcal{H}\otimes\mathbb{C}^{r_N}}(\gamma F[F, P_N]) \\ &= ... \\ &= N\end{aligned}$$

The number depends only on the restriction of the bundle to the subspace $\mathbb{C}\mathsf{P}^1_q \simeq S^2_q$, (a reason to call it the *monopole charge*)

The instanton charge via a Fredholm module on $\mathcal{A}(\mathbb{C}\mathrm{P}^2_q)$

the Hilbert space $\mathcal{H}_2$ (is two copies of) the linear span of orthonormal vectors $|\ell, m\rangle$, with $\ell \in \frac{1}{2}\mathbb{N}$ and $\ell + m \in \mathbb{N}$

the grading γ and the operator F are like before.

for the representation $\pi_2 = \pi_+ \oplus \pi_-$

$$\pi_+(p_{11}) = \pi_+(p_{12}) = \pi_+(p_{13}) = 0 \; ,$$

$$\pi_+(p_{22})|\ell, m\rangle = \begin{cases} q^{2(\ell+m)}|\ell, m\rangle & \text{if } m \leq \ell \; , \\ 0 & \text{if } m > \ell \; , \end{cases}$$

$$\pi_+(p_{23})|\ell, m\rangle = \begin{cases} q^{\ell+m+1}\sqrt{1 - q^{2(\ell+m+1)}}|\ell, m+1\rangle & \text{if } m \leq \ell - 1 \; , \\ 0 & \text{if } m \geq \ell \; . \end{cases}$$

$$\pi_-(p_{11}) = \pi_-(p_{12}) = \pi_-(p_{13}) = 0 \,,$$
$$\pi_-(p_{22})|\ell, m\rangle = q^{2(\ell+m)}|\ell, m\rangle \,,$$
$$\pi_-(p_{23})|\ell, m\rangle = q^{\ell+m+1}\sqrt{1 - q^{2(\ell+m+1)}}|\ell, m+1\rangle \,.$$

$\pi_+(p_{ij}) - \pi_-(p_{ij})$ is trace class for all i, j

the Fredholm module is 1-summable; a cyclic 0-cocycle:

$$\tau_2 : \mathcal{A}(\mathbb{C}\mathsf{P}^2_q) \to \mathbb{C}, \qquad \tau_2(a) := \tfrac{1}{2}\mathsf{Tr}_{\mathcal{H}}(\gamma F[F, a])$$

then

$$\begin{aligned} \langle[\tau_2]|[P_N]\rangle :&= \tau_2(\mathsf{ch}_0(P_N)) = \mathsf{Tr}_{\mathcal{H}_2 \otimes \mathbb{C}^m}(\gamma F[F, P_N]) \\ &= \ldots \\ &= \tfrac{1}{2}N(N+1) \end{aligned}$$

For any $N \in \mathbb{Z}$, the (right) module $\Sigma_{0,N}$ has

'rank' 1
'monopole charge' N
'instanton number' $\frac{1}{2}N(N+1)$

The three generators $\{e_1, e_2, e_3\}$ of $K_0(\mathcal{A}(\mathbb{C}\mathsf{P}^2_q))$ are:

$e_1 = [1]$ is the class of the rank one free $\mathcal{A}(\mathbb{C}\mathsf{P}^2_q)$-module $\Sigma_{0,0}$

e_2 is the class of $\Sigma_{0,-1}$ (the dual of the tautological bundle)

and e_3 is the class of $\Sigma_{0,1}$ (the tautological bundle)

The three generators of $K^0(\mathcal{A}(\mathbb{C}\mathsf{P}^2_q))$ are the classes of the Fredholm modules $(\pi_i, \mathcal{H}_i, F_i)$, $i = 0, 1, 2$, given before

Quantum invariants

Classically, invariants of vector bundles are computed by integrating powers of the curvature of a connection on the bundle, the result being independent of the particular chosen connection.

In order to integrate the curvature of a connection on the quantum projective space $\mathbb{C}\mathrm{P}^2_q$ one needs 'twisted integrals'; the result, is no longer an integer but rather its q-analogue

Invariants for $\mathbb{C}\mathsf{P}^2_q$

The Haar state of $\mathcal{A}(SU_q(3))$

$$\varphi(ab) = \varphi\Big((K \triangleright b \triangleleft K)a\Big) , \qquad \text{for} \quad a, b \in \mathcal{A}(SU_q(3)) ,$$

that when $a, b \in \mathcal{A}(\mathbb{C}\mathsf{P}^2_q)$ means

$$\varphi(ab) = \varphi\Big((K \triangleright b)a\Big) = \varphi\Big(\eta(b)a\Big) .$$

η-twisted cyclic $2n$-cocycles: $\qquad \tau_n : \mathcal{A}^{2n+1} \to \mathbb{C}$

$$\tau_n(a_0, a_1, \ldots, a_{2n}) = \tau_n(\eta(a_{2n}), a_0, \ldots, a_{n-1}) \ ,$$

and Hochschild boundary closed: $\qquad b_\eta \, \tau_n = 0$

$$b_\eta \, \tau_n(a_0, \ldots, a_{2n+1}) := \sum_{j=0}^{2n} (-1)^j \tau_n(a_0), \ldots, a_j a_{j+1}, \ldots, a_{2n+1})$$
$$- \tau(\eta(a_{2n+1}) a_0, a_1, \ldots, a_{2n})$$

The restriction of the Haar state to $\mathcal{A}(\mathbb{C}\mathsf{P}^2_q)$ is the representative of a class in the cohomology $[\tau_0] \in {HC_\eta}^0(\mathcal{A}(\mathbb{C}\mathsf{P}^2_q))$

An element $[\tau_4] \in {HC_\eta}^4(\mathcal{A}(\mathbb{C}\mathsf{P}^2_q))$ is constructed as

$$\tau_4(a_0, \ldots, a_4) := \fint a_0 \mathrm{d}a_1 \wedge_q \ldots \wedge_q \mathrm{d}a_4 \ .$$

A 2-cocycle can be defined in a similar way.
Elements of $\Omega^{1,1}(\mathbb{C}\mathsf{P}^2_q)$ are $\omega = (\alpha, \alpha_4)$, with $\alpha_4 \in \mathcal{A}(\mathbb{C}\mathsf{P}^2_q)$.
Let $\pi : \Omega^{1,1}(\mathbb{C}\mathsf{P}^2_q) \to \mathcal{A}(\mathbb{C}\mathsf{P}^2_q)$ be such that $\pi(\omega) = \alpha_4$;
extend it to a projection $\pi : \Omega^2(\mathbb{C}\mathsf{P}^2_q) \to \mathcal{A}(\mathbb{C}\mathsf{P}^2_q)$
by $\pi(\omega) = 0$ if $\omega \in \Omega^{0,2}$ or $\omega \in \Omega^{2,0}$.

The map $$\tau_2(a_0, a_1, a_2) := \varphi \circ \pi(a_0 \mathrm{d}a_1 \wedge_q \mathrm{d}a_2)$$

is the representative of a class $[\tau_2] \in {HC_\eta}^2(\mathcal{A}(\mathbb{C}\mathsf{P}^2_q))$.

Both classes $[\tau_4]$ and $[\tau_2]$ are proven to be not trivial by pairing them with the monopole projections $P_N = \Psi_N \Psi_N^\dagger$

The pairing of $[\tau_4]$ with P_N :

$$\begin{aligned}\langle [\tau_4] | [P_N] \rangle &= \int\!\!\!\!\!- \, \mathsf{Tr}\Big(P_N (\mathsf{d} P_N)^4 \sigma^N (K_1^{-4} K_2^{-4})^t\Big) \\ &= q^{-2N} \int\!\!\!\!\!- \, \nabla_N^2 \wedge_q \nabla_N^2 \\ &\sim \; [N]^2\end{aligned}$$

For $q = 1$, the integral of the square of the curvature is the *instanton number* of the bundle.

The pairing of $[\tau_2]$ with P_N :

$$\begin{aligned}\langle[\tau_2]|[P_N]\rangle &= \varphi \circ \mathsf{Tr}\Big(\pi(P_N \mathsf{d} P_N \wedge_q \mathsf{d} P_N)\sigma^N(K_1^{-4}K_2^{-4})^t\Big) \\ &= q^{-2N}\varphi \circ \pi(\nabla_N^2) \\ &\sim [N]\end{aligned}$$

At $q = 1$ the integral of the curvature is the *monopole number* of the bundle.

The pairing of $[\tau_0]$ with P_N

$$\begin{aligned}\langle[\tau_0]|[P_N]\rangle &= \varphi \circ \mathsf{Tr}\Big(P_N)\sigma^N(K_1^{-4}K_2^{-4})^t\Big) \\ &= q^{-2N}\end{aligned}$$

At $q = 1$ this is the rank of the bundle.

If q is trascendental, this means that all $[P_N]$ are independent, i.e. the equivariant K_0-group is infinite dimensional

Indeed, were the classes $[P_N]$ not independent, there would exist a sequence $\{k_N\}$ of integers – all zero but for finitely many – such that $\sum_N k_N q^{-2N} = 0$, and q^{-1} would be the root of a non-zero polynomial with integer coefficients.

thank you !!