Central Extensions of Lax Operator Algebras

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Noncommutativity and physics:
Quantum Geometries and Gravity
Bayrischzell May 2009

INTRODUCTION

- algebras of current type, only recently introduced
- generalisations of affine Lie algebras of Krichever-Novikov type – they are generalisations of classical affine Lie algebras
- related to integrable systems
- related to the moduli space of bundles over compact Riemann surfaces

Goal: Classify almost-graded central extensions of these algebras

Joint work with Oleg Sheinman (appeared in Russ. Math. Surveys 63(4), 727-766 (2008)

Geometric Set-Up

Σ a compact Riemann surface,
$A = I \cup O$ disjoint union of finitely many points, I and O non-empty (here only $I = \{P_+\}$ and $O = \{P_-\}$)

Tyurin data: $n \cdot g$ points ($n \in \mathbb{N}$, g genus of Σ)

$$W := \{\gamma_s \in \Sigma \setminus \{P_+, P_-\} \mid s = 1, \ldots, ng\}.$$

$$\gamma_s \mapsto \alpha_s \in \mathbb{C}^n, \qquad T := \{(\gamma_s, \alpha_s) \in \Sigma \times \mathbb{C}^n \mid s = 1, \ldots, ng\}$$

relation to the moduli space of semi-stable framed algebraic vector bundles of rank n and degree $n \cdot g$

fix local coordinates $z_\pm$ at $P_\pm$ and z_s at γ_s, $s = 1, \ldots, ng$

$\mathfrak{g}$ be one of the matrix algebras $\mathfrak{gl}(n)$, $\mathfrak{sl}(n)$, $\mathfrak{so}(n)$, $\mathfrak{sp}(2n)$, or $\mathfrak{s}(n)$ (the algebra of scalar matrices)
Consider meromorphic functions (more precisely trivialisations of sections of a bundle)

$$L : \Sigma \rightarrow \mathfrak{g},$$

which are

1. holomorphic outside $W \cup \{P_+, P_-\}$,
2. have atmost poles of order one (resp. of order two for $\mathfrak{sp}(2n)$) at the points in W,
3. and fulfill certain conditions at W depending on T and $\mathfrak{g}$.

The singularities at W are called weak singularities.

What are the additional properties? (Here only for $\mathfrak{gl}(n)$)
For $s = 1, \ldots, ng$ there exist $\beta_s \in \mathbb{C}^n$ and $\kappa_s \in \mathbb{C}$ such that we get the expansion at $\gamma_s \in W$

$$L(z_s) = \frac{L_{s,-1}}{z_s} + L_{s,0} + \sum_{k>0} L_{s,k} z_s^k$$

with

$$L_{s,-1} = \alpha_s {}^t\beta_s, \quad \operatorname{tr}(L_{s,-1}) = {}^t\beta_s \alpha_s = 0, \quad L_{s,0}\, \alpha_s = \kappa_s \alpha_s.$$

In particular,if $\alpha_s \neq 0$ $L_{s,-1}$ is a rank 1 matrix, and α_s is an eigenvector of $L_{s,0}$.

Algebraic Set-Up

$\mathfrak{sl}(n)$ matrices are trace-less
$\mathfrak{s}(n)$ matrices are scalar matrices
$\mathfrak{so}(n)$ and $^{(}n)$ matrices of the corresponding type, with modified additional conditions.

Theorem

Under the pointwise matrix commutator these objects constitute a Lie algebra, denoted by $\bar{\mathfrak{g}}$ if the finite Lie algebra is denoted by $\mathfrak{g}$.

ALGEBRAIC SET-UP

If all $\alpha_s = 0$ classical KN current algebras.
If $g = 0$ then classical current algebras.

$\mathcal{A}$ associative algebra of meromorphic functions on Σ holomorphic outside of A
$\mathcal{L}$ Lie algebra of meromorphic vector fields on Σ holomorphic outside of A

classical KN current algebra:

$$\overline{\mathfrak{g}} = \mathfrak{g} \otimes \mathcal{A}, \quad [x \otimes f, y \otimes g] := [x, y] \otimes fg$$

$g = 0$, $\Sigma = \mathbb{P}^1(\mathbb{C})$, points $0, \infty$,

$$\mathcal{A} = \mathbb{C}[z, z^{-1}], \quad \overline{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[z, z^{-1}].$$

ALMOST-GRADED STRUCTURE

Grading is important for infinite dimensional Lie algebras
but a weaker concept almost-grading will do

DEFINITION

V an arbitrary Lie algebra is called almost-graded if
(1) $V = \bigoplus_{n\in\mathbb{Z}} V_n$, $\dim V_n < \infty$ as vector space
(2) There exists $L_1, L_2 \in \mathbb{Z}$ such that

$$[V_n, V_m] \subseteq \bigoplus_{h=n+m+L_1}^{n+m+L_2} V_h, \qquad \forall n, m$$

$\mathcal{A}$, $\mathcal{L}$, and the current algebras of KN type are almost-graded.

ALMOST-GRADED STRUCTURE

THEOREM

$\overline{\mathfrak{g}}$ is almost-graded, i.e. $\overline{\mathfrak{g}} = \bigoplus \overline{\mathfrak{g}}_m$, $\dim \overline{\mathfrak{g}}_m = \dim \mathfrak{g}$, *and*

$$[\overline{\mathfrak{g}}_m, \overline{\mathfrak{g}}_n] \subseteq \bigoplus_{h=m+n}^{n+m+M} \overline{\mathfrak{g}}_h$$

The generic bound is $M = g$, the genus of Σ.

Given $X \in \mathfrak{g}$: there exists a unique $X_m \in \overline{\mathfrak{g}}_m$ such that $X_m = Xz_+^m + O(z_+^{m+1})$.

classical situation: we get the well-known grading

CENTRAL EXTENSIONS

- Goal: Construct and classify central extensions of the Lax operator algebras
- Why: Needed by the applications, like regularisation, 2nd quantization, etc.
- Mathematical back-ground: by regularisation we obtain only projective action of $\overline{\mathfrak{g}}$, they correspond to linear actions of a central extension $\widehat{\mathfrak{g}}$
- Strictly speaking: from these application we need only central extensions of $\overline{\mathfrak{g}}$ which allow to extend the almost-grading to $\widehat{\mathfrak{g}}$.

How are central extensions constructed?

$\widehat{\mathfrak{g}} = \overline{\mathfrak{g}} \oplus \mathbb{C}\, t$ as vector space (t is the central element)

$$[\widehat{L_1}, \widehat{L_2}] = \widehat{[L_1, L_2]} + \psi(L_1, L_2)t$$

$\widehat{\mathfrak{g}}$ is a Lie algebra if and only if ψ is a Lie algebra 2-cocycle, i.e.
(1) ψ is antisymmetric
(2) $\psi([L_1, L_2], L_3) + \psi([L_2, L_3], L_1) + \psi([L_3, L_1], L_2) = 0$.

Two different central extensions are equivalent iff difference of the two 2-cocycles is a coboundary (ϕ a linear form)

$$\psi_1(L_1, L_2) - \psi_2(L_1, L_2) = \phi([L_1, L_2])$$

Hence, we need 2-cocycles

For current type KN algebras: $(x, y \in \mathfrak{g}, g, h \in \mathcal{A})$

$$\psi(x \otimes g, y \otimes h) = \langle x, y \rangle \int_C g dh.$$

$\langle ., . \rangle$ invariant symmetric bilinear form,
C a closed contour on $\Sigma \setminus A$.

For Lax operator algebras we do not have such a splitting
our *functions* are not really functions but sections,
before defining a differentiation we need to choose a
connection.

Central extensions

The connection ∇^ω is defined with the help of ω

(1) a $\mathfrak{g}$-valued meromorphic 1-form

(2) holomorphic outside of A and W

(3) obey certain conditions at the weak singularity points:

points $\gamma_s \in W$ with $\alpha_s = 0$: ω is regular there

points γ_s with $\alpha_s \neq 0$: the expansion

$$\omega(z_s) = \left(\frac{\omega_{s,-1}}{z_s} + \omega_{s,0} + \sum_{k \geq 1} \omega_{s,k} z_s^k \right) dz_s.$$

there exist $\tilde{\beta}_s \in \mathbb{C}^n$ and $\tilde{\kappa}_s \in \mathbb{C}$ such that

$$\omega_{s,-1} = \alpha_s {}^t\tilde{\beta}_s, \quad \omega_{s,0}\,\alpha_s = \tilde{\kappa}_s {}^t\alpha_s, \quad \mathrm{tr}(\omega_{s,-1}) = {}^t\tilde{\beta}_s \alpha_s = 1.$$

Such ω exist
we can choose an ω holomorphic at P_+

$$\nabla^{(\omega)} = d + [\omega, .]$$

covariant derivative

$$\nabla_e^{(\omega)} = dz(e)\frac{d}{dz} + [\omega(e), .], \quad e \in \mathcal{L}$$

THEOREM

The covariant derivative makes $\overline{\mathfrak{g}}$ to an almost-graded Lie module over $\mathcal{L}$.

Define

$$\gamma_{1,\omega,C}(L,L') = \frac{1}{2\pi \mathrm{i}} \int_C \mathrm{tr}(L \cdot \nabla^{(\omega)} L'), \qquad L, L' \in \overline{\mathfrak{g}},$$

and

$$\gamma_{2,\omega,C}(L,L') = \frac{1}{2\pi \mathrm{i}} \int_C \mathrm{tr}(L) \cdot \mathrm{tr}(\nabla^{(\omega)} L'), \qquad L, L' \in \overline{\mathfrak{g}}.$$

indeed these are cocycles
$\gamma_{2,\omega,C}$ does not depend on ω, vanishes for $\mathfrak{g} \neq \mathfrak{gl}(n), \mathfrak{s}(n)$
$\gamma_{1,\omega,C}$ for different ω are cohomologous
cocycles depend on the integration path

Geometric Cocycles

Definition

A cocycle γ for $\overline{\mathfrak{g}}$ is called $\mathcal{L}$*-invariant* (with respect to ω) if

$$\gamma(\nabla_e^{(\omega)} L, L') + \gamma(L, \nabla_e^{(\omega)} L') = 0, \qquad \forall e \in \mathcal{L}, \quad \forall L, L' \in \overline{\mathfrak{g}}.$$

Definition

A cocycle γ for $\overline{\mathfrak{g}}$ is called *local* if there exists $M_1, M_2 \in \mathbb{Z}$ such that for all n, m

$$\gamma(\overline{\mathfrak{g}}_m, \overline{\mathfrak{g}}_n) \neq 0, \implies M_1 \leq n + m \leq M_2.$$

Almost-grading can be extended to the central extension if and only if the defining cocycle is local.

Geometric Cocycles

For cohomology classes use the definition if one representative is of this type.
Warning: not all elements in the class of certain type are of this type.

Theorem

The cocycles $\gamma_{1,\omega,C}$ and $\gamma_{2,C}$ are $\mathcal{L}$-invariant.

locality is in general not true.
essentially different integration cycles yield essentially different 2-cocycle classes $\implies$ a lot of non-equivalent central extensions appear
but, denote by C_S an integration cycle separating the point in I from the points in O.

THEOREM

The cocycles $\gamma_{1,\omega,C}$ and $\gamma_{2,C}$ with integration over a separating cycle $C = C_S$ are local.

Question: is the opposite true?
Essentially uniqueness of almost-graded central extensions
Answer:
In the simple case: yes
In the $\mathfrak{gl}(n)$ case: we have to add $\mathcal{L}$-invariance and then obtain a two-dimensional family of central extensions.

$$\gamma_{1,\omega} := \gamma_{1,\omega,C_S}, \qquad \gamma_2 := \gamma_{2,C_S}$$

Main Results

Theorem

If $\mathfrak{g}$ is simple (i.e. $\overline{\mathfrak{g}} = \mathfrak{sl}(n), \mathfrak{so}(n), \mathfrak{sp}(2n)$) then the space of local cohomology classes is one-dimensional. The space will be generated by the class of $\gamma_{1,\omega}$. Every $\mathcal{L}$-invariant local cocycle is a scalar multiple of $\gamma_{1,\omega}$.

Theorem

For $\overline{\mathfrak{g}} = \overline{\mathfrak{gl}}(n)$ the space of local cohomology classes which are $\mathcal{L}$-invariant having been restricted to the scalar subalgebra is two-dimensional. The space will be generated by the classes of the cocycles $\gamma_{1,\omega}$ and γ_2. Every $\mathcal{L}$-invariant local cocycle is a linear combination of $\gamma_{1,\omega}$ and γ_2.

SOME WORDS ON THE PROOF

- start with local and $\mathcal{L}$-invariant cocycle
- use almost-graded structure to show that everything can be reduced to level zero – $\gamma(L_n, L_m)$ is of level $n+m$
- reduced means: $\gamma(L_n, L_m) = 0$ if $n+m > 0$ and is fixed by knowing the values of γ at level zero
- Hence, we only have to show that at level zero it is of the required form.
- Now: we have to get rid of the condition of $\mathcal{L}$-invariance
- abelian part it is o.k. as there we put it into the requirements

SOME WORDS ON THE PROOF

- simple part: we show that in every class there is a $\mathcal{L}$ invariant representative
- for this: consider the Chevalley generators of the finite-dimensional simple Lie algebra
- use almost-gradedness inside of $\overline{\mathfrak{g}}$ and boundedness from above of the cocycle to make cohomologous changes - stay in the same class
- show that for the modified cocycle everything depends on one cocycle value evaluated for a fixed pair of elements
- hence, the cohomogy space is at most one-dimensional
- $\gamma_{1,\omega}$ is a local cocycle which is not a coboundary, hence it is a generator and the space is one-dimensional
- but $\gamma_{1,\omega}$ is also $\mathcal{L}$-invariant.
- gives the proof

SOME WORDS ON THE PROOF

Remark:

For the abelian part $\mathcal{L}$-invariance is really needed. Otherwise, uniqueness can never be true.

Coming from applications (e.g. regularisation of fermionic Fock space representations) $\mathcal{L}$-invariance of the defining 2-cocycle is very often automatic

Reason is that the representation there is in fact a representation of an by the vector field augmented algebra.

First part (start with $\mathcal{L}$-invariance):

$e_p \in \mathcal{L}$, $e_p = z_+^{p+1} \frac{d}{dz_+}$ of degree p

$L_m^r, L_n^s \in \overline{\mathfrak{g}}$ of degree m and n

recall that for $X \in \mathfrak{g}$ we have a unique $X_m \in \overline{\mathfrak{g}}_m$ with
$X_m = X z_+^m + O(z_+^{m+1})$

almost-graded action:
$\nabla_{e_p} L_m^r = m L_{p+m}^r + L'$ with L' of higher order

the $\mathcal{L}$-invariance

$$\gamma(\nabla_{e_p} L_m^r, L_n^s) + \gamma(L_m^r, \nabla_{e_p} L_n^s) = 0$$

implies

$$m\gamma(L_{p+m}^r, L_n^s) + n\gamma(L_m^r, L_{n+p}^s) = \text{cocycle value at higher level}$$

In particular, for $p = 0$

$$(m+n)\gamma(L^r_m, L^s_n) \text{ is of higher level}$$

This shows that for $(m+n) \neq 0$ the value is given by higher level values.
Hence, everything reduces to level zero.

Further analysis shows

$$\gamma(L^r_0, L^s_0) = 0, \qquad \gamma(L^r_n, L^s_{-n}) = n\gamma(L^r_1, L^s_{-1}) + \text{higher level}.$$

Hence, $\gamma(L^r_1, L^s_{-1})$ fixes everything.

Given γ consider the map

$$\psi_\gamma : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}, \quad \psi_\gamma(X, Y) = \gamma(X_1, Y_{-1}).$$

ψ_γ is a symmetric, invariant bilinear form on $\mathfrak{g}$.
For $\mathfrak{g}$ simple it is a multiple of the Cartan-Killing form.

For $\mathfrak{gl}(n) = \mathfrak{s}(n) \oplus \mathfrak{sl}(n)$ the cocycle splits.

As $\mathfrak{s}(n) \cong \mathcal{A}$ we can use a earlier result of mine on the uniqueness of $\mathcal{L}$-invariant cocycles for the abelian part.

for $\mathfrak{g}$ simple in every class there is an $\mathcal{L}$ invariant one:

$E^{\alpha}, E^{-\alpha}, H^{\alpha}$ Chevalley generators of $\mathfrak{g}$

Chevalley-Serre relations for the finite-dimensional $\mathfrak{g}$.

these structure equations are also structure equations in $\overline{\mathfrak{g}}$ modulo higher level terms (comes from the almost-graded structure)

Some more details

by almost-gradedness and boundedness of the cocycle we can change the cocycle in a cohomologous way such that finally zero is an upper bound for nonvanishing level and that for the level < 0 everything is fixed by level zero

this modified cocycle is called normalized cocycle

at level zero all cocycle values (normalized) can be expessed in relation to the cocycle value $\gamma(H_1^\alpha, H_{-1}^\alpha)$ for a single fixed simple root

hence up to coboundary all local cocycles are multiples of a single one