Bayrischzell 15.5.2010

Noncommutativity and Physics: Spacetime Quantum Geometry

Twist deformation quantization, gravity and Einstein spaces

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I recall the main aspects of twist deformation quantization and the asso_ ciated deformed symmetries (e.g. Lorentz covariance, general covariance).

Recall the construction of noncommutative gravity. [Wess group]**

This will be done explicitly using local coordinates.

Show how to glue the local constructions and thus obtain *the* global (coordinate independent) Riemannian connection and its curvature.

Application: Describe a class of noncommutative gravity solutions (NC Einstein spaces)

[P.A. , L. Castellani]
hep-th/0906.2774
J.Geom. Phys 2010

**[P.A.,Blohmann, Dimitrijevic, Meyer, Schupp, Wess]

Motivations:

-The impossibility to test (also with ideal experimens) the structure of space-time at infinitesimal distances leads to relax the usual assumtion of spacetime as a smooth manifold (a continuum of points) and to conceive a more general structure like a lattice or a noncommutative spacetime that naturally encodes a discretized or cell-like structure.

-In a noncommutative geometry a dynamical aspect of spacetime is encoded at a more basic kinematical level.

-It is interesting to understand if on this spacetime one can consistently formulate a gravity theory. I see nc grvity as an effective theory This theory may capture some aspects of a quantum gravity theory.

-It is then also interesting to study solutions of this deformed gravity theory, e.g. NC black holes and cosmological solutions.

NC geometry approaches

• Generators and relations. For example

$$[\widehat{x}^i, \widehat{x}^j] = i\theta^{ij} \qquad \textit{canonical}$$

$$[\widehat{x}^i, \widehat{x}^j] = i f^{ij}{}_k \widehat{x}^k \qquad \textit{Lie algebra}$$

$$\widehat{x}^i \widehat{x}^j - q \widehat{x}^j \widehat{x}^i = 0 \qquad \textit{quantum plane} \tag{1}$$

Quantum groups and quantum spaces are usually described in this way.

• C *algebra completion; representation as operators in Hilbert space

• $\star$-product approach, a new product is considered in the usual space of functions, it is given by a bi-differential operator $B(f,g) \equiv f \star g$ that is associative, $f \star (g \star h) = (f \star g) \star h$. (More precisely we need to extend the space of functions by considering formal power series in the deformation parameter λ).

Example:

$$(f \star h)(x) = \mathrm{e}^{-\frac{i}{2}\lambda\theta^{\mu\nu}\frac{\partial}{\partial x^\mu}\otimes\frac{\partial}{\partial y^\nu}} f(x)h(y)\Big|_{x=y} \, .$$

Notice that if we set

$$\mathcal{F}^{-1} = \mathrm{e}^{-\frac{i}{2}\lambda\theta^{\mu\nu}\frac{\partial}{\partial x^\mu}\otimes\frac{\partial}{\partial y^\nu}}$$

then

$$(f \star h)(x) = \mu \circ \mathcal{F}^{-1}(f \otimes h)(x)$$

The element $\mathcal{F} = \mathrm{e}^{\frac{i}{2}\lambda\theta^{\mu\nu}\frac{\partial}{\partial x^\mu}\otimes\frac{\partial}{\partial y^\nu}}$ is called a twist. We have $\mathcal{F} \in U\Xi \otimes U\Xi$ where $U\Xi$ is the universal enveloping algebra of vectorfields.

[Drinfeld '83, '85]

A more general example of a twist associated to a manifold M is given by:

$$\mathcal{F} = e^{-\frac{i}{2}\lambda\theta^{ab}X_a\otimes X_b}$$

where $[X_a, X_b] = 0$. θ^{ab} is a constant (antisymmetric) matrix.

Another example : a nonabelian example, Jordanian deformations, [Ogievetsky, '93]

$$\mathcal{F} = e^{\frac{1}{2}H\otimes\log(1+\lambda E)}\,, \qquad\qquad [H, E] = 2E$$

We do not consider the most general $\star$-products [Kontsevich], but those that factorize as $\star = \mu \circ \mathcal{F}^{-1}$ where $\mathcal{F}$ is a general Drinfeld twist.

In general, given a noncommutative algebra, we can ask if there is a notion of differential calculus associated to it. We can also try to construct the associated exterior algebra.

Similarly in the star product case it is interesting to consider not only the NC algebra of functions, but also the corresponding deformed tensor algebra and exterior algebra. This is a difficult task for general star products that are quantization of a given Poisson structure.

In case the star product is given by a twist $\mathcal{F}$ then the construction can be done, this is so because of the factorization $\star = \mu \circ \mathcal{F}^{-1}$. More eplicitly, while the Poisson structure associated to a given star product is a 2-polivector field (a tensorfield), the Poisson structure associated to $\star = \mu \circ \mathcal{F}^{-1}$ comes from elements in $U\Xi$.

$[\widehat{x}^i, \widehat{x}^j] = i\theta^{ij}$	*canonical*
$[\widehat{x}^i, \widehat{x}^j] = i f^{ij}{}_k \widehat{x}^k$	*Lie algebra*
$\widehat{x}^i \widehat{x}^j - q \widehat{x}^j \widehat{x}^i = 0$	*quantum plane*

$\star$-Noncommutative Manifold $(M, \mathcal{F})$

M smooth manifold

$\mathcal{F} \in U\Xi \otimes U\Xi$

$\mathcal{A}$ algebra of smooth functions on M

$\Downarrow$

$\mathcal{A}_\star$ noncommutative algebra

Usual product of functions:

$$f \otimes h \xrightarrow{\mu} fh$$

$\star$-Product of functions

$$f \otimes h \xrightarrow{\mu_\star} f \star h$$

$$\mu_\star = \mu \circ \mathcal{F}^{-1}$$

A and $A_\star$ are the same as vector spaces, they have different algebra structure

Example: $M = \mathbf{R}^4$

$$\mathcal{F} = \mathrm{e}^{-\frac{i}{2}\lambda\theta^{\mu\nu}\frac{\partial}{\partial x^\mu}\otimes\frac{\partial}{\partial x^\nu}}$$

$$(f \star h)(x) = \mathrm{e}^{-\frac{i}{2}\lambda\theta^{\mu\nu}\frac{\partial}{\partial x^\mu}\otimes\frac{\partial}{\partial y^\nu}} f(x)h(y)\Big|_{x=y}$$

Notation

$$\mathcal{F} = \mathrm{f}^\alpha \otimes \mathrm{f}_\alpha \qquad\qquad \mathcal{F}^{-1} = \bar{\mathrm{f}}^\alpha \otimes \bar{\mathrm{f}}_\alpha$$

$$f \star h = \bar{\mathrm{f}}^\alpha(f)\,\bar{\mathrm{f}}_\alpha(h)$$

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Notation

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$$f \star h = \overline{\mathrm{f}}^\alpha(f)\,\overline{\mathrm{f}}_\alpha(h)$$

Define $\mathcal{F}_{21} = \mathrm{f}_\alpha \otimes \mathrm{f}^\alpha$ and define the universal R-matrix:

$$\mathcal{R} := \mathcal{F}_{21}\mathcal{F}^{-1} := \mathrm{R}^\alpha \otimes \mathrm{R}_\alpha \quad , \qquad \mathcal{R}^{-1} := \overline{\mathrm{R}}^\alpha \otimes \overline{\mathrm{R}}_\alpha$$

It measures the *noncommutativity* of the $\star$-product:

$$f \star g = \overline{\mathrm{R}}^\alpha(g) \star \overline{\mathrm{R}}_\alpha(f)$$

Noncommutativity is controlled by $\mathcal{R}^{-1}$. The operator $\mathcal{R}^{-1}$ gives a representation of the permutation group on $A_\star$.

$\mathcal{F}$ deforms the geometry of M into a noncommutative geometry.

Guiding principle:

given a bilinear map

$$\begin{aligned} \mu \,:\, X \times Y &\rightarrow Z \\ (\mathsf{x}, \mathsf{y}) &\mapsto \mu(\mathsf{x}, \mathsf{y}) = \mathsf{xy} \,. \end{aligned}$$

deform it into $\mu_\star := \mu \circ \mathcal{F}^{-1}$,

$$\begin{aligned} \mu_\star \,:\, X \times Y &\rightarrow Z \\ (\mathsf{x}, \mathsf{y}) &\mapsto \mu_\star(\mathsf{x}, \mathsf{y}) = \mu(\bar{\mathsf{f}}^{\alpha}(\mathsf{x}), \bar{\mathsf{f}}_{\alpha}(\mathsf{y})) = \bar{\mathsf{f}}^{\alpha}(\mathsf{x})\bar{\mathsf{f}}_{\alpha}(\mathsf{y}) \,. \end{aligned}$$

The action of $\mathcal{F} \in U\Xi \otimes U\Xi$ will always be via the Lie derivative.

$\star$-Tensorfields

$$\tau \otimes_\star \tau' = \bar{\mathsf{f}}^{\alpha}(\tau) \otimes \bar{\mathsf{f}}_{\alpha}(\tau') \,.$$

In particular $h \star v$, $h \star df$, $df \star h$ etc...

$\star$-Forms

$$\vartheta \wedge_\star \vartheta' = \bar{\mathsf{f}}^\alpha(\vartheta) \wedge \bar{\mathsf{f}}_\alpha(\vartheta') \ .$$

Exterior forms are totally $\star$-antisymmetric.

For example the 2-form $\omega \wedge_\star \omega'$ is the $\star$-antisymmetric combination

$$\omega \wedge_\star \omega' = \omega \otimes_\star \omega' - \bar{\mathsf{R}}^\alpha(\omega') \otimes_\star \bar{\mathsf{R}}_\alpha(\omega) \ .$$

The undeformed exterior derivative

$$\mathsf{d} : A \to \Omega$$

satisfies (also) the Leibniz rule

$$\mathsf{d}(h \star g\ ) = \mathsf{d}h \star g\ + h \star \mathsf{d}g\ ,$$

and is therefore also the $\star$-exterior derivative.

The de Rham cohomology ring is undeformed.

$\star$-Action of vectorfields, i.e., $\star$-Lie derivative

$$\mathcal{L}_v(f) = v(f)$$ usual Lie derivative

$$\begin{array}{rcl} \mathcal{L} \ : \Xi \times A & \longrightarrow & A \\ (v, f) & \longmapsto & v(f)\ . \end{array}$$

deform it into $\mathcal{L}^\star := \mathcal{L} \circ \mathcal{F}^{\,-1}$,

$$\mathcal{L}^\star_v(f) = \bar{\mathsf{f}}^\alpha(v)\Big(\bar{\mathsf{f}}_\alpha(f)\Big)$$ $\star$-Lie derivative

Deformed Leibniz rule

$$\mathcal{L}^\star_v(f \star h) = \mathcal{L}^\star_v(f) \star h + \bar{\mathsf{R}}^\alpha(f) \star \mathcal{L}^\star_{\bar{\mathsf{R}}_\alpha(v)}(h)\ .$$

$\star$-Lie algebra of vectorfields $\Xi_\star$ [P.A., Dimitrijevic, Meyer, Wess, ’06]

Theorem 2. The bracket

$$[u,v]_\star = [\bar{\mathsf{f}}^\alpha(u), \bar{\mathsf{f}}_\alpha(v)]$$

gives the space of vectorfields a $\star$-Lie algebra structure (quantum Lie algebra in the sense of S. Woronowicz):

$$[u,v]_\star = -[\bar{\mathsf{R}}^\alpha(v), \bar{\mathsf{R}}_\alpha(u)]_\star \qquad \star\text{-antisymmetry}$$

$$[u,[v,z]_\star]_\star = [[u,v]_\star, z]_\star + [\bar{\mathsf{R}}^\alpha(v), [\bar{\mathsf{R}}_\alpha(u), z]_\star]_\star \qquad \star\text{-Jacoby identity}$$

$$[u,v]_\star = \mathcal{L}^\star_u(v) \qquad \text{the } \star\text{-bracket is the } \star\text{-adjoint action}$$

Theorem 3. $(U\Xi, \star, \Delta_\star, S_\star, \varepsilon)$ is the universal enveloping algebra of the $\star$-Lie algebra of vectorfields $\Xi_\star$. In particular $[u,v]_\star = u \star v - \bar{\mathsf{R}}^\alpha(v) \star \bar{\mathsf{R}}_\alpha(u)$.

To every undeformed infinitesimal transformation there correspond one and only one deformed infinitesimal transformation: $\mathcal{L}_v \leftrightarrow \mathcal{L}^\star_v$.

Application: Twisted Poincare' symmetry versus spontaneously broken Poincare' symmetry

[Wess], [Chaichian, Kulish, Tureanu]

[Gonera, Kosinski, Maslanka, Giller]
[Meyer, Vazquez-Mozo, Alvarez-Gaume']

[P.A. Dimitrijevic, Kulish, Lizzi, Wess Springer LNP 774, ch. 8]

There are two persepctives on twist deformed NC field theories:

• spontaneously broken symmetry

• deformed (Hopf algebra) symmetry

In the first approach we say that Poincaré symmetry is spontaneously broken to the translations group (the subgroup that leaves invariant the Poisson tensor $\theta^{\mu\nu}\partial_\mu \otimes \partial_\nu$.

In the second approach the breaking of the usual Poincaré symmetry is reinterpreted as the presence of a deformed Poincaré symmetry.

This second approach is more powerful, because among the possible ways we can spontaneaously break a symmetry it singles out the ones that preserve the deformed Poincaré symmetry: the classical symmetry breaking terms must appear only in the $\star$-product.

In the case of gravity we have diffeomorphisms symmetry rather than Poincaré symmetry. This deformed diffeomorphisms symmetry allows us to construct a NC gravity theory.

We can now proceed in our study of differential geometry on noncommutative manifolds.

Covariant derivative:

$$\nabla^\star_{h\star u} v = h \star \nabla^\star_u v \,,$$

$$\nabla^\star_u (h \star v\,) \;=\; \mathcal{L}^\star_u(h) \star v + \bar{\mathsf{R}}^\alpha(h) \star \nabla^\star_{\bar{\mathsf{R}}_\alpha(u)} v$$

our coproduct implies that $\bar{\mathsf{R}}_\alpha(u)$ is again a vectorfield

On tensorfields:

$$\nabla^\star_u (v \otimes_\star z) = \bar{\mathsf{R}}^\alpha(\nabla^\star)_u(\bar{\mathsf{R}}_\alpha(v)) \otimes_\star z + \bar{\mathsf{R}}^\alpha(v) \otimes_\star \nabla^\star_{\bar{\mathsf{R}}_\alpha(u)}(z) \;.$$

The torsion $\mathsf{T}^\star$ and the curvature $\mathsf{R}^\star$ associated to a connection $\nabla^\star$ are defined by

$$\begin{aligned} \mathsf{T}^\star(u,v) &:= \nabla^\star_u v - \nabla^\star_{\bar{\mathsf{R}}^\alpha(v)} \bar{\mathsf{R}}_\alpha(u) - [u,v]_\star \,, \\ \mathsf{R}^\star(u,v,z) &:= \nabla^\star_u \nabla^\star_v z - \nabla^\star_{\bar{\mathsf{R}}^\alpha(v)} \nabla^\star_{\bar{\mathsf{R}}_\alpha(u)} z - \nabla^\star_{[u,v]_\star} z \,, \end{aligned}$$

for all $u, v, z \in \Xi_\star$. Where $[u,v]_\star \equiv \mathcal{L}^\star_u(v)$.

$A_\star$-linearity shows that $\mathsf{T}^\star$ and $\mathsf{R}^\star$ are well defined tensors:

$$\mathsf{T}^\star(f \star u, v) = f \star \mathsf{T}^\star(u, v) \quad , \quad \mathsf{T}^\star(u, f \star v) = \bar{\mathsf{R}}^\alpha(f) \star \mathsf{T}^\star(\bar{\mathsf{R}}_\alpha(u), v)$$

and similarly for the curvature. We also have $\star$-antisymmetry property of $\mathsf{T}^\star(u, v)$ and $\mathsf{R}^\star(u, v, z)$ in u and v.

Local coordinates description

We denote by $\{e_i\}$ a local frame of vectorfields (subordinate to an open $U \subset M$) and by $\{\theta_j\}$ the dual frame of 1-forms:

$$\langle e_i \,,\, \theta^j \rangle_\star = \delta_i^j \ .$$

$$\langle e_i \,,\, \theta^j \rangle_\star = \langle \bar{\mathsf{f}}^\alpha(e_i) \,,\, \bar{\mathsf{f}}_\alpha \theta^j) \rangle \ .$$

Connection coefficients ${\Gamma_{ij}}^k$,

$$\nabla^\star_{e_i} e_j = \Gamma^k_{ij} \star e_k$$

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Connection coefficients ${\Gamma_{ij}}^k$,

$$\nabla^\star_{e_i}e_j=\Gamma_{ij}^k\star e_k$$

$$\mathsf{T}^\star=\frac{1}{2}\theta^j\wedge_\star\theta^i\star{\mathsf{T}^\star}_{ij}{}^l\otimes_\star e_l \ ,$$
$$\mathsf{R}^\star=\frac{1}{2}\theta^k\otimes_\star\theta^j\wedge_\star\theta^i\star{\mathsf{R}^\star}_{ijk}{}^l\otimes_\star e_l \ .$$

Example, $\mathcal{F} = \mathrm{e}^{-\frac{i}{2}\theta^{\mu\nu}\partial_\mu \otimes \partial_\nu}$

$$\nabla^\star_{\partial_\mu}(\partial_\nu) = \Gamma^{\star\rho}_{\mu\nu} \star \partial_\rho$$

$$\mathsf{T}^{\star\rho}{}_{\mu\nu} = \Gamma^{\star\rho}_{\mu\nu} - \Gamma^{\star\rho}_{\nu\mu}$$

$$\mathsf{R}^{\star\sigma}{}_{\rho\mu\nu} = \partial_\mu \Gamma^{\star\sigma}_{\nu\rho} - \partial_\nu \Gamma^{\star\sigma}_{\mu\rho} + \Gamma^{\star\beta}_{\nu\rho} \star \Gamma^{\star\sigma}_{\mu\beta} - \Gamma^{\star\beta}_{\mu\rho} \star \Gamma^{\star\sigma}_{\nu\beta}\,.$$

$$\mathsf{Ric}^\star_{\mu\nu} = \mathsf{R}^{\star\sigma}{}_{\mu\sigma\nu}.$$

$\star$-Riemaniann geometry

$\star$-symmetric elements:

$$\omega \otimes_\star \omega' + \bar{\mathsf{R}}^\alpha(\omega') \otimes_\star \bar{\mathsf{R}}_\alpha(\omega) \ .$$

Any symmetric tensor in $\Omega \otimes \Omega$ is also a $\star$-symmetric tensor in $\Omega_\star \otimes_\star \Omega_\star$, proof: expansion of above formula gives $\bar{\mathsf{f}}^\alpha(\omega) \otimes \bar{\mathsf{f}}_\alpha(\omega') + \bar{\mathsf{f}}_\alpha(\omega') \otimes \bar{\mathsf{f}}^\alpha(\omega)$.

$$g = g_{\mu\nu} \star dx^\mu \otimes_\star dx^\nu$$

$$\Gamma^{\star\rho}_{\mu\nu} = \frac{1}{2}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}) \star g^{\star\sigma\rho}$$

$$g_{\nu\sigma} \star g^{\star\sigma\rho} = \delta^\rho_\nu.$$

Noncommutative Gravity

$$\mathrm{Ric}^\star = 0$$

[P.A., Blohmann, Dimitrijevic, Meyer, Schupp, Wess]

[P.A., Dimitrijevic, Meyer, Wess]

Metric Connections on NC Manifolds

[**P.A., Castellani** J. Geom. Phys. 2010]

Generalize the construction of a metric connection to any NC manifold M with abelian twist

$$\mathcal{F} = e^{-\frac{i}{2}\lambda\theta^{ab}X_a\otimes X_b}$$

where $[X_a, X_b] = 0$. θ^{ab} is a constant (antisymmetric) matrix.

Let $|span\{X_a\}|$ be the dimension of the vectors space spanned by the X_a.

-If $|span\{X_a\}| = const$ then we are in the previous situation. Use $X_a = \frac{\partial}{\partial x^a}$.

-If locally $|span\{X_a\}| = const$ locally we are in the previous situation.

-Problems may arise when one of the vectorfields X_a vanishes, or more in general when $|span\{X_a\}|$ changes. This is a very common case!

Ex. Manin plane

$$x \star y \ = qy \star x$$

is given by the twist $\mathcal{F} = e^{-\frac{i}{2}\theta(x\partial_x \otimes y\partial_y - y\partial_y \otimes x\partial_x)}$.

We call regular points of $\mathcal{F}$ those points $P \in M$ such that there exists an open neighbourhood of P where $|span\{X_a\}|$ is constant.
As in Poisson geometry regular points of $\mathcal{F}$ are an open dense submanifold M_{reg} of M.

Thm. 1 There is a unique noncommutative Levi-Civita (i.e. metric compatible and torsion free) connection $\nabla^\star$ on M_{reg}. Hint: Glue local connections. Transition functions can be chosen to be insensitive to $\star$-product. Christoffel symbols transforms as in the commutative case.

Thm. 2 The noncommutative Levi-Civita connection $\nabla^\star$ extends by continuity from M_{reg} to M.

Rmk.: This is the fundamental theorem of NC Riemannian Geometry!
In other words:

Thm. 2 Given a smooth manifold M with metric g and arbitrary abelian twist $\mathcal{F} = e^{-\frac{i}{2}\lambda\theta^{ab}X_a \otimes X_b}$, there exists a unique noncommutative Levi-Civita connection $\nabla^\star$ on M.

Noncommutative Einstein Manifolds (Gravity solutions)

[**P.A., Castellani** J. Geom. Phys. 2010]

$$\mathrm{Ric}^\star = \Lambda g$$

Example $\mathcal{F} = \mathrm{e}^{-\frac{i}{2}\theta^{\mu\nu}\partial_\mu\otimes\partial_\nu}$

$$\mathrm{Ric}^\star_{\mu\nu} = \Lambda g_{\mu\nu}$$

$$\Gamma^{\star\rho}_{\mu\nu} = \frac{1}{2}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}) \star g^{\star\sigma\rho}$$

$$g_{\nu\sigma} \star g^{\star\sigma\rho} = \delta^\rho_\nu .$$

$$\mathrm{R}^{\star\sigma}{}_{\rho\mu\nu} = \partial_\mu \Gamma^{\star\sigma}_{\nu\rho} - \partial_\nu \Gamma^{\star\sigma}_{\mu\rho} + \Gamma^{\star\beta}_{\nu\rho} \star \Gamma^{\star\sigma}_{\mu\beta} - \Gamma^{\star\beta}_{\mu\rho} \star \Gamma^{\star\sigma}_{\nu\beta} .$$

$$\mathrm{Ric}^\star_{\mu\nu} = \mathrm{R}^{\star\sigma}{}_{\mu\sigma\nu} .$$

Approaches:

- perturbative solutions in power of the deformation parameter

-exact solutions, using symmetry arguments.

-topological solutions (e.g. gravitational instantons).

Motivations:

-understand noncommutative Einstein Manifolds

-Cosmological solutions. Black hole solutions.

We consider here exact solutions. These solutions are present when the NC and the metric structures are compatible. Inspired by P. Schupp talks [Bayrischzell 2007, Vienna 2007] where NC and metric structures were related so that the $\star$-product would "drop out" of NC equations. This has led to study exact NC black hole solutions. Symmetry considerations and compatibility conditions between metric and twist structures are also present in [Ohl, Schenkel] JHEP 2008.

Related work has simultaneously appeared in [Schupp, Solodukhin 0906.2724]
[Ohl, Schenkel 0906.2730]

Solutions I. Moyal-Weyl $\star$-product

Thm. Given the twist $\mathcal{F} = \mathrm{e}^{-\frac{i}{2}\theta^{\mu\nu}\partial_\mu\otimes\partial_\nu}$ consider all undeformed metrics g such that the associated Killing Lie algebra g_K has the twist compatibility property

$$\theta^{\mu\nu}\partial_\mu \otimes \partial_\nu \in \Xi \otimes g_K + g_K \otimes \Xi$$

Then:

i) the NC Riemann tensor and the NC Ricci tensor of the NC Levi-Civita connection are the undeformed ones.

ii) If these metric are Einstein metrics then they are also NC Einstein metrics.

Key point: the star product disappears from the NC Einstein equation.

Notice that even if curvature and Einstein tensors coincide with undeformed ones, we have $\nabla^\star \neq \nabla$. (Only the NC and commutative Christoffel symbols in the special "Moyal-Weyl" basis are equal).

Solutions II. Generalize the previous construction to any NC manifold M with abelian twist

$$\mathcal{F} = e^{-\frac{i}{2}\lambda\theta^{ab}X_a \otimes X_b}$$

where $[X_a, X_b] = 0$. θ^{ab} is a constant (antisymmetric) matrix.

Thm. 3 Given the NC manifold M with abelian twist $\mathcal{F} = \mathrm{e}^{-\frac{i}{2}\theta^{ab}X_a\otimes X_b}$, consider all metrics g such that the associated Killing Lie algebra g_K has the twist compatibility property

$$\theta^{ab}X_a \otimes X_b \in \Xi \otimes g_K + g_K \otimes \Xi$$

Then:

i) the NC Riemann tensor and the NC Ricci tensor of the NC Levi-Civita connection are the undeformed ones, (while $\nabla^\star \neq \nabla$).

ii) If these metric are Einstein metrics then they are also NC Einstein metrics.

Special case: Isospectral deformations obtained via an isometric torus action [Connes, Landi], [Connes, Dubois-Violette].

Solutions III. The general twist case.

Thm. Given the noncommutative manifold $(M, \mathcal{F})$, consider all undeformed Einstein metrics g such that the associated Killing Lie algebra g_K has the twist compatibility property

$$\mathcal{F} \in U g_K \otimes U g_K \tag{4}$$

Then these undeformed Einstein metrics are also noncommutative Einstein metrics.

Proof. In this case $\nabla^\star = \nabla$. Then it follows that the NC curvature = undeformed curvature.

We can more in general consider g_K the Lie algebra of conformal Killing vector fields.

Rmk. We have studied gravity solutions corresponding to three cases of twists I (Moyal Weyl), II (arbitrary abelian) , III (general). Similar arguments apply also to NC gravity solutions in the first order formalism studied in [P.A., Castellani JHEP 2009].