Schwarzschild Geometry Emerging from Matrix Models

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Outline

Matrix models of Yang-Mills type

$$S_{YM} = -\mathrm{Tr}[X^a, X^b][X^c, X^d]\eta_{ac}\eta_{bd}$$
$$\rightarrow \text{ e.o.m: } [X^a, [X^b, X^c]]\eta_{ab} = 0$$

- X^a are Hermitian matrices acting on a Hilbert space $\mathcal{H}$, and η_{ab} is D dimensional flat background metric — fixes signature
- simplest solution of e.o.m.: $[X^a, X^b] = i\theta^{ab} =$ constant $\Rightarrow$ flat Groenewold-Moyal space $\mathbb{R}_\theta$
- $X^a = \left(X^\mu, \Phi^i\right)$, $\mu = 1, \ldots, 2n$, $i = 1, \ldots, D-2n$, so that $\Phi^i(X) \sim \phi^i(x)$ define embedding $\mathcal{M}^{2n} \hookrightarrow \mathbb{R}^D$ (in semi-classical limit)

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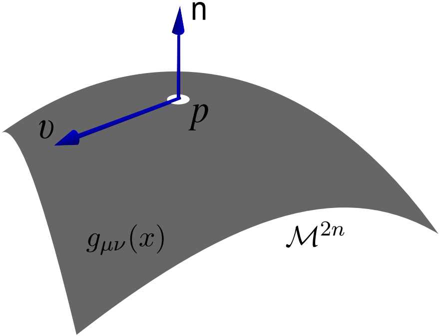
Yang-Mills Matrix models II



- induced metric of $2n$ dimensional submanifold $\mathcal{M}^{2n} \in \mathbb{R}^D$

$$
\begin{aligned}
g_{\mu\nu}(x) &= \partial_\mu x^a \partial_\nu x^b \eta_{ab} \\
&= \eta_{\mu\nu} + \partial_\mu \phi^i \partial_\nu \phi^j \eta_{ij}
\end{aligned}
$$

Yang-Mills Matrix models III

- $\mathcal{M}^{2n}$ endowed with a Poisson structure
  $-i[X^\mu, X^\nu] \sim \{x^\mu, x^\nu\}_{PB} = \theta^{\mu\nu}(x)$
  $\Rightarrow$ "effective" metric

$$G^{\mu\nu} = e^{-\sigma}\theta^{\mu\rho}\theta^{\nu\sigma}g_{\rho\sigma} = -(\mathcal{J}^2)^\mu_\rho g^{\rho\nu} \quad , \qquad e^{-\sigma} \equiv \frac{\sqrt{\det \theta^{-1}_{\mu\nu}}}{\sqrt{\det G_{\rho\sigma}}}$$

- special case: $2n = 4 \qquad \Rightarrow \qquad \det G_{\mu\nu} = \det g_{\mu\nu}$
- opens possibility for special class of geometries where $G_{\mu\nu} = g_{\mu\nu} \quad \leftrightarrow \quad \mathcal{J}^2 = -1$
- corresponds to a self-dual symplectic form $\theta^{-1}_{\mu\nu}$, i.e. $\Theta = \frac{1}{2}\theta^{-1}_{\mu\nu}dx^\mu \wedge dx^\nu \,, \qquad \star\Theta = \pm i\Theta$
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Yang-Mills Matrix models IV

- example: scalar field ϕ on $\mathcal{M}^4$ in the semi-classical limit where $X^a \sim x^a$ are mere coordinates

$$
\begin{aligned}
S[\phi] &= -\mathrm{Tr}[X^a, \phi][X^c, \phi]\eta_{ac} \\
&\sim \int d^4x \sqrt{\det \theta^{-1}_{\mu\nu}}\, \{X^a, \phi\}_{PB} \{X^c, \phi\}_{PB}\, \eta_{ac} \\
&= \int d^4x \sqrt{\det G_{\mu\nu}}\, e^{-\sigma}\, \theta^{\mu\nu} \underline{\partial_\mu x^a} \partial_\nu \phi\, \theta^{\rho\sigma} \underline{\partial_\rho x^c} \partial_\sigma \phi\, \underline{\eta_{ac}} \\
&= \int d^4x \sqrt{\det G_{\mu\nu}}\, G^{\nu\sigma} \partial_\nu \phi \partial_\sigma \phi\,,
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- natural vector fields: $e^a(f) := -i[X^a, f] \sim \theta^{\mu\nu}\partial_\mu x^a \partial_\nu f$

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Yang-Mills Matrix models V

- Also possible to add $U(N)$ gauge fields A to the matrix model (for simplicity, consider only 4 dimensions):

$$Y^\mu = X^\mu - \theta^{\mu\nu} A_\nu \qquad \text{“covariant coordinates”}$$

where the A_μ are some $U(N)$ valued fields.

- Field strength tensor appears in semiclassical limit of commutator:

$$[Y^\mu, Y^\nu] \sim i \left(1 - \theta^{\rho\sigma} A_\sigma \partial_\rho\right) \theta^{\mu\nu} - i\theta^{\mu\rho}\theta^{\nu\sigma} F_{\rho\sigma}$$

$\rightarrow S_{YM} \sim$ Yang-Mills action

- Turns out, that this describes $SU(N)$ gauge fields coupled to gravity — where $U(1)$ gauge field becomes geometrical d.o.f. (see e.g. review of H. Steinacker, arXiv:1003.4134)

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Energy-momentum tensor when $G_{\mu\nu} = g_{\mu\nu}$

$S_{YM} \propto \mathrm{Tr}[X^a, X^b]^2$:

$$T^{ab} = H^{ab} - \frac{H}{4}\eta^{ab}\,,$$
$$H^{ab} = \frac{1}{2}\Big[[X^a, X^c], [X^b, X^{c'}]\Big]_+ \eta_{cc'}\,,$$
$$H = H^{ab}\eta_{ab}\,,$$

matrix ward-identity: $[X^a, T^{a'b}]\eta_{aa'} = 0$

- *semiclassical limit:*

$$T^{ab} \sim e^{\sigma}\,\mathcal{P}_N^{ab}\,, \qquad H^{ab} \sim -e^{\sigma}\,\mathcal{P}_T^{ab}\,,$$
$$\mathcal{P}_T^{ab} = g^{\mu\nu}\partial_\mu x^a \partial_\nu x^b\,, \qquad \mathcal{P}_N^{ab} = \eta^{ab} - \mathcal{P}_T^{ab}$$

where $\mathcal{P}_{N,T}$ are the projectors on the normal resp. tangential space at $p \in \mathcal{M}^4$. This means that

$$\mathcal{P}_T^{ab}\partial_\mu x_b = \partial_\mu x^a\,, \qquad \mathcal{P}_N^{ab}\partial_\mu x_b = 0\,,$$
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Curvature

Can use projector $\mathcal{P}_N$ to write down covariant derivatives $\nabla \equiv \nabla_g$, i.e.

$$\begin{aligned}\mathcal{P}_N^{ab}\partial_\mu\partial_\nu x_b = (\eta^{ab} - \mathcal{P}_T^{ab})\partial_\mu\partial_\nu x_b &= (\partial_\mu\partial_\nu - \Gamma^\rho_{\mu\nu}\partial_\rho)x^a \\ &= \nabla_\mu\nabla_\nu x^a\end{aligned}$$

from which follows $\nabla_\mu x^a \nabla_\nu\nabla_\rho x_a = 0$ and $\mathcal{P}_N^{ab}\nabla_\mu\nabla_\nu x_b = \nabla_\mu\nabla_\nu x^a$.

$\Rightarrow$ Riemann tensor:

$$\begin{aligned}R_{\rho\sigma\nu\mu} = R_{\rho\sigma\nu}{}^\tau\partial_\tau x^a\partial_\mu x_a &= [\nabla_\rho, \nabla_\sigma]\nabla_\nu x^a\nabla_\mu x_a \\ &= \nabla_\sigma\nabla_\mu x^a\nabla_\rho\nabla_\nu x_a - \nabla_\sigma\nabla_\nu x^a\nabla_\mu\nabla_\rho x_a \quad (\rightarrow \text{G.-C. theo.}) \\ &= \mathcal{P}_N^{ab}\left(\partial_\sigma\partial_\mu x_a\partial_\rho\partial_\nu x_b - \partial_\sigma\partial_\nu x_a\partial_\mu\partial_\rho x_b\right)\end{aligned}$$

$$R = \Box_g x^a\Box_g x_a - \nabla_\mu\nabla_\nu x^a\nabla^\mu\nabla^\nu x_a$$

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One-loop effective action

D. Klammer and H. Steinacker, *JHEP* **02** (2010) 074:

$$S_\Psi = -\mathrm{Tr}\left(\frac{1}{4}[X^a, X^b][X_a, X_b] + \frac{1}{2}\bar{\psi}\gamma_a[X^a, \Psi]\right)$$

$$\Rightarrow \Gamma_\Psi = \frac{k}{16\pi^2}\int\sqrt{g}\left[4\Lambda^4 + \Lambda^2\Big(-\frac{1}{3}R + \frac{1}{4}\partial^\mu\sigma\partial_\mu\sigma + \frac{1}{8}e^{-\sigma}\theta^{\mu\nu}\theta^{\rho\sigma}R_{\mu\nu\rho\sigma} + \frac{1}{4}\Box_g x^a\Box_g x_a\Big) + \mathcal{O}(\log\Lambda)\right]$$

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Extensions to the matrix model action

- compare with:

$$S_6 = \mathrm{Tr}\left(\alpha \Box X^a \Box X_a + \frac{\beta}{2}[X^c,[X^a,X^b]][X_c,[X_a,X_b]]\right)$$
$$\sim \frac{\alpha+\beta}{(2\pi)^2}\int \sqrt{g}\, e^{\sigma} \Box_g x^a \Box_g x_a$$
$$+ \frac{\beta}{(2\pi)^2}\int \sqrt{g}\left(\frac{1}{2}\theta^{\mu\rho}\theta^{\eta\alpha}R_{\mu\rho\eta\alpha} - 2R + e^{\sigma}\partial^{\mu}\sigma\partial_{\mu}\sigma\right)$$

with $\Box X^a \equiv [X^b,[X_b,X^a]]$.

- extensions of order 10

$$S_{\text{E-H}} = \mathrm{Tr}\left(2T^{ab}\Box X_a \Box X_b - T^{ab}\Box H_{ab}\right)$$
$$\sim \frac{2}{(2\pi)^2}\int \sqrt{g}\, e^{2\sigma} R$$

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Deviations from $G = g$

- order 10-terms lead to E-H action only for $G = g$, but variation requires $G_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}$
- d.o.f: ϕ^i and A_μ, i.e.

$$g_{\mu\nu} = \eta_{\mu\nu} + \partial_\mu \phi^i(x) \partial_\nu \phi^j(x) \eta_{ij} \,,$$
$$\theta^{-1}_{\mu\nu} = \bar{\theta}^{-1}_{\mu\nu} + F_{\mu\nu} \,, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

- variations

$$\delta_\phi g_{\mu\nu} = \delta_\phi G_{\mu\nu} =: h^{(\phi)}_{\mu\nu} \,,$$
$$\delta_A \theta^{-1}_{\mu\nu} = \partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu \,, \qquad \to G_{\mu\nu} = g_{\mu\nu} + \delta_A G_{\mu\nu}$$

- Several matrix terms are semiclassically equivalent for $G = g$, but not for $G \neq g$.
  $\to$ further terms possible (work in progress)

Deviations from $G = g$

- order 10-terms lead to E-H action only for $G = g$, but variation requires $G_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}$
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Embedding of Schwarzschild metric

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$$ds^2 = -\left(1 - \frac{r_c}{r}\right) dt_S^2 + \left(1 - \frac{r_c}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

Consider Eddington-Finkelstein coordinates and define:

$$t = t_S + (r^* - r)\,, \qquad r^* = r + r_c \ln\left|\frac{r}{r_c} - 1\right|\,,$$

$$\Rightarrow ds^2 = -\left(1 - \frac{r_c}{r}\right) dt^2 + \frac{2r_c}{r} dt dr + \left(1 + \frac{r_c}{r}\right) dr^2 + r^2 d\Omega^2$$

need 3 extra dimensions:

$$\phi_1 + i\phi_2 = \phi_3 e^{i\omega(t+r)}\,,$$

$$\phi_3 = \frac{1}{\omega}\sqrt{\frac{r_c}{r}}\,, \qquad \text{where } \phi_3 \text{ is time-like}$$

Embedding of Schwarzschild metric

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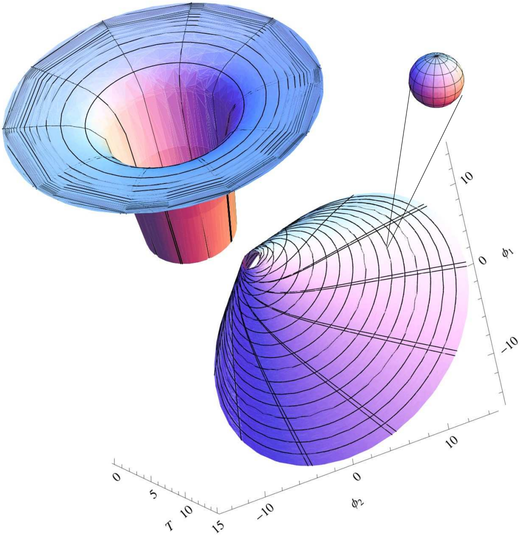
Embedding of Schwarzschild metric II

7-dim. embedding given by

$$x^a = \begin{pmatrix} t \\ r\cos\varphi\sin\vartheta \\ r\sin\varphi\sin\vartheta \\ r\cos\vartheta \\ \frac{1}{\omega}\sqrt{\frac{r_c}{r}}\cos\left(\omega(t+r)\right) \\ \frac{1}{\omega}\sqrt{\frac{r_c}{r}}\sin\left(\omega(t+r)\right) \\ \frac{1}{\omega}\sqrt{\frac{r_c}{r}} \end{pmatrix}$$

with background metric $\eta_{ab} = \text{diag}(-,+,+,+,+,+,-)$.

Embedded Schwarzschild black hole



Symplectic form

Require $\star\Theta = i\Theta$, so that $G^{\mu\nu} = e^{\sigma}\theta^{\mu\rho}\theta^{\nu\sigma}g_{\rho\sigma} = g^{\mu\nu}$ and $\lim_{r\to\infty} e^{-\sigma} = \text{const.} \neq 0$.

Solution:

$$\begin{aligned}
&\Theta = iE\wedge dt_S + B\wedge d\varphi\,,\\
&E = c_1\left(\cos\vartheta dr - r\gamma\sin\vartheta d\vartheta\right) = d(f(r)\cos\vartheta)\,,\\
&B = c_1\left(r^2\sin\vartheta\cos\vartheta d\vartheta + r\sin^2\vartheta dr\right) = \frac{c_1}{2}d(r^2\sin^2\vartheta)\,,\\
&\gamma = \left(1-\frac{r_c}{r}\right)\,, \qquad f(r) = c_1 r\gamma\,, \qquad f' = c_1 = \text{const.}\,,
\end{aligned}$$

from which follows

$$e^{-\sigma} = c_1^2\left(1-\frac{r_c}{r}\sin^2\vartheta\right) \equiv c_1^2 e^{-\bar\sigma}\,.$$

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Darboux coordinates

$x^\mu_D = \{H_{ts}, t_S, H_\varphi, \varphi\}$ corresponding to Killing vector fields $V_{ts} = \partial_{t_s}$, $V_\varphi = \partial_\varphi$ where the symplectic form Θ is constant:

$$\begin{aligned}\Theta &= ic_1 dH_{ts} \wedge dt_S + c_1 dH_\varphi \wedge d\varphi \,, \\ &= c_1 d\left(iH_{ts} dt_S + H_\varphi d\varphi\right) \,,\end{aligned}$$

$$H_{ts} = r\gamma \cos\vartheta \,, \qquad H_\varphi = \frac{1}{2} r^2 \sin^2\vartheta$$

Relations to the Killing vector fields:

$$E = c_1 dH_{ts} = c_1 E_\mu dx^\mu = i_{V_{ts}}\Theta \,, \qquad E_\mu = V^\nu_{ts}\theta^{-1}_{\nu\mu} \,,$$

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Star product

A Moyal type star product can easily be defined as

$$(g \star h)(x_D) = g(x_D) e^{-\frac{i}{2}\left(\overleftarrow{\partial}_\mu \theta_D^{\mu\nu} \overrightarrow{\partial}_\nu\right)} h(x_D)\,,$$

with

$$\theta_D^{\mu\nu} = \epsilon \begin{pmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$

where $\epsilon = 1/c_1 \ll 1$ denotes the expansion parameter.

Star product II

…or in embedding coordinates:

$$(g\star h)(x) = g(x)\exp\left[\frac{i\epsilon}{2}\left(\left(\left(\overleftarrow{\partial}_t\frac{ir_c z e^{\bar\sigma}}{r^2\gamma}+\overleftarrow{\partial}_z i e^{\bar\sigma}\right)\wedge\overrightarrow{\partial}_t\right.\right.\right.$$

$$\left.\left.\left.+\left(\left(\overleftarrow{\partial}_t-\overleftarrow{\partial}_z\tfrac{z}{r}\right)\tfrac{r_c e^{\bar\sigma}}{r^2}+\left(\overleftarrow{\partial}_x x+\overleftarrow{\partial}_y y\right)\tfrac{1}{x^2+y^2}\right)\wedge\left(x\overrightarrow{\partial}_y-y\overrightarrow{\partial}_x\right)\right)\right]h(x)$$

where care must be taken with the sequence of operators and the side they act on.

Higher orders in this star product lead to non-commutative corrections to the embedding geometry, e.g.:

$$\phi_1\star\phi_1+\phi_2\star\phi_2\neq\phi_3\star\phi_3$$

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$$\left.\left. + \left(\left(\overleftarrow{\partial}_t - \overleftarrow{\partial}_z \tfrac{z}{r} \right) \tfrac{r_c e^{\bar{\sigma}}}{r^2} + \left(\overleftarrow{\partial}_x x + \overleftarrow{\partial}_y y \right) \tfrac{1}{x^2+y^2} \right) \wedge \left(x \overrightarrow{\partial}_y - y \overrightarrow{\partial}_x \right) \right) \right] h(x)$$

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Star commutators for Schwarzschild geometry

$$-i\left[x^a \stackrel{\star}{,} x^b\right] = \epsilon e^{\bar\sigma}$$

$$\begin{pmatrix}
0 & -\frac{r_c y}{r^2} & \frac{r_c x}{r^2} & -i & \frac{iz f^+_{12}(1)}{r} & \frac{iz f^-_{21}(1)}{r} & \frac{iz\phi_3}{2r^2} \\
\frac{r_c y}{r^2} & 0 & e^{-\bar\sigma} & -\frac{r_c yz}{r^3} & \frac{-y f^+_{12}(\gamma)}{r} & \frac{-y f^-_{21}(\gamma)}{r} & -\frac{y\gamma\phi_3}{2r^2} \\
-\frac{r_c x}{r^2} & -e^{-\bar\sigma} & 0 & \frac{r_c xz}{r^3} & \frac{x f^+_{12}(\gamma)}{r} & \frac{x f^-_{21}(\gamma)}{r} & \frac{x\gamma\phi_3}{2r^2} \\
i & \frac{r_c yz}{r^3} & -\frac{r_c xz}{r^3} & 0 & -i\omega\phi_2 & i\omega\phi_1 & 0 \\
\frac{-iz f^+_{12}(1)}{r} & \frac{y f^+_{12}(\gamma)}{r} & \frac{-x f^+_{12}(\gamma)}{r} & i\omega\phi_2 & 0 & -\frac{i\omega z\phi_3^2}{2r^2} & \frac{-i\omega z\phi_3\phi_2}{2r^2} \\
\frac{-iz f^-_{21}(1)}{r} & \frac{y f^-_{21}(\gamma)}{r} & \frac{-x f^-_{21}(\gamma)}{r} & -i\omega\phi_1 & \frac{i\omega z\phi_3^2}{2r^2} & 0 & \frac{i\omega z\phi_3\phi_1}{2r^2} \\
-\frac{iz\phi_3}{2r^2} & \frac{y\gamma\phi_3}{2r^2} & -\frac{x\gamma\phi_3}{2r^2} & 0 & \frac{i\omega z\phi_3\phi_2}{2r^2} & \frac{-i\omega z\phi_3\phi_1}{2r^2} & 0
\end{pmatrix}$$

$$+\,\mathcal{O}(\epsilon^3)\,,$$

with

$$f^{\pm}_{ij}(Y) = \left(\frac{Y}{2r}\phi_i \pm \omega\phi_j\right) .$$

Embedding of Reissner-Nordström metric

RN metric in spherical coordinates $x^\mu = \{t, r, \vartheta, \varphi\}$:

$$ds^2 = -\left(1 - \frac{2m}{r} + \frac{q^2}{r^2}\right) d\tilde{t}^2 + \left(1 - \frac{2m}{r} + \frac{q^2}{r^2}\right)^{-1} dr^2 + r^2 d\Omega$$

which has two concentric horizons at

$$r_h = \left(m \pm \sqrt{m^2 - q^2}\right)$$

Shift the time-coordinate according to

$$t = \tilde{t} + (r^* - r)\,, \qquad \text{with } dr^* \equiv \left(1 - \tfrac{2m}{r} + \tfrac{q^2}{r^2}\right)^{-1} dr\,,$$

and arrive at

$$ds^2 = -\left(1 - \frac{2m}{r} + \frac{q^2}{r^2}\right) dt^2 + 2\left(\frac{2m}{r} - \frac{q^2}{r^2}\right) dtdr + \left(1 + \frac{2m}{r} - \frac{q^2}{r^2}\right) dr^2 + r^2 d\Omega\,.$$

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Embedding of RN metric II

10-dimensional embedding $\mathcal{M}^{1,3} \hookrightarrow \mathbb{R}^{4,6}$ with additional coordinates ϕ_i given by

$$\phi_1 + i\phi_2 = \phi_3 e^{i\omega(t+r)}\,, \qquad \phi_3 = \frac{1}{\omega}\sqrt{\frac{2m}{r}}\,,$$

$$\phi_4 + i\phi_5 = \phi_6 e^{i\omega(t+r)}\,, \qquad \phi_6 = \frac{q}{\omega r}$$

ϕ_3, ϕ_4 and ϕ_5 are *time-like* coordinates.

Symplectic form and Darboux coordinates

$$\Theta = \frac{1}{\epsilon}\left(i dH_{\tilde{t}} \wedge d\tilde{t} + dH_\varphi \wedge d\varphi\right) ,$$

$$H_{\tilde{t}} = \gamma\, r \cos\vartheta\,, \qquad H_\varphi = \frac{r^2}{2}\left(1 - \frac{q^2}{r^2}\right)\sin^2\vartheta\,,$$

$$\gamma = \left(1 - \frac{2m}{r} + \frac{q^2}{r^2}\right) ,$$

$$e^{-\bar{\sigma}} = \gamma \sin^2\vartheta + \left(1 - \frac{q^2}{r^2}\right)^2 \cos^2\vartheta$$

$$ds^2_D = -\gamma d\tilde{t}^2 + \frac{e^{\bar{\sigma}}}{\gamma} dH^2_{\tilde{t}} + r^2 \sin^2\vartheta d\varphi^2 + \frac{e^{\bar{\sigma}}}{r^2 \sin^2\vartheta} dH^2_\varphi$$

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$$ds_D^2 = -\gamma d\tilde{t}^2 + \frac{e^{\bar{\sigma}}}{\gamma} dH_{\tilde{t}}^2 + r^2 \sin^2\vartheta d\varphi^2 + \frac{e^{\bar{\sigma}}}{r^2 \sin^2\vartheta} dH_\varphi^2$$

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Star product for RN geometry

A Moyal type star product can again be defined as

$$(g \star h)(x_D) = g(x_D) e^{-\frac{i}{2}\left(\overleftarrow{\partial}_\mu \theta_D^{\mu\nu} \overrightarrow{\partial}_\nu\right)} h(x_D)\,,$$

with the same block-diagonal $\theta^{\mu\nu}$ as before.

. . . and once more, higher orders in the star product lead to non-commutative corrections to the embedding geometry, e.g.:

$$\phi_1 \star \phi_1 + \phi_2 \star \phi_2 \neq \phi_3 \star \phi_3\,,$$
$$\phi_4 \star \phi_4 + \phi_5 \star \phi_5 \neq \phi_6 \star \phi_6\,.$$

Star product for RN geometry

A Moyal type star product can again be defined as

$$(g \star h)(x_D) = g(x_D) e^{-\frac{i}{2}\left(\overleftarrow{\partial}_\mu \theta_D^{\mu\nu} \overrightarrow{\partial}_\nu\right)} h(x_D)\,,$$

with the same block-diagonal $\theta^{\mu\nu}$ as before.

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Star commutators for RN geometry

$$-i\left[x^\mu \stackrel{\star}{,} x^\nu\right] \approx \theta^{\mu\nu} = \epsilon e^{\bar\sigma}$$

$$\begin{pmatrix} 0 & \frac{-(1-\gamma)y}{r} + \frac{iq^2xz}{r^4} & \frac{(1-\gamma)x}{r} + \frac{iq^2yz}{r^4} & -i\beta \\ \frac{(1-\gamma)y}{r} & 0 & e^{-\varsigma} & \frac{-yz\eta}{r^2} \\ \frac{-(1-\gamma)x}{r} & -e^{-\varsigma} & 0 & \frac{xz\eta}{r^2} \\ i\beta & \frac{yz\eta}{r^2} & \frac{-xz\eta}{r^2} & 0 \end{pmatrix}$$

$$-i\left[\phi_i \stackrel{\star}{,} x^\mu\right] \approx \epsilon e^{\bar\sigma}$$

$$\begin{pmatrix} \frac{-iz\alpha f^+_{12}\left(\frac{1}{2}\right)}{r} & \frac{y f^+_{12}\left(\frac{\gamma}{2}\right)}{r} - \frac{iq^2xz\omega\phi_2}{r^4} & \frac{-x f^+_{12}\left(\frac{\gamma}{2}\right)}{r} - \frac{iq^2yz\omega\phi_2}{r^4} & i\omega\phi_2\beta \\ \frac{-iz\alpha f^-_{21}\left(\frac{1}{2}\right)}{r} & \frac{y f^-_{21}\left(\frac{\gamma}{2}\right)}{r} + \frac{iq^2xz\omega\phi_1}{r^4} & \frac{-x f^-_{21}\left(\frac{\gamma}{2}\right)}{r} + \frac{iq^2yz\omega\phi_1}{r^4} & -i\omega\phi_1\beta \\ \frac{-iz\phi_3\alpha}{2r^2} & \frac{y\gamma\phi_3}{2r^2} & \frac{-x\gamma\phi_3}{2r^2} & 0 \\ \frac{-iz\alpha f^+_{45}(1)}{r} & \frac{y f^+_{45}(\gamma)}{r} - \frac{iq^2xz\omega\phi_5}{r^4} & \frac{-x f^+_{45}(\gamma)}{r} - \frac{iq^2yz\omega\phi_5}{r^4} & i\omega\phi_5\beta \\ \frac{-iz\alpha f^-_{54}(1)}{r} & \frac{y f^-_{54}(\gamma)}{r} + \frac{iq^2xz\omega\phi_4}{r^4} & \frac{-x f^-_{54}(\gamma)}{r} + \frac{iq^2yz\omega\phi_4}{r^4} & -i\omega\phi_4\beta \\ \frac{-iz\phi_6\alpha}{r^2} & \frac{y\gamma\phi_6}{r^2} & \frac{-x\gamma\phi_6}{r^2} & 0 \end{pmatrix}$$

Star commutators for RN geometry II

$$-i\left[\phi_i \stackrel{\star}{,} \phi_j\right] \approx \epsilon e^{\bar{\sigma}}$$

$$\begin{pmatrix}
0 & \frac{-i\omega z\phi_3^2\alpha}{2r^2} & \frac{-i\omega z\phi_3\phi_2\alpha}{2r^2} & \frac{-i\omega z\phi_1\phi_5\alpha}{2r^2} & \frac{-i\omega z\alpha g_\phi}{2r^2} & \frac{-i\omega z\phi_3\phi_5\alpha}{r^2} \\
\frac{i\omega z\phi_3^2\alpha}{2r^2} & 0 & \frac{i\omega z\phi_3\phi_1\alpha}{2r^2} & \frac{-i\omega z\alpha g_\phi}{2r^2} & \frac{i\omega z\phi_2\phi_4\alpha}{2r^2} & \frac{i\omega z\phi_3\phi_4\alpha}{r^2} \\
\frac{i\omega z\phi_3\phi_2\alpha}{2r^2} & \frac{-i\omega z\phi_3\phi_1\alpha}{2r^2} & 0 & \frac{i\omega z\phi_3\phi_5\alpha}{2r^2} & -\frac{i\omega z\phi_3\phi_4\alpha}{2r^2} & 0 \\
\frac{i\omega z\phi_1\phi_5\alpha}{2r^2} & \frac{i\omega z\alpha g_\phi}{2r^2} & \frac{-i\omega z\phi_3\phi_5\alpha}{2r^2} & 0 & \frac{-i\omega z\phi_6^2\alpha}{r^2} & \frac{-i\omega z\phi_5\phi_6\alpha}{r^2} \\
\frac{i\omega z\alpha g_\phi}{2r^2} & \frac{-i\omega z\phi_2\phi_4\alpha}{2r^2} & \frac{i\omega z\phi_3\phi_4\alpha}{2r^2} & \frac{i\omega z\phi_6^2\alpha}{r^2} & 0 & \frac{i\omega z\phi_4\phi_6\alpha}{r^2} \\
\frac{i\omega z\phi_3\phi_5\alpha}{r^2} & \frac{-i\omega z\phi_3\phi_4\alpha}{r^2} & 0 & \frac{i\omega z\phi_5\phi_6\alpha}{r^2} & \frac{-i\omega z\phi_4\phi_6\alpha}{r^2} & 0
\end{pmatrix}$$

with

$$f_{ij}^{\pm}(Y) = \left(\frac{Y}{r}\phi_i \pm \omega\phi_j\right), \qquad \alpha = \left(1 - \frac{q^2}{r^2}\right),$$

$$e^{-\varsigma} = \left(\gamma + 2\frac{z^2}{r^2}\left(\frac{m}{r} - \frac{q^2}{r^2}\right)\right), \qquad \beta = \left(1 - \frac{q^2 z^2}{r^4}\right),$$

$$g_\phi = (\phi_3\phi_6 + \phi_1\phi_5) = (\phi_3\phi_6 + \phi_2\phi_4)\,.$$

Conclusion and Outlook

- Have shown, that E-H action can emerge in the framework of matrix models.
- Discussed explicit embeddings of Schwarzschild and RN geometries including self-dual symplectic forms.
- Open questions: deviations from $G = g$, higher order quantum effects, etc. (work in progress).

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Thank you for your attention!