Bosonic Spectral Action from Anomaly Cancellation

Tales from the “other” side

Fedele Lizzi

Università di Napoli Federico II

and Insitut de Ciencies del Cosmo, Universitat de Barcelona

Work in collaboration with A.A. Andrianov (St. Petersburg)

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As you all know there are roughly two (entangled) sides to the noncommutative story

On one side there is the "almost commutative story", attempts to see the standard model, and namely the Higgs, as coming from a noncommutative structure of spacetime, Connes, Lott, Dubois-Violette, Madore, Kerner . . .

In these models spacetime is the usual one, but there is an "internal" noncommutative structure which carries information on the nontrivial symmetries of the model

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And then there is the Wess Side Story

The physics of the standard model plus gravity is inserted in more general programme aimed at translating all concepts of ordinary geometry in an algebraic framework, which opens the possibility to generalize all concepts in the noncommutative framework.

Three main ingredients form the Spectral Triple (plus some seasonings)

- A $\boxed{C^*}$-algebra $\boxed{\mathcal{A}}$ encodes the topology of spacetime
- A Hilbert space $\boxed{\mathcal{H}}$ on which the algebra is represented as bounded operators, and which gives the matter content of the theory
- A Generalization of the Dirac operator $\boxed{D}$ which gives the differential and metric structures, and whose fluctuations give the action
- The seasoning are the chiral structure $\boxed{\gamma}$, and the real structure $\boxed{J}$ given by the generalization of the charge conjugation operator

Connes' approach the standard model is aimed at understanding the geometry of it. To use his words one has to "twist" the geometry to make it fit the standard model and gravity.

The game is then to see which sets of data (an algebra, a Hilbert space, a Dirac operator) reproduce the standard model

The algebra is the product of the algebra of functions on space-time times a finite dimensional matrix algebra

$$\boxed{\mathcal{A} = C(\mathbb{R}^4) \otimes \mathcal{A}_F}$$

Likewise the Hilbert space is the product of fermions times a finite dimensional space which contains all matter degrees of freedom, and also the Dirac operator contains a continuous part and a discrete one

$$\boxed{\mathcal{H} = \mathsf{Sp}(\mathbb{R}^4) \otimes \mathcal{H}_F}$$

$$\boxed{D = \gamma^\mu \partial_\mu \otimes \mathbb{I} + \gamma \otimes D_F}$$

In its most recent form (Chamseddine-Connes-Marcolli) a crucial role is played by the mathematical requirements that the non-commutative algebra satisfies the requirements to be a manifold

Then the internal algebra, is almost uniquely derived to be

$$\boxed{\mathcal{A}_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})}$$

For example if the spacetime is two copies of a manifold the generalization of the electrodynamics action gives a $\boxed{U(1) \times U(1) \to U(1)}$ Higgs mechanism

Instead for an algebra given by functions on spacetime with values in $\boxed{\mathbb{C} \otimes \mathbb{H} \otimes M_3}$, (complex numbers, quaternions, three by three matrices) we obtain the standard model

The finite part of the Dirac operator $\boxed{D_F}$ contains all informations about fermion masses and coupling

Central to this construction is the action, purely based on spectral properties of a covariant Dirac operator

Consider the covariant operator

$$\boxed{D_A = D + A}$$

Where $\boxed{A}$ is the connection which naturally comprises all the fluctuations of the "metric". The internal part of the algebra gives the inner gauge group, while the fluctuations of the continuous part give the Levi-Civita conection

The action is

$$\boxed{S = S_B + S_F = \text{Tr}\,\chi\left(\frac{D_A^2}{\Lambda^2}\right) + \langle\Psi|\, D_A\,|\Psi\rangle}$$

With $\boxed{\Lambda}$ a cutoff in Wilsonian sense, and $\boxed{\chi}$ some possibly smoothened version of the step function

The bosonic action, in the case of $\boxed{\chi}$ the step function, is just the number of eigenvalues smaller than the cutoff

It can be evaluated using heath kernel techniques and the final result gives the action of the standard model coupled with gravity.

The fascinating aspect of this theory is that the Higgs appears naturally as the "vector" boson of the internal noncommutative degrees of freedom. In the process of writing the action all masses and coupling are used as inputs, but one saves one parameter.

The Higgs mass is predicted, in the present form of the model, to be $\boxed{\sim 170\text{GeV}}$. A value too small and experimentally disfavoured.

Technically the bosonic spectral action is a sum of residues and can be expanded in a power series in terms of Λ^{-1} as

$$S_B = \sum_n f_n \, a_n(D^2/\Lambda^2)$$

where the f_n are the momenta of χ

$$\begin{aligned} f_0 &= \int_0^\infty \mathrm{d}x \, x\chi(x) \\ f_2 &= \int_0^\infty \mathrm{d}x \, \chi(x) \\ f_{2n+4} &= (-1)^n \partial_x^n \chi(x) \Big|_{x=0} \quad n \geq 0 \end{aligned}$$

the a_n are the Seeley-de Witt coefficients which vanish for n odd. For D^2 of the form

$$D^2 = g^{\mu\nu}\partial_\mu\partial_\nu \mathbb{1} + \alpha^\mu \partial_\mu + \beta$$

defining

$$\begin{aligned}
\omega_\mu &= \frac{1}{2} g_{\mu\nu} \left(\alpha^\nu + g^{\sigma\rho} \Gamma^\nu_{\sigma\rho} \mathbb{1}\right) \\
\Omega_{\mu\nu} &= \partial_\mu \omega_\nu - \partial_\nu \omega_\mu + [\omega_\mu, \omega_\nu] \\
E &= \beta - g^{\mu\nu} \left(\partial_\mu \omega_\nu + \omega_\mu \omega_\nu - \Gamma^\rho_{\mu\nu} \omega_\rho\right)
\end{aligned}$$

then

$$\begin{aligned}
a_0 &= \frac{\Lambda^4}{16\pi^2} \int \mathrm{d}x^4 \sqrt{g} \, \mathrm{tr} \, \mathbb{1}_F \\
a_2 &= \frac{\Lambda^2}{16\pi^2} \int \mathrm{d}x^4 \sqrt{g} \, \mathrm{tr} \left(-\frac{R}{6} + E\right) \\
a_4 &= \frac{1}{16\pi^2} \frac{1}{360} \int \mathrm{d}x^4 \sqrt{g} \, \mathrm{tr} \, (-12 \nabla^\mu \nabla_\mu R + 5R^2 - 2R_{\mu\nu} R^{\mu\nu} \\
&\quad + 2R_{\mu\nu\sigma\rho} R^{\mu\nu\sigma\rho} - 60RE + 180E^2 + 60\nabla^\mu \nabla_\mu E + 30\Omega_{\mu\nu}\Omega^{\mu\nu})
\end{aligned}$$

tr is the trace over the inner indices of the finite algebra $\mathcal{A}_F$ and in Ω and E are contained the gauge degrees of freedom including the gauge stress energy tensors and the Higgs, which is given by the inner fluctuations of D

There are other problems with this approach, it is Euclidean, fine tuning is needed, the coupling constants all meet in one point the quantization is done in the "commutative" way, which is somehow anticlimactic.

The fact that the model is "ad hoc" and in the end it writes a known action is not a problem. The programme was to fit the standard model into a more general framework, not to derive it form an higher theory

Once the framework is known one can try to understand where it comes from.

The model is probably not yet ready to give trustful experimental predictions, but it is important to constantly update these prediction to understand in which direction the refinements are needed.

The reason to be of the construction is that it is made using the spectral properties of the noncommutative geometry, and as such is it immediately ready for noncommutative generalizations or deformations of spacetime

Wilson's renormalization becomes the fact that the cutoff is just the truncations of the higher eigenvalues of $\boxed{D_A}$, that is the ultraviolet components of the geometry. Since the standard model may be an effective theory, the cutoff may have a physical meaning of the limit of validity of this almost commutative geometry, leading to a fully noncommutative one.

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Or may be the other way around...

I will present a simple calculation which shows how the bosonic part of the spectral action can be obtained as the contribution to the action necessary to cancel the scale anomaly

Notice that in the usual treatment of the spectral action the bosonic part is, given $\boxed{\Lambda}$ already finite, while the fermionic action must be regularised, and this is done using standard techniques

Let me start with just a theory in which some fermions are coupled to some background, this background however is fixed because I have not considered the bosonic part of the action.

I may take the background to be flat, but this is not necessary. Note the similarities with Sakharov emergent gravity at one loop, and Steinacker emergent gravity from matrix models.

The fermionic action appears in the partition function of the theory:

$$\boxed{Z(D) = \int [\mathrm{d}\psi][\mathrm{d}\bar{\psi}]e^{-S_\psi} = \det(D)}$$

where the last equality is formal because the expression is divergent and needs regularizing

The regularization can be done in several ways. In the spirit of noncommutative geometry the most natural one is a truncation of the spectrum of the Dirac operator. This was considered long ago by Andrianov, Bonora, Novozhilov, Vassilevich

The energy cutoff is enforced by considering only the first $\boxed{N}$ eigenvalues of $\boxed{D}$

Consider the projector $\boxed{P_N = \sum_{n=0}^{N} |\lambda_n\rangle \langle\lambda_n|}$ with $\boxed{\lambda_n}$ and $\boxed{|\lambda_n\rangle}$ the eigenvalues and eigenvectors of $\boxed{D}$

The integer $\boxed{N}$ is a function of the cutoff and is defined as

$$\boxed{N = \max n \text{ such that } \lambda_n \leq \Lambda}$$

We effectively use the N^{th} eigenvalue as cutoff

The fermionic action is invariant under a scale transformation:

$$\boxed{x^\mu \to \mathrm{e}^{\phi} x^\mu \ , \psi \to \mathrm{e}^{-\frac{3}{2}\phi}\psi \ , D \to \mathrm{e}^{-\frac{1}{2}\phi} D \mathrm{e}^{-\frac{1}{2}\phi}}$$

for this talk $\boxed{\phi}$ is a constant

The invariance for this scale transformation means that with a redefinition of the units of measurement we can freely change the units of the $\boxed{x}$'s

However (fortunately!) the measure of the partition function is not invariant. This means that we have an anomaly.

The impossibility to go to the shortest distances is a quantum effect, due to the appearance of $\boxed{\hbar}$ in the theory

Hence the fermionic theory alone is anomalous, and we need to add extra ingredients. The anomaly can however be cancelled adding another term to the action

We define the regularized partition function

$$\boxed{Z_\Lambda(D) = \prod_{n=0}^{N} \lambda_n = \det\left(\mathbb{1} - P_N + P_N \frac{D}{\Lambda} P_N\right)}$$

$\boxed{Z_\Lambda}$ has a well defined meaning setting $\boxed{\psi = \sum a_n \,|\lambda_n\rangle \quad , \bar{\psi} = \sum b_n \,|\lambda_n\rangle}$ with $\boxed{a_n, b_n}$ anticommuting (Grassman) quantities, we have

$$\boxed{Z_\Lambda(D) = \int \prod_{n=0}^{N} \mathrm{d}a_n \mathrm{d}b_n \mathrm{e}^{-\sum_{n=0}^{N} b_n \frac{\lambda_n}{\Lambda} a_n} = \det\left(D_N\right)}$$

where we defined $\boxed{D_N = 1 - P_N + P_N \frac{D}{\Lambda} P_N}$ which corresponds to set to $\boxed{1}$ all eigenvalues larger than $\boxed{1}$.

Note that $\boxed{D_N}$ is dimensionless and depends on $\boxed{\Lambda}$ both explicitly and intrinsically via the dependence of $\boxed{N}$ and $\boxed{P_N}$

We can express the spectral action (number of eigenvalues smaller than $\boxed{\Lambda}$) as

$$\boxed{\mathrm{Tr}\,\chi\left(\frac{D^2}{\Lambda^2}\right) = \mathrm{Tr}\,P_N = N}$$

Recall that the $\boxed{N}$ depends on $\boxed{\Lambda}$, on $\boxed{D}$ and also on the function $\boxed{\chi}$, which we have chosen to be a sharp cutoff.

The compensating term, the effective action, is

$$\boxed{Z_{\mathrm{inv}\Lambda}(D) = Z_{\Lambda}(D)\int \mathrm{d}\phi\, \mathrm{e}^{-S_{\mathrm{anom}}}}$$

The calculation is standard and not difficult:

Define

$$Z^{-1}_{\mathrm{inv}\Lambda}(D) = \int \mathrm{d}\phi Z^{-1}_{\Lambda}(\mathrm{e}^{-\frac{1}{2}\phi} D \mathrm{e}^{-\frac{1}{2}\phi})$$

therefore

$$S_{\mathrm{anom}} = \log Z_{\Lambda}(D) Z^{-1}_{\mathrm{inv}\,N}(D)$$

Let us indicate

$$Z_t = Z_{\Lambda}(\mathrm{e}^{-\frac{t}{2}\phi} D \mathrm{e}^{-\frac{t}{2}\phi})$$

therefore $Z_0 = Z_{\Lambda}(D)$ and

$$Z_{\Lambda}(D) Z^{-1}_{\mathrm{inv}\,N}(D) = \int \mathrm{d}\phi \frac{Z_0}{Z_1}$$

and hence

$$S_{\mathrm{eff}} = \int_0^1 \mathrm{d}t \partial_t \log Z_t = \int_0^1 \mathrm{d}t \frac{\partial_t Z_t}{Z_t}$$

We have the following relation that can easily proven:

$$D_N^{-1} = (1 - P_N + P_N D P_N)^{-1} = 1 - P_N + P_N D^{-1} P_N$$

and

$$\begin{aligned}
\partial_t Z_t &= \partial_t \det(\mathrm{e}^{-\frac{t}{2}\phi} D \mathrm{e}^{-\frac{t}{2}\phi})_N \\
&= \partial_t \mathrm{e}^{\mathrm{tr}\,\log(1 - P_N + \mathrm{e}^{-\frac{t}{2}\phi} D_N \mathrm{e}^{-\frac{t}{2}\phi})} \\
&= \mathrm{Tr}\,(\partial_t \log(1 - P_N + \mathrm{e}^{-\frac{t}{2}} D_N \mathrm{e}^{-\frac{t}{2}\phi})) Z_t \\
&= \mathrm{Tr}\,((1 - P_N + \mathrm{e}^{-\frac{t}{2}\phi} D_N \mathrm{e}^{-\frac{t}{2}\phi})^{-1} \phi \mathrm{e}^{-\frac{t}{2}\phi} D_N \mathrm{e}^{-\frac{t}{2}\phi}) Z_t \\
&= \phi Z_t \,\mathrm{tr}\, P_N
\end{aligned}$$

In the end

$$\boxed{S_{\text{anom}} = \int_0^1 \mathrm{d}t\, \phi\, \mathrm{tr}\, P_N}$$

and repeating the calculations above

$$\boxed{S_{\text{anom}} = \int_0^{\phi} \mathrm{d}t' \sum_n \mathrm{e}^{(4-n)t'} a_n f_n = \frac{1}{8}(e^{4\phi} - 1)a_0 + \frac{1}{2}(e^{2\phi} - 1)a_2 + \phi a_4.}$$

For $\boxed{\phi}$ constant this is basically the Spectral action with some numerical corrections to the first two Seeley-de Witt coefficients due to the integration in $\boxed{t\phi}$

The case of $\boxed{\phi}$ noncostant is in progress

So we started from an action which described the behaviour of matter in a gauge and gravitational background, and let the background fluctuate, but not self-interact. The introduction of quantization, and the subsequent anomaly forced us to add another term to the action

The extra term we add turn out to be a (version of) the spectral action, which gives the interaction of matter and gauge fields with itself

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Did God create matter before light?