Geometrical action of the modular group for disjoint intervals in 2D boundary conformal theory.

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work in collaboration with
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Outline:

1. Modular group as a flow of time
   Tomita-Takesaki's theory, KMS condition
2. Algebraic quantum field theory
   Wedges and doubles-cones in Minkowski spacetime
3. Double-cones in 2d boundary conformal field theory
   Conformal field and Longo's ad-hoc state
   Free Fermi fields and the vacuum state

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This often writes

$$\omega(\sigma_s(a)b) = \omega(b\sigma_{s-i}(a)),$$

and this characterizes a thermal equilibrium state at temperature -1.

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But the tangent vector ∂_s to these orbits must be normalised,

$$\partial_t \doteq \frac{\partial_s}{\beta} \text{ with } \beta \doteq \|\partial_s\| = \left\|\partial_t \frac{dt}{ds}\right\| = |\frac{dt}{ds}|.$$

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Defining $\alpha_{-\beta s} \doteq \sigma_s$, the KMS condition yields

$$\omega((\alpha_{-\beta s} a) b) = \omega(b(\alpha_{-\beta s + i\beta} a)),$$

- ω is an equilibrium state at temperature β^{-1} with respect to the time evolution $t = -\beta s$.

2. Algebraic Quantum Field Theory

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A net of algebras of local observables is a map

$$\mathcal{O} \in \mathcal{B}(\text{Minkovski}) \to \mathcal{A}(\mathcal{O})$$

where $\mathcal{A}(\mathcal{O})$'s are C^*-algebras fulfilling

- isotony: $\mathcal{O}_1 \subset \mathcal{O}_2 \Longrightarrow \mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2)$,
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together with an irreducible representation π on an Hilbert space $\mathcal{H}$ such that

- Poincaré covariance: There is a unitary rep. U of the Poincaré group G s.t.

$$U(\Lambda)\pi(\mathcal{A}(\mathcal{O}))U^*(\Lambda) = \pi(\mathcal{A}(\Lambda\mathcal{O}))$$

-Positive energy: P^μ has spectrum in the forward light cone: $p^0 =\geq 0, p^2 \geq 0$.
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Ω defines the *vacuum state* $\omega : a \mapsto \langle \Omega, , a\Omega \rangle$. In the associated GNS representation (*the vacuum representation*) one defines

$$\mathcal{M}(\mathcal{O}) = \pi(\mathcal{A}(\mathcal{O}))''$$

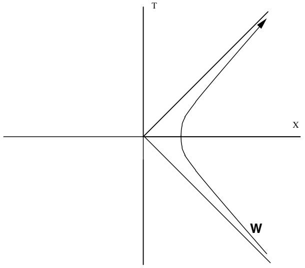
which is *the von Neumann algebra of local observables associated to* $\mathcal{O}$.

Wedge and Unruh temperature

Bisognano, Wichman, Sewell

$$W \longrightarrow \begin{cases} \text{algebra of observables } \mathcal{M}(W) \\ \text{vacuum modular group } \sigma_s^W \to \text{ boosts} \to \text{geometrical action} \end{cases}$$

$$\begin{matrix} \text{uniformly accelerated observer's trajectory} \\ \tau \in]-\infty, +\infty[\end{matrix} \quad = \quad \begin{matrix} \text{orbit of the modular group} \\ s \in]-\infty, +\infty[\end{matrix}$$



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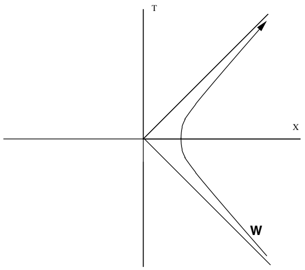

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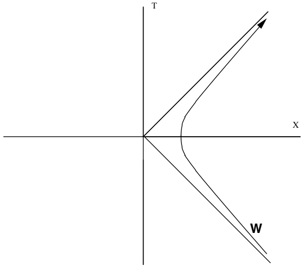

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- The temperature is constant along a given trajectory, and vanishes as $a \rightarrow 0$.

Double-cone in Minkowski space

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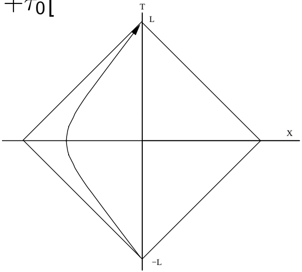

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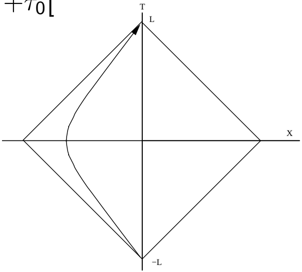

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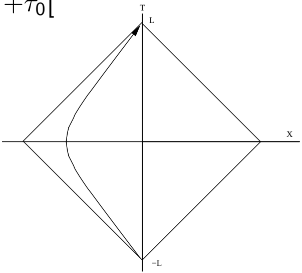

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- $T_D \doteq \frac{1}{\beta}$ is not constant along the orbit, and does not vanish for $a = 0$: $T_D(L)_{a=0} = \frac{\hbar}{\pi k_b L} \simeq \frac{10^{-11}}{L} K \rightarrow$ thermal effect for inertial observer.

3. Double-cone in 2d boundary CFT

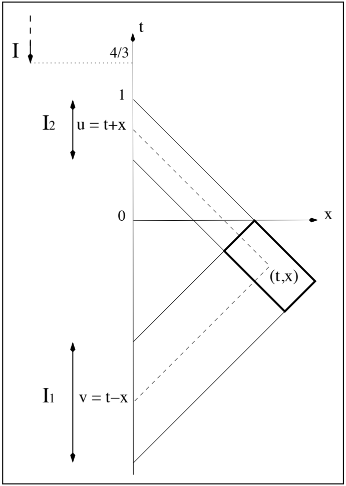
Boundary CFT

Stress energy tensor $T_{ab}, a, b = 1, 2$ is conserved,

$$\partial_0 T_{00} - \partial_1 T_{10} = \partial_0 T_{10} - \partial_1 T_{11} = 0$$

and has vanishing trace. Hence

$$\frac{1}{2}(T_{00} + T_{01}) = T_L(t + x),$$
$$\frac{1}{2}(T_{00} - T_{01}) = T_R(t - x).$$



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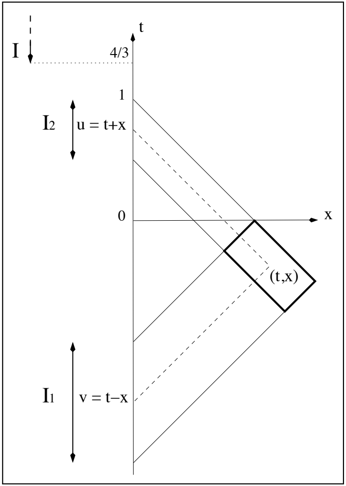

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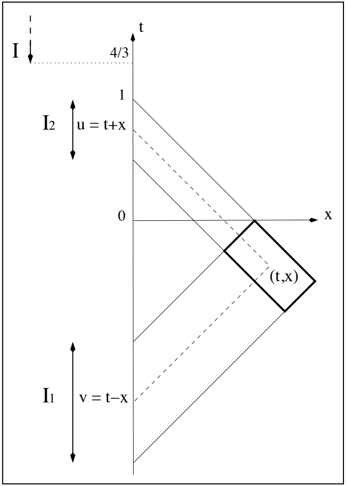
T yields a chiral net of local v.Neumann algebras



$$\mathcal{I} = (A, B) \subset \mathbb{R} \mapsto \mathcal{A}(\mathcal{I}) := \{T(f), T(f)^* : \text{supp } f \subset \mathcal{I}\},$$

and a net of double-cones algebras:

$$\mathcal{O} = I_1 \times I_2 \mapsto \mathcal{M}(\mathcal{O}) \doteq \mathcal{M}(I_1) \vee \mathcal{M}(I_2) \quad \text{where} \quad \mathcal{M}(I_k) = \mathcal{A}(I_k)''.$$

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In Minkowski space, the Poincaré group is both the covariance automorphism group and the group of invariance of the vacuum. Here $\mathcal{A}(\mathcal{I})$ is covariant under an action of Diff(S^1). But the vacuum is only Möbius invariant where

$$\text{Möbius} = PSL(2,\mathbb{R}) = SL(2,\mathbb{R})/\{-1,1\}$$

acts on $\bar{\mathbb{R}}$ as

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}: \quad x \mapsto gx = \frac{ax+b}{cx+d}.$$

From the boundary to the circle

Square and square root:

$$z \mapsto z^2 \quad \Longleftrightarrow \quad x \mapsto \sigma(x) \doteq \frac{2x}{1-x^2},$$

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A pair of symmetric intervals:

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- Two equivalent points of view for the Möbius group: S^1 or $\bar{\mathbb{R}}$:

$$R(\varphi) = \begin{pmatrix} \cos\frac{\varphi}{2} & \sin\frac{\varphi}{2} \\ -\sin\frac{\varphi}{2} & \cos\frac{\varphi}{2} \end{pmatrix}, \ \delta(s) = \begin{pmatrix} e^{\frac{s}{2}} & 0 \\ 0 & e^{\frac{s}{2}} \end{pmatrix}, \ \tau(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix},$$

acting as

$$R(\varphi)z = e^{i\varphi}z \text{ on } S^1, \quad \delta(s)x = e^s x \text{ on } \bar{\mathbb{R}}, \quad \tau(t)x = x + t \text{ on } \bar{\mathbb{R}}.$$

Nets that are not Möbius, but only translational covariant (Longo-Witten 2010).

Modular group

Given a pair of symmetric intervals I_1, I_2 such that $I_1 \cap I_2 = \emptyset$. Consider the state

$$\varphi = (\varphi_1 \otimes \varphi_2) \circ \chi$$

of the algebra $\mathcal{M}(\mathcal{O}) = \mathcal{M}(I_1) \vee \mathcal{M}(I_2)$ where

$\chi : \mathcal{M}(I_1) \vee \mathcal{M}(I_2) \to \mathcal{M}(I_1) \otimes \mathcal{M}(I_2)$ (split property),

$\varphi_k = \omega \circ \mathrm{Ad}U(\gamma_k)$ with ω the vacuum and γ_k a diffeomorphism of S^1 such that $z \mapsto z^2$ on I_k.

Modular group

Given a pair of symmetric intervals I_1, I_2 such that $I_1 \cap I_2 = \emptyset$. Consider the state

$$\varphi = (\varphi_1 \otimes \varphi_2) \circ \chi$$

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The associated modular group has a geometrical action

$$(u, v) \in \mathcal{O} \ \mapsto \ (u_s, v_s) \in \mathcal{O} \qquad s \in \mathbb{R},$$

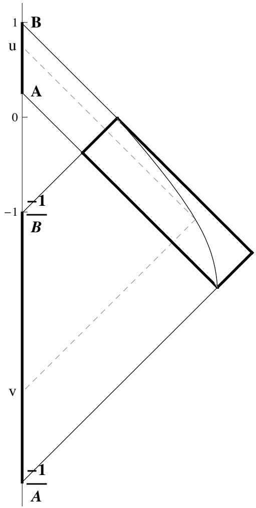
with orbits

$$\begin{aligned} u_s &= \rho_+ \circ m \circ \lambda_s \circ m^{-1} \circ \sigma(u) \in I_2, \\ v_s &= \rho_- \circ m \circ \lambda_s \circ m^{-1} \circ \sigma(v) \in I_1, \end{aligned}$$

where $\lambda_s(x) = e^s x$ is the dilation of $\mathbb{R}$, and m is a Möbius transformation which maps $\mathbb{R}_+$ to $I = \sigma(I_1) = \sigma(I_2)$.

Implicit equation of the orbits:

$$\frac{(u_s - A)(Au_s + 1)}{(u_s - B)(Bu_s + 1)} \cdot \frac{(v_s - B)(Bv_s + 1)}{(v_s - A)(Av_s + 1)} = \text{const},$$



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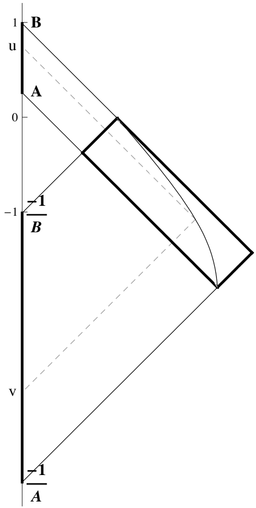

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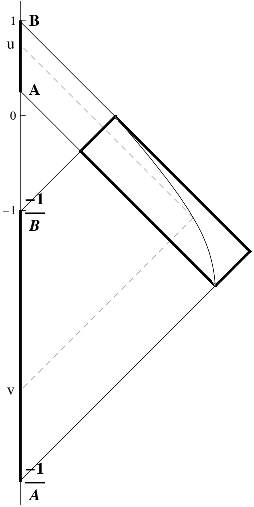

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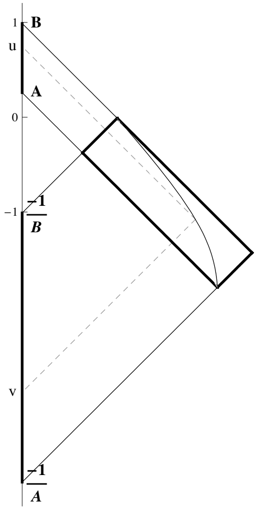

- This equation only depends on the end points of $I_2 = (A, B)$, $I_1 = (-\frac{1}{A}, -\frac{1}{B})$.
- All orbits are time-like, hence $\beta = |\frac{d\tau}{ds}|$ makes sense as a temperature.
- One and only one orbit is a boost (const $= 1$) and thus is the trajectory of a uniformly accelerated observer.

Explicit equation of the orbits:

$I \in \mathbb{R}^+ \Longrightarrow I_2 = (A, B) \subset (0, 1) \Longrightarrow A = \tanh \frac{\lambda_A}{2},\ B = \tanh \frac{\lambda_B}{2}.$

$$u \in\ (A, B)\ = \tanh \frac{\lambda}{2} \quad \text{for } \lambda_A < \lambda < \lambda_B, \quad \sigma(u) = \sinh \lambda,$$

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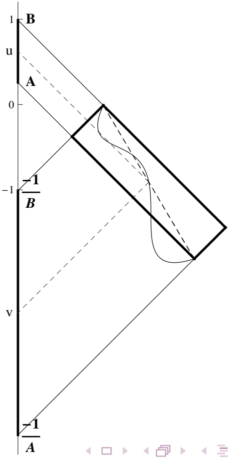

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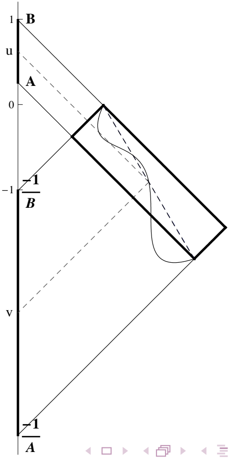

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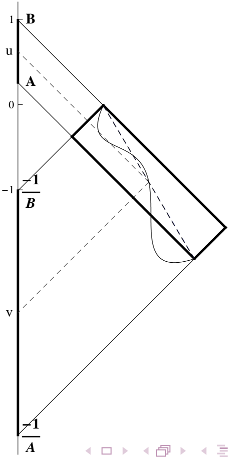
- complicated dynamics (e. g. the sign of the acceleration may change).
- difficult to parametrize such a curve by its proper length τ, hence difficult to find the temperature $\frac{ds}{d\tau}$.



Temperature on the boost trajectory

Constant acceleration: $d\tau^2 = du\,dv$ hence

$$\beta = \frac{d\tau}{ds} = \sqrt{u'v'}$$

with $' = \frac{d}{ds}$. On the boost orbit, $v_s = -\frac{1}{u_s}$ hence

$$\beta = \frac{u'}{u} = \frac{d}{ds}\ln u_s \Longrightarrow \tau(s) = \ln u_s - \ln u_0 \Longrightarrow u_s = u_o e^{\tau(s)}.$$

Knowing

$$u_s' = f_{AB}(u_s) \doteq \frac{(u_s - A)(Au_s + 1)(B - u_s)(Bu_s + 1)}{(B - A)(1 + AB)\cdot(1 + u_s^2)}.$$

one finally gets

$$\beta(\tau) = \frac{f_{AB}(u_o e^{\tau})}{u_o e^{\tau}}.$$

Vacuum modular group for free Fermi fields

A pair of intervals $I_1 = (A_1, B_1)$, $I_2 = (A_2, B_2)$, with $x_1 = v \in I_1$, $x_2 = u \in I_2$. The action of the modular group σ_s of the vacuum, on monomials $\psi(x_i)$ is

$$\sqrt{\frac{dx_i}{d\zeta}}\,\sigma_s(\psi(x_i)) = \sum_{k=1,2} O_{ik}(s)\sqrt{\frac{dx_k}{d\zeta}}\,\psi(x_k(t)), \quad i = 1, 2,$$

where the geometrical action is

$$-\frac{x_i(\zeta) - A_1}{x_i(\zeta) - B_1} \cdot \frac{x_i(\zeta) - A_2}{x_i(\zeta) - B_2} = e^{\zeta}$$

with $\zeta(s) = \zeta_0 - 2\pi s$, and the "mixing" action is determined by the differential equation

$$\dot{O}(s) = K(s)O(s)$$

with

$$K_{ik}(s) = 2\pi \frac{\sqrt{\frac{dx_i}{d\zeta}}\sqrt{\frac{dx_k}{d\zeta}}}{x_i(s) - x_k(s)} \text{ for } i \neq k, \quad K_{ii}(s) = 0.$$

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- The geometrical action is the same as the one in BCFT. The new feature is the mixing between the intervals.

Independant proof:

- because of the unicity of the KMS flow: enough to check that the vacuum is KMS with respect to σ_s.

- because the vacuum is quasi-free, enough to check on the 2-point functions, i.e. compute

$$\omega\left(\sigma_t(\psi(x_i))\sigma_s(\psi(y_j))\right)$$

using the propagator $\omega(\psi(x)\psi(y)) = \frac{-i}{x-y-i\epsilon}$.

One finds

$$\omega(\psi(x_i)\sigma_{-\frac{i}{2}}(\psi(y_j))) = \omega(\psi(y_j)\sigma_{-\frac{i}{2}}(\psi(x_i)))$$

Conclusion

BCFT with Longo's state φ: modular action on disjoint intervals is purely geometric.

free Fermi field with vacuum state ω: modular action on disjoint intervals is a combination of the geometrical action of BCFT and some "mixing terms".

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- Connes cocycle $U^{\omega,\varphi}$ between the vacuum and Longo's ad-hoc state is purely non-geometric.

One of the first examples in which there is an explicit control on the non-geometric part of the modular action.

Hint for modular action in double-cones for non-conformal theories (e.g. massive ones) ?