“Oscillating” quantum groups, pentagon equations and multiplicative unitaries

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joint work with Philippe Bonneau, Francesco D’Andrea and Victor Gayral

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Weights on Lie groups

Definition G : connected real Lie group G with Lie algebra $\mathfrak{g}$.
$\mu \in C^\infty(G, \mathbb{R}_0^+)$ is a **weight** if
(i) $\forall X \in \mathcal{U}(\mathfrak{g}), \exists C_L, C_R > 0$: $|\widetilde{X}.\mu| \leq C_L\,\mu$ and $|X^\star.\mu| \leq C_R\,\mu$.
(ii) $\exists L, R \in \mathbb{N}$ and $C > 0$: $\forall g, h \in G$, $\mu(gh) \leq C\,\mu^L(g)\,\mu^R(h)$.

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$$\left\{F \in C^\infty(G, \mathcal{E}) \;:\; \forall X \in \mathcal{U}(\mathfrak{g}), \forall j \in \mathbb{N}, \exists C > 0 \;:\; \|\widetilde{X}F\|_j \leq C\,\mu_j\right\} .$$

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Lemma (i) $\mathcal{B}^{\{\mu_j\}}(G, \mathcal{E})$ is Fréchet.

(ii) $\mu_j \succ \mu'_j \;\forall j \in \mathbb{N} \Rightarrow \mathrm{closure}_{\mathcal{B}^{\{\mu_j\}}(G,\mathcal{E})}(\mathcal{D}(G, \mathcal{E})) \supset \mathcal{B}^{\{\mu'_j\}}(G, \mathcal{E})$

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$$dS : G \to \mathfrak{g}^\star : x \mapsto \left[\mathfrak{g} \to \mathbb{R} : X \mapsto \mathrm{d}S_x(\widetilde{X}) = (\widetilde{X}.S)(x)\right]$$

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Vector space decomposition: $\mathfrak{g} = \bigoplus_{n=0}^{N} V_n$

$\forall n$: basis $\{e_j^n\}_{j=1,\dots,\dim(V_n)}$ of $V_n \rightsquigarrow x_n^j := (\widetilde{e_j^n}.S)(x)$

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Definition Tempered pair is **admissible**, if $\exists\, X_n \in \mathfrak{S}(V_n) \subset \mathcal{U}(\mathfrak{g})$ with $\widetilde{X}_n e^{iS} =: \alpha_n e^{iS}$ such that

(i) $\exists\ C_n, \rho_n > 0$: $|\alpha_n| \geq C_n\left(1 + |x_n|_n^{\rho_n}\right) \quad (x_n := (x_n^j))$.

(ii) $\exists\ \mu_n \in C^\infty(G, \mathbb{R}_0^+)$ tempered:

(ii.1) $\forall\ A \in \mathrm{alg}_{\mathcal{U}(\mathfrak{g})}(\oplus_{k=0}^{n} V_k)$: $\left|\widetilde{A}\,\alpha_n\right| \leq C_A\,|\alpha_n|\,\mu_n$.

(ii.2) $\forall\ r \leq n$: $\frac{\partial \mu_n}{\partial x_r^j} = 0$.

Oscillating twists

Theorem (G, S): admissible tempered pair, μ: tempered weight, $\mathbf{m} \in \mathcal{B}^{\mu}(G)$ and $\{\mu'_j\}_{j\in\mathbb{N}} \succ \{\mu_j\}_{j\in\mathbb{N}}$: tempered weights. Then $\mathcal{D}(G, \mathcal{E}) \to \mathcal{E} : F \mapsto \int_G \mathbf{m} e^{iS} F$ uniquely continuously extends to

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$(\mathcal{A},\{\|.\|_j\}_{j\in\mathbb{N}})$: Fréchet algebra.

$$\mathcal{R}\otimes\mathcal{R}:C^\infty(G,\mathcal{A})\times C^\infty(G,\mathcal{A})\to C^\infty\big(G\times G,C^\infty(G,\mathcal{A})\big):$$

$$(F,F')\longmapsto\Big[(x,y)\mapsto(R_x^\star F)(R_y^\star F'):=\big[g\mapsto F(gx)F'(gy)\big]\Big]\ .$$

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Theorem $(G\times G,S)$: admissible. $\mathbf{m}\in\mathcal{B}^\mu(G\times G,\mathbb{C})$. Set

$\star_S:=\Big[(F,F')\mapsto\widetilde{\int_{G\times G}}\mathbf{m}e^{iS}\circ\mathcal{R}\otimes\mathcal{R}(F,F')\Big]\,.$ Then:

$$\star_S:\mathcal{B}^{\{\mu_j\}}(G,\mathcal{A})\times\mathcal{B}^{\{\mu'_j\}}(G,\mathcal{A})\to\mathcal{B}^{\{\mu_j^{L_j}\mu_j'^{L'_j}\}_{j\in\mathbb{N}}}(G,\mathcal{A})$$

is a jointly continuous bilinear map.

Schwartz spaces

Definition (G,S): tempered. $\alpha_X := e^{-iS}\,\widetilde{X}\,e^{iS}$.
$\mathcal{S}^S(G) := \left\{ f \in C^\infty(G) \;:\; \left|\alpha_X^n(x)\left(\widetilde{Y}\,f\right)(x)\right| < C \right\}$.

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Proposition Weak associativity $\Rightarrow$ Associativity on $\mathcal{B}^{\{\mu_j\}}(G,\mathcal{A})$.

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Consider $\mathbf{A}_j \in \mathcal{B}^{\mu_j}_{LR}(G\times G)$ $(j=1,2)$ (μ_j tempered).
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Schwartz co-algebras

Lemma Co-product: $\Delta : \mathcal{S}(G) \to \mathcal{M}(\mathcal{S}(G)\hat{\otimes}\mathcal{S}(G))$: $\mathcal{S}(G)$ is a (commutative) multiplier Fréchet-Hopf algebra.

Remark $\Delta : \mathcal{S}^S(G) \to \mathcal{B}^{\mu}_{LR}(G \times G) \subset \mathcal{M}_{\star}(\mathcal{S}(G)\hat{\hat{\otimes}}\mathcal{S}(G))$.

Proposition The condition:

$\int_{G\times G} \Delta^R(\xi\eta)\, K_1(x\xi, y\eta)\, K_2(\xi,\eta)\, \mathrm{d}\xi\, \mathrm{d}\eta \;=\; \delta_{(e,e)}(x,y)$ (INV)

$(\mathrm{d}(x\xi) \;=:\; \Delta^R(\xi)\, \mathrm{d}x)$

implies $\Delta(\varphi \star \psi) \;=\; \Delta(\varphi) \star\!\otimes\!\star\, \Delta(\psi)$ on $\mathcal{S}(G \times G)$.

Deformed Kac-Takesaki operators and pentagon equation

Kac-Takesaki operator:

$W := (1 \otimes m_0)(\Delta \otimes 1) \quad \text{on} \quad \mathcal{S}(G \times G)\ .$

Property: [*pentagon equation*] $W_{12} W_{13} W_{23} = W_{23} W_{12}\ .$

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Definition (G, S): admissible. **Deformed Kac-Takesaki operator**:
$W_\star(\varphi_1 \otimes \varphi_2)(x, y) := \lim_{\psi_n \to 1} (\Delta(\varphi_1) \star (\psi_n \otimes \varphi_2)(x, y))$
(limit in $\mathcal{B}^\mu_{LR}(G \times G)$).

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(limit in $\mathcal{B}^{\mu}_{LR}(G \times G)$).
Theorem Assume:
- $\star_j \ := \ \mu_0 \circ \mathcal{L}_{\mathbf{A}_j}$: weakly associative $(j = 1, 2)$
- Condition (INV).

Then: on $\mathcal{S}(G) \otimes \mathcal{S}(G) \otimes \mathcal{S}(G)$, the operator $W_\star$ satisfies the pentagon equation.

Oscillating twists for Kahlerian Lie groups I: old result

Definition Let $N \in \mathbb{N}_0$.
$\mathbf{\Theta}_N := \{\tau \in C^\infty(\mathbb{R}^N, \mathbb{C}) \mid \exp(\pm\tau) \in \mathcal{O}_C(\mathbb{R}^N, \mathbb{C})\}$
Theorem $\mathbb{B}$: negatively curved rank N Kahlerian Lie group. $\exists$

- $\hat{S} \in C^\infty(\mathbb{B} \times \mathbb{B} \times \mathbb{B}, \mathbb{R})^{\mathbb{B}}$
- $\mathbf{\Theta}_N \to \mathcal{B}^\mu(\mathbb{B} \times \mathbb{B} \times \mathbb{B})^{\mathbb{B}} \; : \; \tau \mapsto \hat{\mathbf{A}}_\tau$
- a left-invariant function sub-space $\mathcal{D}(\mathbb{B}) \subset \mathcal{A}_\tau \subset C_0^\infty(\mathbb{B})$

such that:

(i) the formula ($\theta \in \mathbb{R}_0$):

$$\varphi_1 \star_{\theta,\tau} \varphi_2(x) := \frac{1}{\theta^{\dim \mathbb{B}}} \int_{\mathbb{B}\times\mathbb{B}} \hat{\mathbf{A}}_\tau(x,y,z) e^{\frac{2i}{\theta}\hat{S}(x,y,z)} \varphi_1(y)\varphi_2(z) \mathrm{d}y\mathrm{d}z$$

extends from $\mathcal{D}(\mathbb{B}) \times \mathcal{D}(\mathbb{B})$ to a left-invariant associative algebra structure on $\mathcal{A}_\tau$.

(ii) $\varphi_1 \star_{\theta,\tau} \varphi_2 \;\sim\; \varphi_1\varphi_2 + \frac{\theta}{2i}\{\varphi_1 \, , \, \varphi_2\} + \ldots.$ (formal star-product).

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Theorem Define:

$$S(x,y) := \hat{S}(x,y,e)\ ,\ \mathbf{A}_\tau(x,y) := \hat{\mathbf{A}}_\tau(x,y,e)\ ,\ K_\tau := \frac{1}{\theta^{\dim \mathbb{B}}}\mathbf{A}_\tau e^{\frac{2i}{\theta}S}\ .$$

Then
(i) $(\mathbb{B}\times\mathbb{B}, S)$ is an admissible tempered pair.
(ii) $\tau \in \mathcal{B}^{\mu_0}(\mathbb{R}^N) \ \Rightarrow\ \mathbf{A}_\tau \in \mathcal{B}^{\mu}_{LR}(\mathbb{B}\times\mathbb{B})$.
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Corollary $\star^\tau_\theta\ :=\ \mu_0\circ\mathcal{L}_{\mathbf{A}_\tau}\circ\mathcal{R}_{\mathbf{A}_{-\overline{\tau}}}\ \Rightarrow\ W_{\theta,\tau}\ :=\ W_{\star^\tau_\theta}$ satisfies pentagon equation on $\mathcal{S}(\mathbb{B})\otimes\mathcal{S}(\mathbb{B})\otimes\mathcal{S}(\mathbb{B})$.

Oscillating twists for Kahlerian Lie groups III: Multiplicative unitaries

Kac-Takesaki $W : L^2_r(G) \otimes L^2_r(G) \to L^2_r(G) \otimes L^2_r(G)$ unitary.

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Proposition Assume: τ odd. Then $\forall \theta \in \mathbb{R}^+$, $\exists$ strictly positive pseudo-differential operator P_θ such that:

$$< \varphi_1, \varphi_2 >_{\theta,\tau} := \int_{\mathbb{B}} \overline{\varphi_1} \star^\tau_\theta \varphi_2(x) \, \mathrm{d}_r x \; = \int_{\mathbb{B}} \overline{P_\theta \varphi_1} \, P_\theta \varphi_2(x) \, \mathrm{d}_r x$$

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Definition $\mathcal{H}_{\theta,\tau}$= Hilbert completion of $(\mathcal{S}(\mathbb{B}), < , >_{\theta,\tau})$.
Theorem The operator $W_{\theta,\tau}$ uniquely extends from $\mathcal{S}(\mathbb{B}) \otimes \mathcal{S}(\mathbb{B})$ to $\mathcal{H}_{\theta,\tau} \otimes \mathcal{H}_{\theta,\tau}$ as a *unitary* operator:

$$W_{\theta,\tau} : \mathcal{H}_{\theta,\tau} \otimes \mathcal{H}_{\theta,\tau} \to \mathcal{H}_{\theta,\tau} \otimes \mathcal{H}_{\theta,\tau} \; .$$

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Proposition Set $\mathcal{H}_\star(G) := \mathcal{H}_{\theta,\tau} \otimes L^2(K)$.
Then, the $C(K)_\infty$-valued deformed Kac-Takesaki operator $W_{\theta,\tau}$ extends as a multiplicative unitary operator on $\mathcal{H}_\star(G) \otimes \mathcal{H}_\star(G)$.

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$\rightsquigarrow$ Deformed Minkowski space:
Semi-classical Poisson structure ($e_0 = (1,0,...,0)$):
$\tilde{x}^1 \; := \; x^1 - x^0 \, , \, \tilde{x}^k \; := \; x^k \; (k \neq 1) \; \rightsquigarrow \; \{ \, \tilde{x}^0 \, , \, \tilde{x}^k \, \} \; = \; \tilde{x}^k \; .$