ON THE CONFORMAL ORIGIN OF MASS

M. Francaviglia
L. Fatibene

Department of Mathematics
University of Torino

INFN - Sezione Torino
Iniziativa Specifica Na12

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From Newton to Einstein SR and to GR

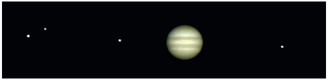
Electromagnetic field is the basis of observation protocols

- Synchronization for ($v << c$): classical mechanics, astronomy
- Synchronization for ($v \sim c$): SR
- Synchronization in GR

From Newton to Einstein SR and to GR

There is no purely mechanical experiment.
In any experiment we use E.M. field (or other physics) to support our observational protocols.

Even when light time travel was practically irrelevant we still use in principle light to reveal, synchronize and often detect events.



Even in XVIII century, one had to account for finite speed of light to synchronize when observing Jupiter's satellites and their eclipses.

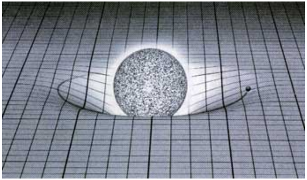
In XX century the fundamental role of synchronization is finally acknowledged in SR and GR.

Newton to Einstein SR and to GR

SR is a good model for *empty* spacetime and *electromagnetic field*

Matter is allowed only as test particles

As soon as a matter particle is added to SR,
it generates gravitational field and it curves spacetime



Einstein took the only way available:

gravity is the metric curvature of spacetime

GR is a theory of spacetime,
electromagnetic field and *matter*.

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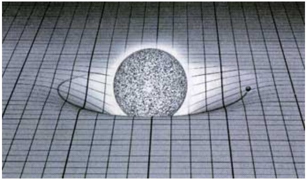

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GR is a theory of spacetime,
electromagnetic field and *matter*.

It is only one possible theory of this kind!

GR corresponds to the choice of a metric connection

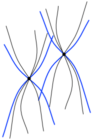
$$\Gamma^{\alpha}_{\mu\nu} = \{g\}^{\alpha}_{\mu\nu}$$

to describe the curvature of spacetime.

Ehlers-Pirani-Schild (EPS)

The curvature of space has to be determined on the bases of astronomical observations.

B. Riemann, *On the Hypotheses that lie at the Foundations of Geometry*, Göttingen 1854



One observes *lightrays* and *particles*, analyzes their properties and derive from them the geometrical structure of spacetime.

Lightrays determine a *Lorentzian conformal structure* $(M, [g])$:

$$[g] = \{\tilde{g} \in \mathrm{Lor}(M) : \exists \varphi,\ \tilde{g} = \varphi(x) \cdot g\} = \{\text{lightcones}\}$$

Particles determine a *projective structure* $(M, [\Gamma])$

$$[\Gamma] = \{\tilde{\Gamma} \in \mathrm{Con}(M) : \exists V_\mu, \tilde{\Gamma}^\alpha_{\mu\nu} = \Gamma^\alpha_{\mu\nu} + \delta^\alpha_{(\mu} V_{\nu)} = \{\Gamma\text{-geodesic trajectories}\}$$

Projective structure

The reason to consider projective structure is that in any homogeneous formalism different parametrizations of the same worldline do represent the same physical motion.

Hence only geodesic *trajectories* are endowed with a direct physical meaning.

One needs a connection to define geodesic motions but only a projective structure is needed for geodesic trajectories.

One can prove that a projective structure is identified by the free falling congruence.

Geodesics

In this context also strong equivalence principle has to be reviewed.

One usually has two structures (one for lightcones and one for free fall) and one should specify for which one SR is expected to be recovered.

Or define geodesics without resorting to SR.

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Just keep the terms that cannot be eliminated in a covariant way independent of the parametrization

One can prove that the equation for Γ-geodesic follows:

$$\ddot{q}^\mu + \Gamma^\mu_{\alpha\beta}\dot{q}^\alpha\dot{q}^\beta = \lambda\dot{q}^\mu$$

Ehlers-Pirani-Schild (EPS)

Conformal and projective structures cannot be independent.

Since the time of Eddington, it is known that photons feel the gravitational field and they follow curved trajectories when masses are present.

This implies that photon trajectories, i.e. the null geodesics of g (or any other conformally equivalent metric), should bear explicit relation with parallel lines of connection Γ determining free fall.

Ehlers-Pirani-Schild (EPS)

Conformal and projective structure must be *compatible*:

Particles can chase lightrays as close as one wishes.
Equivalently:
the g-lightlike g-geodesics are also Γ-geodesics
or: Γ *is "tangent" to lightcones*

An *EPS structure* is a triple $(M, [g], [\Gamma])$ *($[g]$ and $[\Gamma]$ compatible)*

In any EPS structure one can fix canonically the connection Γ for the projective structure such that $\exists A$ a covector:

$$\overset{\Gamma}{\nabla} g = 2A \otimes g$$

A *Weyl geometry* is the triple $(M, [g], \Gamma)$ ($\simeq$ a triple $(M, [g], A)$)

the conformal structure determines lightcones and lightrays

Γ determines particle free fall

Ehlers-Pirani-Schild (EPS)

Non-metric connections have problems of holonomies

EPS tried to add an axiom to constrain the connection to be metric

"Atomic time" and "gravitational time" do coincide

The physical content of this axiom is somehow obscure

But how compelling is time-equality postulate?

Ehlers-Pirani-Schild, 1972

Th: If Γ is metric $\Rightarrow$ it is metric for a conformal metric to g.

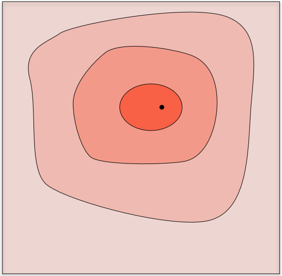
EPS $\Rightarrow$

- Palatini formalism
- Field equations must imply EPS-compatibility
- in $f(R)$ models dynamics $\Rightarrow$ EPS-compatibility (and metricity)
- In vacuum, standard GR is recovered

Relativistic theories, Extended theories, and GR

EPS allows to extend gravitational theories from GR to a much wider class.
At the same time it imposes constraints to possible generalized dynamics.

- Relativistic Theories
- Extended Theories of Gravitation (dynamics $\Rightarrow$ EPS compatibility)
- Extended Metric Theories (dynamics $\Rightarrow$ EPS metric connection)
- $f(R)$ theories
- GR



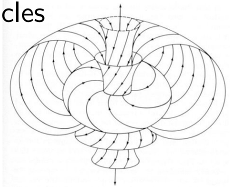
Matter in SR

EPS shows there is much more freedom in choosing free fall than in standard GR on $(M, [g])$.

On Minkowski space $(\mathbb{R}^4, \eta)$ only straight lines are geodesics.

In a Weyl-Minkowski geometry $(\mathbb{R}^4, [\eta], \Gamma)$ one can fix Γ so that any congruence of timelike curves is Γ-geodesic.

One can model a dust (or fluid) of massive particles in a Minkowski background!



Metric informations encoded in $[\eta]$

Gravity encoded in the connection Γ (or A)

Matter in SR

On $(\mathbb{R}^4, \eta)$ consider **any** timelike unit vector n.

$$A := n^\mu \overset{\eta}{\nabla}_\mu n_\nu \, dx^\nu$$

$$\Gamma^\alpha_{\beta\mu} := \{\eta\}^\alpha_{\beta\mu} + \left(\eta^{\alpha\epsilon}\eta_{\beta\mu} - 2\delta^\alpha_{(\beta}\delta^\epsilon_{\mu)}\right) A_\epsilon$$

The integral curves of n are Γ-geodesics.

If A is exact then Γ is metric.

Conservation Laws

Fluids have two conservation laws to be considered

Energy-momentum: $\nabla_\nu T^{\mu\nu} = 0$

$$T_{\mu\nu} = p g_{\mu\nu} + (p + \rho) n_\mu n_\nu$$

Number of particles: $\nabla_\mu J^\mu = 0$

$$J^\mu = \sqrt{g}\, T^{\mu\nu} n_\nu$$

In standard GR: $\nabla_\nu T^{\mu\nu} = 0$ + n is Killing $\Rightarrow \nabla_\mu J^\mu = 0$

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But generically n is not Killing!

Conservation Laws

In Weyl geometry:
energy-momentum conservation implies number of particles conservation iff

$$T^{\mu\nu}\overset{\Gamma}{\nabla}_{\mu}n_{\nu}=0 \qquad T^{\mu\nu}A_{\mu}n_{\nu}=0$$

There is a conformal frame in which these hold

In a sense matter brakes conformal invariance and fixes a conformal gauge in which

$$\nabla_{\nu}T^{\mu\nu}=0 \quad \Rightarrow \nabla_{\mu}J^{\mu}=0$$

even if n is not a Killing vector.

Matter Lagrangians and Conformal Transformations

Klein-Gordon matter field:

$$L_{KG}(g,\varphi) = \frac{\sqrt{g}}{2}\left(\nabla_\mu\varphi\nabla^\mu_{\cdot}\varphi - m^2\varphi^2\right) \qquad \Rightarrow \begin{cases} T_{\mu\nu}(g,\varphi) \\ \Box\varphi + m^2\varphi = 0 \end{cases}$$

Conformal transformation:

$$\tilde{g} = \Phi^2 \cdot g \qquad \tilde{\varphi} = \Phi^{-1}\cdot\varphi$$

$$L_{KG}(\tilde{g},\tilde{\varphi}) = \frac{\sqrt{\tilde{g}}}{2}\left(\tilde{\nabla}_\mu\tilde{\varphi}\tilde{\nabla}^\mu_{\cdot}\tilde{\varphi} - \tilde{m}^2\tilde{\varphi}^2\right) + \mathrm{Div} \qquad \Rightarrow \begin{cases} \tilde{T}_{\mu\nu}(\tilde{g},\tilde{\varphi}) \\ \tilde{\Box}\tilde{\varphi} + \tilde{m}^2\tilde{\varphi} = 0 \end{cases}$$

where we set

$$\tilde{m}^2 = \frac{m^2}{\Phi^2} + \frac{1}{2\Phi^4}\Box\Phi^2 - \frac{1}{4\Phi^4}\tilde{\nabla}_\mu\Phi^2\tilde{\nabla}^\mu_{\cdot}\Phi^2$$

Is the Conformal Frame Observable?

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Could we see the mass of a particle being pointwise?

For example, in $f(R)$-theories in cosmology the conformal factor Φ depends on scalar curvature which depends on matter through the quantity $\rho + 3p$.
Then it is expected to depend on time only, on scales of millennia it is expected to be constant.

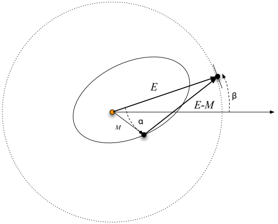
Is the Conformal Frame Observable?

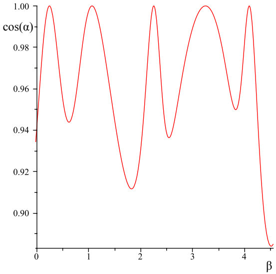
Angles are conformally invariant, distances are not.
EPS framework is conformally invariant ($\tilde{g} = \Phi^2 \cdot g$ and $\tilde{\Gamma} = \Gamma$)

Can a generalized theory pass Solar System tests?
Of course purely electromagnetic tests are conformally invariant.
What about perihelia precession?

Is the Conformal Frame Observable?

Let us graph the angle α between the Sun and Mercury against the angle β between the rotation of the Earth around the Sun ($\cos\alpha = \chi(\beta)$)





Assuming Kepler laws one gets a prediction to be tested

Is the Conformal Frame Observable?

The curve ($\cos\alpha = \chi(\beta)$) is conformally invariant.
By fitting data, one can measure:

- the eccentricity of Mercury orbit,
- the ratio between the distances of Mercury and Earth from the Sun
- the angle θ of the orbit of Mercury with a fixed direction

By keeping observing mercury for centuries one can see the angle θ preceeding as prescribed by GR.
That is true in any $f(R)$-theory!

No scale or mass can be fixed conformally.
The conformal factor can be observed only fixing scales or when it is not constant.

Conclusions

If Einstein had known general connections maybe GR would have developed along different lines

Actually he was well aware that gravity was connected to connections more than to metric
(though he never had an affine relativistic formulation)

Mass is related to conformal invariance