Construction of a 4 D nc ϕ^4 model

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Introduction

There are no nontrivial local QFTs in 4 D. Candidates are:

- trivial (free fields, Φ^4), or
- perturbative renormalizable (Landau ghost,...), or
- not understood (confinement, renormalons, YM).

We show: 4D-ϕ^4-model can be constructed on a NC MF.

- Has broken symmetries, is constructed on Euclidean ST.
- Infinitely many divergent renormalisable Feynman graphs.
- The guiding mathematical structure waits for discovery.

Minkowski space

- Rank 2: Analytic continuation of the one-loop contributions.
- Fixed point occurs in Minkowski ST too. HG + Wohlgenannt
- HG, Lechner, Ludwig, Verch The analytic continuation from Euclidean to Minkowski deformed QFT is possible
- Rank 4 Minkowski??? Fischer, Szabo; Zahn; Bahns

Renormalised noncommutative ϕ^4_4-theory

Moyal space: $(a \star b)(x) = \int d^D y d^D k a(x + \frac{1}{2}\Theta \cdot k) b(x+y)\, e^{iky}$

Renormalisation leads to IR/UV - Mixing, cure it...

ϕ^4-theory on 4D-Moyal space + harmonic oscillator potential

$$S[\phi] = \int d^4x \Big(\frac{Z}{2} \phi \star (-\Delta + \Omega^2 (2\Theta^{-1}x)^2 + \mu_0^2)\phi + \frac{\lambda_0 Z^2}{4} \phi \star \phi \star \phi \star \phi \Big)(x)$$

- renormalisable as formal power series in λ_0 G-Wulkenhaar, 04
  (renormalisation of $\mu_0^2, \lambda_0, Z \in \mathbb{R}_+$ and $\Omega \in [0,1]$)
  means: well-defined perturbative quantum field theory
- Langmann-Szabo duality: theories at Ω and $\Omega^* = \frac{1}{\Omega}$ are the same;
  self-dual case $\Omega = 1$ is matrix model
- β-function vanishes to all orders in λ_0 for $\Omega = 1$
  Disertori-Gurau-Magnen-Rivasseau 06 means: almost scale-invariant

Is the self-dual (critical) model integrable?

Matrix Model

- Action in matrix base at $\Omega = 1$
- Action functionals for *bare* mass μ_{bare}
- Wave function renormalisation $\phi \mapsto Z^{\frac{1}{2}}\phi$.

Fix $\theta = 4$, $\phi_{mn} = \overline{\phi_{nm}}$ real:

$$S = \sum_{m,n \in \mathbb{N}^2_\Lambda} \frac{1}{2}\phi_{mn}H_{mn}\phi_{nm} + V(\phi)\,,$$

$$H_{mn} = Z\left(\mu^2_{bare} + |m| + |n|\right)\,, \qquad V(\phi) = \frac{Z^2\lambda}{4}\sum_{m,n,k,l \in \mathbb{N}^2_\Lambda} \phi_{mn}\phi_{nk}\phi_{kl}\phi_{lm}\,,$$

- Λ is cut-off. μ_{bare}, Z divergent
- No infinite renormalisation of coupling constant

$m, n, \ldots$ belong to $\mathbb{N}^2$, $|m| := m_1 + m_2$.

Ward identity

- inner automorphism $\phi \mapsto U\phi U^\dagger$ of M_Λ, infinitesimally $\phi_{mn} \mapsto \phi_{mn} + \mathrm{i}\sum_{k\in\mathbb{N}^2_\Lambda}(B_{mk}\phi_{kn} - \phi_{mk}B_{kn})$
- not a symmetry of the action, but invariance of measure $\mathcal{D}\phi = \prod_{m,n\in\mathbb{N}^2_\Lambda} d\phi_{mn}$ gives

$$0 = \frac{\delta W}{\mathrm{i}\delta B_{ab}} = \frac{1}{\mathcal{Z}}\int \mathcal{D}\phi \,\Big(-\frac{\delta S}{\mathrm{i}\delta B_{ab}} + \frac{\delta}{\mathrm{i}\delta B_{ab}}(\mathrm{tr}(\phi J))\Big) e^{-S+\mathrm{tr}(\phi J)}$$
$$= \frac{1}{\mathcal{Z}}\int \mathcal{D}\phi \sum_n \Big((H_{nb} - H_{an})\phi_{bn}\phi_{na} + (\phi_{bn}J_{na} - J_{bn}\phi_{na})\Big) e^{-S+\mathrm{tr}(\phi J)}$$

where $W[J] = \ln \mathcal{Z}[J]$ generates connected functions

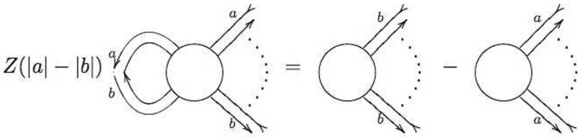
trick $\phi_{mn} \mapsto \frac{\partial}{\partial J_{nm}}$

$$0 = \Big\{ \sum_n \Big((H_{nb} - H_{an})\frac{\delta^2}{\delta J_{nb}\,\delta J_{an}} + \Big(J_{na}\frac{\delta}{\delta J_{nb}} - J_{bn}\frac{\delta}{\delta J_{an}}\Big)\Big) \times \exp\Big(-V\Big(\frac{\delta}{\delta J}\Big)\Big) e^{\frac{1}{2}\sum_{p,q} J_{pq}H^{-1}_{pq}J_{qp}} \Big\}_c$$

Interpretation

The insertion of a special vertex $V_{ab}^{ins} := \sum_n (H_{an} - H_{nb})\phi_{bn}\phi_{na}$

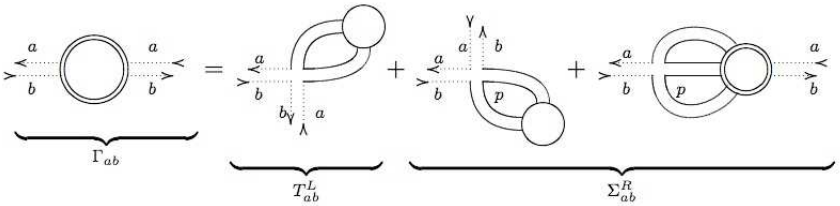
into an external face of a ribbon graph is the same as the difference between the exchanges of external sources $J_{nb} \mapsto J_{na}$ and $J_{an} \mapsto J_{bn}$



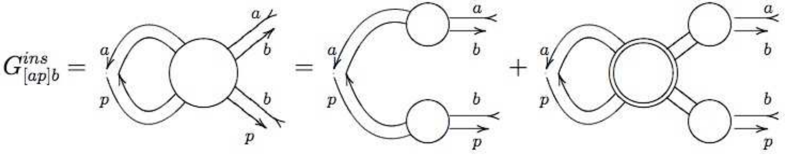
The dots stand for the remaining face indices.

$$Z(|a| - |b|)G^{0,ins}_{[ab]\dots} = \quad G^0_{b\dots} \qquad - \qquad G^0_{a\dots}$$

SD equation 2



- vertex is $Z^2\lambda$, connected two-point function is G^0_{ab}:
  first graph equals $Z^2\lambda\sum_q G^0_{aq}$
- open p-face in Σ^R and compare with insertion into connected two-point function; insert either into 1P reducible line or into 1PI function:



SD equation 2

Amputate upper G^0_{ab} two-point function, sum over p, multiply by vertex $Z^2\lambda$, obtain: Σ^R_{ab}:

$$\Sigma^R_{ab} = Z^2\lambda\sum_p (G^0_{ab})^{-1} G^{0,ins}_{[ap]b} = -Z\lambda\sum_p (G^0_{ab})^{-1}\frac{G^0_{bp}-G^0_{ba}}{|p|-|a|} .$$

Use $(G^0_{ab})^{-1} = H_{ab} - \Gamma_{ab}$ and $T^L_{ab} = Z^2\lambda\sum_q G^0_{aq}$

gives for 2 point function:

$$Z^2\lambda\sum_q G^0_{aq} - Z\lambda\sum_p (G^0_{ab})^{-1}\frac{G^0_{bp}-G^0_{ba}}{|p|-|a|} = H_{ab} - (G^0_{ab})^{-1} .$$

Symmetry $\Gamma_{ab} = \Gamma_{ba}$ is not manifest!

Renormalisation: pass to 1PI functions

$G^0_{ab} =: (Z(a+b+\mu_0^2) - \Gamma_{ab})^{-1}$ and Taylor-expand

$\Gamma_{ab} = Z\mu_0^2 - \mu^2 + (Z-1)(a+b) + \Gamma^{ren}_{ab}$

normalisation conditions: $\Gamma^{ren}_{00} = 0$ and $(\partial\Gamma^{ren})_{00} = 0$

revert to $G^{0,ren}_{ab} = \frac{G_{\alpha\beta}}{x_a+x_b+\mu^2}$, $\alpha = \frac{x_a}{x_a+\mu^2}$, $\xi = \frac{\Lambda_\infty}{\Lambda_\infty+\mu^2}$

express $a, b, p \in [0, \Lambda_\infty]$ by $\alpha, \beta, \rho \in [0, \xi] \subset [0, 1[$

Cubic equation for (renormalised) $G_{\alpha\beta}$

$$
\begin{aligned}
&(G_{\alpha\beta}-1)-(Z^{-1}-1)+(Z^{-1}-1)G_{\alpha\beta}\frac{(1-\alpha)(1-\beta)}{(1-\alpha\beta)}\\
&=-\lambda G_{\alpha\beta}\frac{(1-\alpha)(1-\beta)}{(1-\alpha\beta)}\frac{1}{Z^{-1}}\int_0^\xi\frac{\rho\,d\rho}{(1-\rho)^2}\Big(\frac{(1-\alpha)G_{\alpha\rho}}{1-\alpha\rho}-G_{0\rho}\Big)\\
&+\lambda\frac{(1-\alpha)(1-\beta)}{(1-\alpha\beta)}\int_0^\xi\frac{d\rho}{(1-\rho)}\Big(\frac{G_{\rho\beta}}{1-\rho\beta}-G_{\alpha\beta}G_{\rho 0}+\alpha\frac{G_{\rho\beta}-G_{\alpha\beta}}{\rho-\alpha}\Big)
\end{aligned}
$$

$$
Z^{-1}=1-\lambda\int_0^\xi d\rho\frac{G_{\rho 0}}{1-\rho}+\lambda\int_0^\xi d\rho\Big(G_{\rho 0}-\frac{\frac{d}{d\sigma}G_{\rho\sigma}\big|_{\sigma=0}}{1-\rho}\Big)
$$

This equation can be solved!

- blue term expressed by eq. at $\beta=0$ ($\Rightarrow$ limit $\xi\to 1$ exists)
- red term by Hilbert transform if $G_{\alpha\beta}$ is Hölder-continuous

Result:

$$\Big(\frac{\beta(1-\alpha)}{\alpha(1-\beta)} + \frac{1+\lambda\mathcal{Y}+\lambda\pi\alpha\mathcal{H}_\alpha\big[G_{\bullet 0}\big]}{\alpha G_{\alpha 0}}\Big) D_{\alpha\beta} - \lambda\pi\mathcal{H}_\alpha\big[D_{\bullet\beta}\big] = -G_{\alpha 0},$$

$$-\lambda\pi\mathcal{H}_0[D_{\bullet 0}] = \frac{\lambda\mathcal{Y}}{1+\lambda\mathcal{Y}}$$

where $D_{\alpha\beta} := \frac{\alpha(1-\beta)}{\beta(1-\alpha)}\Big(\frac{(1-\beta)}{1-\alpha\beta}G_{\alpha\beta} - G_{\alpha 0}\Big)$

Finite Hilbert transform $\mathcal{H}_\alpha[f(\bullet)] := \frac{1}{\pi}\lim_{\epsilon\to 0}\Big(\int_0^{\alpha-\epsilon} + \int_{\alpha+\epsilon}^1\Big)\frac{f(\rho)}{\rho-\alpha}$

- preserves $L^p[0,1]$ for $p>1$, not for $p=1$ [M. Riesz]
- does not preserve $\mathcal{C}[0,1]$
- preserves locally-Hölder* spaces $(L^p \cap H_\eta)(]0,1[)$ [Okada-Elliott]

$f \in H_\eta[a,b] \;\Leftrightarrow\; \|f\|_\eta = \sup_{a\le\alpha\le b}|f(\alpha)| + \sup_{a\le\alpha<\beta\le b}\frac{|f(\beta)-f(\alpha)|}{(\beta-\alpha)^\eta} < \infty$

The Carleman equation

Theorem [Carleman 1922, Tricomi 1957]

The singular linear integral equation

$$a(x)y(x) - \lambda\pi\mathcal{H}_x[y] = f(x)\;, \qquad x \in [-1,1]$$

is for $a(x)$ continuous + Hölder near ± 1 and $f \in L^p$ solved by

$$y(x) = \frac{\sin(\theta(x))}{\lambda\pi}\Big(f(x)\cos(\theta(x)) + e^{\mathcal{H}_x[\theta]}\mathcal{H}_x\Big[e^{-\mathcal{H}_\bullet[\theta]}f(\bullet)\sin(\theta(\bullet))\Big] + \frac{Ce^{\mathcal{H}_x[\theta]}}{1-x}\Big)$$

$$\theta(x) = \arctan_{[0,\pi]}\Big(\frac{\lambda\pi}{a(x)}\Big)\;, \quad \sin(\theta(x)) = \frac{|\lambda\pi|}{\sqrt{(a(x))^2 + (\lambda\pi)^2}}$$

where C is an arbitrary constant.

Assumption: $C = 0$

The breakthrough

Theorem

$$\frac{(1-\beta)}{1-\alpha\beta}\frac{G_{\alpha\beta}}{1+\lambda\mathcal{Y}} = \frac{\sin(\theta_\beta(\alpha))}{|\lambda|\pi\alpha} e^{\mathcal{H}_\alpha[\theta_\beta(\bullet)]-\mathcal{H}_0[\theta_0(\bullet)]+\mathcal{H}_1[\theta_0(\bullet)-\theta_\beta(\bullet)]}$$

$$\frac{\lambda\mathcal{Y}}{1+\lambda\mathcal{Y}} = \int_0^1 d\rho\,\frac{\sin^2(\theta_0(\rho))}{\lambda\pi^2\rho^2}$$

$$\theta_\beta(\alpha) = \underset{[0,\pi]}{\arctan}\left(\frac{\lambda\pi\alpha}{\frac{\beta(1-\alpha)}{1-\beta}+\frac{1+\lambda\mathcal{Y}+\lambda\pi\alpha\mathcal{H}_\alpha[G_{\bullet 0}]}{G_{\alpha 0}}}\right) \qquad (*)$$

Consequence: $G_{\alpha\beta} \geq 0$!

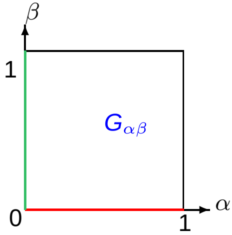
Main steps of the proof:

1. (*) is Carleman eq. $\lambda\pi\cot\theta_0(\alpha)G_{\alpha 0} - \lambda\pi\mathcal{H}_\alpha[G_{\bullet 0}] = \frac{1+\lambda\mathcal{Y}}{\alpha}$
2. Tricomi's identity $e^{-\mathcal{H}_\alpha[\theta_\beta]}\cos(\theta_\beta(\alpha)) + \mathcal{H}_\alpha\left[e^{-\mathcal{H}_\bullet[\theta_\beta]}\sin(\theta_\beta(\bullet))\right] = 1$

The self-consistency equation

Given boundary value $G_{\alpha 0}$,
Carleman computes $G_{\alpha\beta}$,
in particular $G_{0\beta}$

symmetry forces $G_{\beta 0} = G_{0\beta}$



Master equation

The theory is completely determined by the solution of

$$G_{\beta 0} = \frac{1+\lambda\mathcal{Y}}{1+(1-\beta)\lambda\mathcal{Y}} \times \exp\left(-\lambda \int_0^{\frac{\beta}{1-\beta}} dt \int_0^1 \frac{d\rho}{(\lambda\pi\rho)^2 + \left(t(1-\rho) + \frac{1+\lambda\mathcal{Y}+\lambda\pi\rho\mathcal{H}_\rho[G_{\bullet 0}]}{G_{\rho 0}}\right)^2}\right)$$

(provided it exists, together with eq. for $\lambda\mathcal{Y}$)

Some non-perturbative results

Corollary: $\lambda > 0$ $\frac{(1+(1-\beta)\lambda\mathcal{Y})}{1+\lambda\mathcal{Y}} G_{\beta 0} \in \mathcal{C}^1([0,1[)$, monotonously decreasing, positive;
G_{10} exists, $G_{\beta 0} \in \mathcal{C}[0,1]$.-
Let $\lambda > 0$, $G = TG$ the master equation and $F \in H_\lambda[0,1]$.

Recall $Z^{-1}(G) = 1 + \lambda \mathcal{Y}_G - \lambda \int_0^1 d\rho \, \frac{G_{\rho 0}}{1-\rho}$

1. If $F(1) \neq 0$, then $(TF)(1) = 0$.
2. If $Z^{-1}(F) \geq \delta > 0$, then $(TF)(1) \geq \epsilon > 0$.
3. If $Z^{-1}(F) < 0$, then $(TF)(1) = 0$.

Consequently, $G_{10} = 0$ and $Z^{-1}(G) \leq 0$.

But gives $G_{\alpha 0} = 0 \quad \Rightarrow \quad 1 + \lambda\mathcal{Y} + \lambda\pi\alpha\mathcal{H}_\alpha\left[G_{\bullet 0}\right] = 0$

For $\alpha = 1$ this means $Z^{-1}(G) = 0$.

The planar regular four-point function

Schwinger-Dyson equation for $G_{abcd} = G_{ab}G_{bc}G_{cd}G_{da}\Gamma_{abcd}$ becomes again a Carleman equation:

$$\lambda\pi \cot(\theta_\beta(\alpha))\mathcal{G}_{\alpha\beta\gamma\delta} - \lambda\pi\mathcal{H}_\alpha[\mathcal{G}_{\bullet\beta\gamma\delta}] = \lambda f_{\beta\gamma\delta}(\alpha)$$

where

$$\mathcal{G}_{\alpha\beta\gamma\delta} = \alpha G_{\gamma\delta}\frac{(1-\beta)G_{\alpha\beta}}{1-\alpha\beta}\frac{(1-\delta)G_{\delta\alpha}}{1-\delta\alpha}\Gamma_{\alpha\beta\gamma\delta}$$

$$f_{\beta\gamma\delta}(\alpha) = \frac{(1-\gamma)(1-\alpha\delta)G_{\gamma\delta} - (1-\alpha)(1-\gamma\delta)G_{\alpha\delta}}{(1-\delta\alpha)(\alpha-\gamma)}$$

$$\Gamma_{\alpha\beta\gamma\delta} = \frac{\lambda}{(\alpha-\gamma)(\beta-\delta)}\left(\frac{(1-\alpha\beta)}{G_{\alpha\beta}}\frac{(1-\gamma\delta)}{G_{\gamma\delta}} - \frac{(1-\gamma\beta)}{G_{\gamma\beta}}\frac{(1-\alpha\delta)}{G_{\alpha\delta}}\right)$$

Numerical iteration of T

We approximate G by piecewise-linear function sampled at $0 = x_0 < x_1 < \cdots < x_N = 1$ and study $f \mapsto Tf$ numerically.
We find for $\lambda > 0$ independently of starting point f:

1. $\frac{\lambda\mathcal{Y}}{1+\lambda\mathcal{Y}} = \ldots$ has unique solution
2. Tf is monotonously decreasing
3. $Tf \in H_\eta[0,1]$ for $\eta = \min(\lambda, \frac{1}{\pi})$
4. $T^n f$ converges in $H_\eta[0,1]$ to fixed-point solution $G = TG$
5. G is L^2-close to $(1-\beta^{\frac{1}{c}})^c$ for $c \approx \eta$
6. $Z^{-1} > 0$ for $0 < \lambda < \frac{1}{\pi}$, but $Z^{-1} \to 0$ for more sample points
   $Z^{-1} < 0$ for $\lambda > \frac{1}{\pi}$, stable in number of sample points

Something happens at $\lambda^* = \frac{1}{\pi}$!

The remaining part of the proof

Fact

Let $0 < \lambda \leq \frac{1}{\pi}$ and $K \subset \mathcal{C}[0,1]$ be a convex subset, with

– $\forall F \in K$: $\quad F \in H_\lambda[0,1]$ and $\sqrt{1-\beta^2} \leq F(\beta) \leq 1$

– maybe F non-increasing, convex, ...

Assume for all $F \in K$:

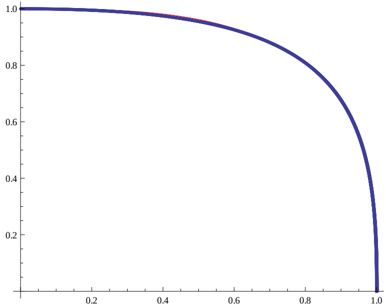
1. $$\frac{Y_F}{1+Y_F} = \int_0^1 d\rho \, \frac{\lambda}{(\lambda\pi\rho)^2 + \left(\frac{1+Y_F+\lambda\pi\rho\mathcal{H}_\rho[F(\bullet)]}{F(\rho)}\right)^2}$$
   has unique solution which depends continuously on F
2. The following function TF is monotonously decreasing:
$$(TF)(\beta) = \frac{1+Y_F}{1+(1-\beta)Y_F} \exp\left(-\int_0^{\frac{\beta}{1-\beta}} dt \int_0^1 \frac{\lambda \, d\rho}{(\lambda\pi\rho)^2 + \left(t(1-\rho) + \frac{1+Y_F+\lambda\pi\rho\mathcal{H}_\rho[F(\bullet)]}{F(\rho)}\right)^2}\right)$$
3. $\|TF\|_\lambda \leq M$ uniformly on K.

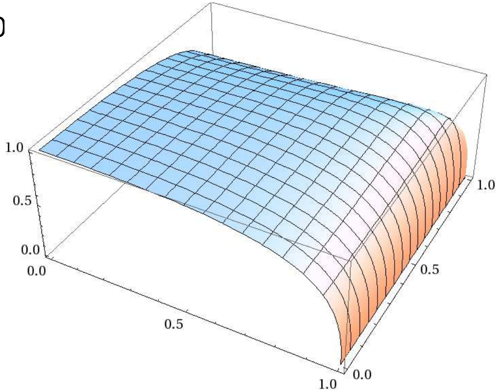
Then there is a fixed-point solution $G = TG \in K$.

Proof. Arzelà-Ascoli + Schauder fixed-point theorem

Some plots: $\lambda = \frac{1}{\pi} = 0.318310\ldots$

$x_k = \sin(\frac{\pi}{2}\sin(\frac{\pi}{2}\frac{k}{N})), N = 2000$





eff. coupling: $\Gamma_{0000} = 0.333359$

$Z^{-1} = 1 * 10^{-5}$

asymmetry

$\max_{\alpha,\beta} |G_{\alpha\beta} - G_{\beta\alpha}| = 4 * 10^{-6}$

$\|G - F\|_2 = 6 * 10^{-3}$,

evidence that $G = TG$ does not solve for $\lambda > \frac{1}{\pi}$...

$F = (1 - \beta^p)^{\frac{1}{p}}, p = 3.1395$

An analogy

2D Ising model	4D nc ϕ^4-theory
temperature T, $K = \frac{J}{k_B T}$	frequency Ω
Kramers-Wannier duality $\sinh(2K)\sinh(2K^*) = 1$	Langmann-Szabo duality $\Omega\Omega^* = 1$
solvable at $K = K^*$ scale-invariant	solvable at $\Omega = \Omega^*$ almost scale-invariant
CFT minimal model ($m = 3$)	matrix model
operator product expansion Virasoro constraints	Schwinger-Dyson equation Ward identities
critical exponents $G^{\sigma\sigma}_{n0} \propto \frac{1}{n^{d-2+\eta}}$, $\eta = \frac{1}{4}$ $G^{\sigma\sigma\sigma\sigma}_{nn00} \propto \frac{1}{n^{2(d-1/\nu)}}$, $\nu = 1$	critical exponents $G^{\phi\phi}_{n0} \propto \frac{1}{n^{1+\lambda}}$, $\lambda \in]0, \frac{1}{\pi}]$ $G^{\phi\phi\phi\phi}_{nn00}$ (soon ?)
Virasoro algebra, CFT, subfactors, . . .	**???**