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Equivariant dimensional reduction over noncommutative spaces

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Dimensional reduction over the quantum sphere
and non-abelian q-vortices
G. L., R.J. Szabo
Commun. Math. Phys. 308 (2011)

Holomorphic structures on the quantum projective line
M. Khalkhali, G. L., W.D. van Suijlekom
Int'l Mathematics Research Notices (2010),

generalizes work by: Alvarez-Consul, Bradlow, Garcia-Prada, ...

Equivariant dimensional reduction :

A systematic procedure for including internal fluxes on S/R
(instantons and/or monopoles of R-fields)
'symmetric' (equivariant) for S

Vortices and gauge fields ; Taubes , ...

The Ginzburg–Landau equations for vortices is related to the four dimensional Yang–Mills equations via reduction:

any $SO(3)$ symmetric solution to the $SU(2)$ Y–M eqs on $\mathbb{R}^2 \times S^2$

yields a solution to the G–L eqs on $\mathbb{R}^2$ and vice versa.

Equivariant dimensional reduction :

R-instantons and/or monopoles ‘symmetric’ (equivariant) for S

S-equivariant complex vector bundles over M_d

$$B \longrightarrow M_d = M_4 \times S/R,$$

correspond (1 to 1) to R-equivariant bundles over M_4,

$$E \longrightarrow M_4,$$

S acts trivially on M_d ; standard left translation action on S/R

In general the reduction yields rise quiver gauge theories on M_4

A simple example: Complex projective line

$G = U(k)$, $\quad S = SU(2)$ and $R = U(1)$ $\quad \Rightarrow S^2 \simeq SU(2)/U(1)$

Embedding $S \hookrightarrow G$ results into decomposing $U(k) \to \prod_{i=0}^{m} U(k_i)$,

$k = \sum_{i=0}^{m} k_i$, associated with the $(m+1)$-dim I.R. of $SU(2)$

Gauge theory on $M \times S^2$, reduces to into $k_i \times k_j$ blocks

$$A(x,y) = A(x) + a(y) + \Phi(x)\overline{\beta}(y) + \Phi^{\dagger}(x)\beta(y),$$

$a = \oplus_{i=0}^{m} a_{m-2i}$, $\quad a_{m-2i}$ charge $m-2i$ monopole connection and $\Phi(x)$ is a collection of Higgs fields

Dimensional reduction generates a 4-dim Higgs potential,

$$V(\Phi) = \frac{g^2}{2}\,\mathrm{tr}_k\left(\frac{1}{4g^2r^2}\begin{pmatrix} m\mathbf{1}_{k_0} & 0 & \cdots & 0 \\ 0 & (m-2)\mathbf{1}_{k_1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & -m\mathbf{1}_{k_m} \end{pmatrix} - [\Phi,\Phi^\dagger]\right)^2,$$

whose minimization gives a vacuum structure depending on the monopole charges $p_i = m - 2i$

For example: the Ginsburg–Landu action functional

$$GL(A, \Phi) \;=\; \int_{\mathbb{R}^2} \mathrm{tr}\Big(\,-\, \frac{1}{4}F^2 \,+\, D\Phi^{\dagger} D\Phi \,+\, \lambda(\Phi^{\dagger}\Phi \,-\, 1)^2 \Big)$$

as mentioned self-duality equation are vortex equations:

$$\star F = \mathrm{id}_{\mathcal{E}_0} - \Phi \circ \Phi^* \qquad \text{and} \qquad D\Phi = 0$$

M a smooth manifold; $\mathbb{C}\mathsf{P}^1_q$ the quantum projective line

Characterize vector bundles over the quantum space

$$\underline{M} := \mathbb{C}\mathsf{P}^1_q \times M$$

equivariant under an action of the quantum group $\mathsf{SU}_q(2)$

These are finitely-generated and projective $\mathsf{SU}_q(2)$-equivariant modules over the algebra of functions

$$\mathcal{A}(\,\underline{M}\,) = \mathcal{A}(\mathbb{C}\mathsf{P}^1_q) \otimes \mathcal{A}(M)$$

Describe the dimensional reduction of invariant connections

In particular, Yang–Mills gauge theory on $\mathcal{A}(\,\underline{M}\,)$ is reduced to

a type of Yang–Mills–Higgs theory on the manifold M

The equations of motion give q-deformations of known vortex equations, whose solutions possess remarkable properties

In particular de-singularization of moduli spaces deformation parameter $q \in \mathbb{R}_{>0}$ $\quad q \simeq q^{-1}$

$\mathcal{A}(\mathrm{SU}_q(2))$:= $*$-algebra generated by a and c, with relations

$$UU^* = U^*U = 1 \qquad U = \begin{pmatrix} a & -qc^* \\ c & a^* \end{pmatrix}$$

$$ac = qca, \qquad ac^* = qc^*a, \qquad cc^* = c^*c,$$

$$a^*a + c^*c = aa^* + q^2cc^* = 1$$

Hopf $*$-algebra structure on $\mathcal{A}(\mathrm{SU}_q(2))$:

$$\Delta U = U \otimes U \qquad S(U) = U^* \qquad \varepsilon(U) = 1$$

These dualize classical operations

$\mathcal{A}_1 = \mathcal{A}(\mathrm{SU}(2))$, polynomial functions on SU(2)

$$\Delta : \mathcal{A}_1 \to \mathcal{A}_1 \otimes \mathcal{A}_1 \qquad (\Delta f)(x \otimes y) = f(xy)$$

$$S : \mathcal{A}_1 \to \mathcal{A}_1 \qquad (Sf)(x) = f(x^{-1})$$

$$\varepsilon : \mathcal{A}_1 \to \mathbb{C} \qquad (\varepsilon f) = f(e)$$

A (right) $*$-action: $\alpha : \mathsf{U}(1) \to \mathsf{Aut}(\mathcal{A}(\mathsf{SU}_q(2)))$

$$\alpha_u \begin{pmatrix} a & -qc^* \\ c & a^* \end{pmatrix} = \begin{pmatrix} a & -qc^* \\ c & a^* \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix}, \qquad \text{for} \quad u \in \mathsf{U}(1).$$

$$\alpha_u \begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix} u, \quad \alpha_u \begin{pmatrix} a^* \\ c^* \end{pmatrix} = u^* \begin{pmatrix} a^* \\ c^* \end{pmatrix}, \quad \text{for} \quad u \in \mathsf{U}(1).$$

The invariant elements form a subalgebra of $\mathcal{A}(\mathsf{SU}_q(2))$,

the coordinate algebra $\mathcal{A}(\mathsf{S}^2_q)$ of the standard Podleś sphere S^2_q

$$\mathcal{A}(\mathsf{S}^2_q) = \mathcal{A}(\mathsf{SU}_q(2))^{\mathsf{U}(1)}$$

the algebra inclusion

$$\mathcal{A}(\mathsf{S}^2_q) \hookrightarrow \mathcal{A}(\mathsf{SU}_q(2))$$

is a noncommutative principal bundle

As a set of generators for $\mathcal{A}(\mathsf{S}^2_q)$ we may take

$$B_- := ac^*, \qquad B_+ := ca^*, \qquad B_0 := cc^*.$$

A natural complex structure on the 2-sphere S^2_q

for the unique 2-dimensional $\mathsf{SU}_q(2)$-covariant calculus;

S^2_q becomes a quantum Riemannian sphere or qpl $\mathbb{C}\mathsf{P}^1_q$

A vector space decomposition

$$\mathcal{A}(\mathsf{SU}_q(2)) = \bigoplus_{n\in\mathbb{Z}} \mathcal{L}_n \ , \qquad (\star)$$

$$\mathcal{L}_n := \mathcal{A}(\mathsf{SU}_q(2)) \boxtimes_{\rho_n} \mathbb{C} \simeq \Big\{ x \in \mathcal{A}(\mathsf{SU}_q(2)) \ \Big| \ \alpha_u(x) = x\,(u^*)^n \Big\}$$

for $u \in \mathsf{U}(1)$

Each $\mathcal{L}_n$ is a finitely-generated projective (right, say) $\mathcal{A}(\mathbb{C}\mathsf{P}^1_q)$-module of rank one

module of $\mathsf{SU}_q(2)$-equivariant sections of a line bundles over the quantum projective line $\mathbb{C}\mathsf{P}^1_q$ with degree (monopole charge) $-n$

Enlarging the space

For a smooth manifold M,

consider $\underline{M} := \mathbb{C}\mathsf{P}^1_q \times M$ with ‘coordinate’ algebra,

$$\mathcal{A}(\underline{M}) := \mathcal{A}(\mathbb{C}\mathsf{P}^1_q) \otimes \mathcal{A}(M) \ .$$

A coaction of $\mathsf{SU}_q(2)$ on $\mathcal{A}(\underline{M})$;

trivially on $\mathcal{A}(M)$ and with canonical coaction Δ_L on $\mathcal{A}(\mathbb{C}\mathsf{P}^1_q)$:

$$\underline{\Delta} \ : \mathcal{A}(\underline{M}) \ \longrightarrow \ \mathcal{A}(\mathsf{SU}_q(2)) \otimes \mathcal{A}(\underline{M})$$

A $\mathsf{SU}_q(2)$-equivariant right $\mathcal{A}(\underline{M})$-module $\underline{\mathcal{E}}$ carries a coaction

$$\delta : \underline{\mathcal{E}} \longrightarrow \mathcal{A}(\mathsf{SU}_q(2)) \otimes \underline{\mathcal{E}}$$

compatible with the coaction $\underline{\Delta}$ of $\mathcal{A}(\mathsf{SU}_q(2))$ on $\mathcal{A}(\underline{M})$,

$$\delta(\varphi \cdot \underline{f}) = \delta(\varphi) \cdot \underline{\Delta}(\underline{f}) \qquad \text{for all} \quad \varphi \in \underline{\mathcal{E}} \;, \;\; \underline{f} \in \mathcal{A}(\underline{M})$$

Relate $\mathcal{A}(\mathsf{SU}_q(2))$-equivariant bundles $\underline{\mathcal{E}}$ on the q. space $\underline{M}$

to U(1)-equivariant bundles E over the manifold M

Proposition 1. *Every finitely-generated* $\mathrm{SU}_q(2)$*-equivariant projective module* $\underline{\mathcal{E}}$ *over* $\mathcal{A}(\underline{M})$ *equivariantly decomposes as*

$$\underline{\mathcal{E}} = \bigoplus_{i=0}^{m} \underline{\mathcal{E}}_i = \bigoplus_{i=0}^{m} \mathcal{L}_{m-2i} \otimes \mathcal{E}_i$$

(and uniquely up to isomorphism), for some $m \in \mathbb{N}_0$*;*

$\mathcal{E}_i$ *are modules of sections of (usual) vector bundles* E_i *over* M *with trivial* $\mathrm{SU}_q(2)$ *coactions;*

$\mathcal{L}_n$ *are the above modules of sections of* $\mathrm{SU}_q(2)$*-equivariant line bundles over* $\mathbb{C}\mathrm{P}^1_q$*.*

(there are also morphisms $\Phi_i \in \mathrm{Hom}_{\mathcal{A}(\underline{M})}(\underline{\mathcal{E}}_{i-1}, \underline{\mathcal{E}}_i)$, of $\mathcal{A}(\underline{M})$-modules, coming from the $\mathrm{SU}_q(2)$-coaction).

Lemma 2. *A unitary connection* $\underline{\nabla}$ *on* $(\underline{\mathcal{E}}, \underline{h})$ *decomposes as*

$$\underline{\nabla} = \sum_{i=0}^{m} \left(\underline{\nabla}_i + \sum_{j<i} \left(\underline{\beta}_{ji} - \underline{\beta}_{ji}^{*} \right) \right) ,$$

where:

1. *Each* $\underline{\nabla}_i$ *is a unitary connection on* $(\underline{\mathcal{E}}_i, \underline{h}_i)$*, i.e.*

$$\underline{h}_i(\underline{\nabla}_i\varphi,\psi) + \underline{h}_i(\varphi,\underline{\nabla}_i\psi) = \underline{\mathsf{d}}\left(\underline{h}_i(\varphi,\psi)\right) \qquad \textit{for} \quad \varphi,\psi \in \underline{\mathcal{E}}_i \ .$$

2. *For* $j \neq i$*,*

$\underline{\beta}_{ji} \in \mathrm{Hom}_{\mathcal{A}(\underline{M})}(\underline{\mathcal{E}}_i, \Omega^1(\underline{\mathcal{E}}_j))$ *is the adjoint of* $-\underline{\beta}_{ij}$*, i.e.*

$$\underline{h}(\underline{\beta}_{ji}\varphi,\psi) + \underline{h}(\varphi,\underline{\beta}_{ij}\psi) = 0 \qquad \textit{for} \quad \varphi \in \underline{\mathcal{E}}_i \ , \ \psi \in \underline{\mathcal{E}}_j \ .$$

Integrable connections

M be a complex manifold, with standard complex structure ;
a complex structure for $\mathbb{C}\mathrm{P}^1_q$

a complex structure for $\mathcal{A}(\underline{M}) = \mathcal{A}(\mathbb{C}\mathrm{P}^1_q) \otimes \mathcal{A}(M)$.

If $\underline{\nabla}$ is a connection, the $(0,2)$-component of the curvature

$$F^{0,2}_{\underline{\nabla}} \in \mathrm{Hom}_{\mathcal{A}(\underline{M})}(\underline{\mathcal{E}}, \Omega^{0,2}(\underline{\mathcal{E}})), \qquad \Omega^{0,2}(\underline{\mathcal{E}}) = \mathcal{E} \otimes \Omega^{0,2}(\underline{M})$$

The connection $\underline{\nabla}$ is then integrable if $F^{0,2}_{\underline{\nabla}} = 0$.

In this case the pair $(\underline{\mathcal{E}}, \underline{\nabla})$ is a holomorphic vector bundle.

Gauge theory

Let $\mathcal{C}(\underline{\mathcal{E}})$ be the space of unitary connections on an $\mathrm{SU}_q(2)$-equivariant hermitian $\mathcal{A}(\underline{M})$-module $(\underline{\mathcal{E}}, \underline{h})$.

The Y–M action functional $\mathrm{YM} : \mathcal{C}(\underline{\mathcal{E}}) \to [0, \infty)$ is as usual

$$\mathrm{YM}(\underline{\nabla}) = \left\| F_{\underline{\nabla}} \right\|_{\underline{h}}^2 \tag{3}$$

from a suitable L^2-norm $\|-\|_{\underline{h}}$ on the space $\mathrm{Hom}_{\mathcal{A}(\underline{M})}(\underline{\mathcal{E}}, \Omega^p(\underline{\mathcal{E}}))$

Dimensional reduction of the Yang–Mills action functional

Proposition 4.
The functional $\mathsf{YM}\big|_{\mathcal{C}(\underline{\mathcal{E}})^{\mathsf{SU}_q(2)}}$ *on the quantum space* $\underline{M}$*, when restricted to* $\mathsf{SU}_q(2)$*-invariant unitary connections coincides with the Y–M–H functional* $\mathsf{YMH}_{q,m}$ *on* M*:*

$$\mathsf{YMH}_{q,m}(\nabla,\phi)=\sum_{i=0}^{m}\Big(\big\|F_{\nabla_i}\big\|^2_{h_i}+\big(q^2+1\big)\big\|\nabla_{i-1,i}(\phi_i)\big\|^2_{h_{i-1,i}}$$

$$+\big\|\phi^*_{i+1}\circ\phi_{i+1}-q^2\,\phi_i\circ\phi^*_i-q^{m-2i+1}\,[m-2i]_q\ \mathrm{id}_{\mathcal{E}_i}\big\|^2_{h_i}\Big)\,,$$

$\phi_0 := 0 =: \phi^*_0$ *and* $\phi_{m+1} := 0 =: \phi^*_{m+1}$

with

- $F_{\nabla_i} = \nabla_i^2$, *the curvature of the connection* $\nabla_i \in \mathcal{C}(\mathcal{E}_i)$ *on* M

- $\nabla_{i-1,i}$ *the connection on* $\mathrm{Hom}_{\mathcal{A}(M)}(\mathcal{E}_{i-1}, \mathcal{E}_i)$ *induced by* ∇_{i-1} *on* $\mathcal{E}_{i-1}$ *and* ∇_i *on* $\mathcal{E}_i$ *and given by*

$$\nabla_{i-1,i}(\phi_i) = \phi_i \circ \nabla_{i-1} - \nabla_i \circ \phi_i \ .$$

Symbol

$$[x]_q = \frac{q^x - q^{-x}}{q - q^1} \qquad\qquad q \neq 1$$

This functional restricts to a map on gauge orbits

$$\mathrm{YMH}_{q,m} : \mathscr{C}(\underline{\mathcal{E}}) \big/ \mathscr{U}(\underline{\mathcal{E}}) \to [0, \infty)$$

Characterize stable critical points of the Y–M functional (3) on $\underline{M}$, and study their reduction to configurations on M.

A Hodge operator (as a bimodule map)

$$\underline{\star} := \widehat{\star} \otimes \star \, : \, \Omega^p(\underline{M}) \longrightarrow \Omega^{2(d+1)-p}(\underline{M})$$

Lemma 5. *Let $\underline{\nabla} \in \mathcal{C}(\underline{\mathcal{E}})$ be a unitary connection such that*

$$\underline{\star}\, F_{\underline{\nabla}} = -F_{\underline{\nabla}} \wedge \Sigma \tag{6}$$

for $\Sigma \in \Omega^{2d-2}(\underline{M})$ a closed form of degree $2d-2$.

Then $\underline{\nabla}$ is a critical point of the Y–M functional and

$$\mathrm{YM}(\underline{\nabla}) = \mathrm{Top}_2(\underline{\mathcal{E}}, \Sigma) := -\Big(F_{\underline{\nabla}}\,,\ \underline{\star}(F_{\underline{\nabla}} \wedge \Sigma)\Big)_{\underline{h}}$$

The functional $\mathrm{Top}_2(\underline{\mathcal{E}}, \Sigma)$ does not depend on the choice of $\underline{\nabla}$

It defines a 'topological action' depending only on the $\mathcal{A}(\underline{M})$-module $\underline{\mathcal{E}}$ and the closed form Σ

Provides an *a priori* lower bound on the Y–M functional

The gauge invariant equation (6) is the Σ-anti-selfduality eqn

The gauge equivalence classes in $\mathcal{C}(\underline{\mathcal{E}}) \big/ \mathcal{U}(\underline{\mathcal{E}})$ of solutions are

generalized instantons or Σ-instantons

1. Deformations of holomorphic triples and stable pairs

A holomorphic triple $(\mathcal{E}_0, \mathcal{E}_1, \phi)$ on a compact Kähler manifold (M, ω) is a pair of holomorphic vector bundles $\mathcal{E}_0, \mathcal{E}_1$ over M and a holomorphic morphism

$$\mathcal{E}_0 \ \xrightarrow{\phi} \ \mathcal{E}_1$$

With $\phi := \phi_1$, we get

$$F^{\omega}_{\nabla_0} = q^2 \left(\mathrm{id}_{\mathcal{E}_0} - q^{-2} \, \phi \circ \phi^* \right) \quad \text{and} \quad F^{\omega}_{\nabla_1} = - \left(\mathrm{id}_{\mathcal{E}_1} - q^2 \, \phi^* \circ \phi \right) \ (\diamondsuit)$$

The degrees of the bundles are related by

$$\deg(\mathcal{E}_0) + q^{-2} \deg(\mathcal{E}_1) = q^2 \operatorname{rank}(\mathcal{E}_0) - q^{-2} \operatorname{rank}(\mathcal{E}_1)$$

Much more stringent than the undeformed stability condition

2. q-instantons

Let (M, ω) be a Kähler surface. Set $\mathcal{E}_0 \simeq \mathcal{E}_1 =: \mathcal{E}$.

Since ϕ is a holomorphic section, $\nabla^{\overline{\partial}}_{0,1}(\phi) = 0$;

we have $\nabla_0 = \nabla_1 =: \nabla$ and both equations in ($\diamondsuit$) simplify to

$$F^{\omega}_{\nabla} = \left(q^2 - 1\right) \mathrm{id}_{\mathcal{E}}$$

a deformation of the hermitian Yang–Mills equation on M, and hence of the standard anti-selfduality equations $\star F_{\nabla} = -F_{\nabla}$
Its gauge equivalence classes of solutions called q-instantons

When $M=\mathbb{C}^2$, the constant shift in the moment map condition

$$\text{from} \quad \mu_{\mathcal{C}}=0 \quad \text{to} \quad \mu_{\mathcal{C}}=(q^2-1)\ \mathrm{id}_{\mathcal{E}}$$

induces a shift in the corresponding real ADHM equation.

NS: this modification arises in the equations which determine instantons on a certain noncommutative deformation of $\mathbb{R}^4$

Here we have the same sort of resolution of instanton moduli space via our q-deformed dimensional reduction procedure over the quantum projective line $\mathbb{C}\mathrm{P}^1_q$.

Summing up:

Characterized vector bundles over the quantum space

$$\underline{M} := \mathbb{C}\mathsf{P}^1_q \times M$$

equivariant under an action of the quantum group $\mathsf{SU}_q(2)$

Described the dimensional reduction of invariant connections

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The equations of motion give q-deformations of known vortex equations, whose solutions possess remarkable properties.