Hawking radiation as a local tunneling process: algebraic QFT viewpoint

Valter Moretti

Department of Mathematics
University of Trento
Italy

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Plan of the talk

- Hawking radiation, Killing horizons, Hadamard states
- Parikh-Wilczek's tunneling interpretation and its problems

Re-formulation in the framework of AQFT

- Geometric and algebraic setup
- Constructions necessary for the computation
- Main results
- Physical interpretation
- Final comments, further developments, open issues

Bibliography

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Hawking radiation, Killing horizons and Hadamard states

73-75 **Bekenstein, Hawking**: BH version of the laws of thermodynamics and BH radiation, with temperature $T_H = (8\pi M)^{-1}$, at future null infinity.

76-80 **Bisognano-Wichmann, Sewell**: Interplay of geometry (existence of bifurcate Killing horizons) and thermal (KMS condition) properties of quantum states invariant under the Killing symmetry.

'90 **Fredenhagen and Haag**: (essentially) Hadamard states show Hawking radiation at future infinity in a spacetime containing spherically symmetric collapsing matter.

'91 **Kay and Wald**: Hadamard states are unique (if invariant under the action of the Killing symmetry) and KMS at $T = T_H$ in spacetimes containing a bKh.

'11 (C.Dappiaggi, V.M., N.Pinamonti: Rigorous construction of Unruh state as a Hadamard state. $\sim$ 90 pages)

Parikh and Wilczek's Tunneling Interpretation

Tunneling interpretation: Parikh and Wilczek [2000]

They compute the **transition probability** P_E through a Schwarzschild BH horizon (**tunneling probability**) for a particle with energy E finding a **thermal** behaviour (it is throughout assumed $k_{Boltzmann} = 1$):

$$P_E \; \alpha \; \exp(-E/T_H) \, .$$

(Up to now, more than 500 citations)

⇒ In principle, Hawking Radiation is now viewed as a **local** phenomenon

- no detection **at future null infinity** seems to be necessary,
- **no global BH structure** seems to be necessary, just a neighborhood of a point on the horizon needs,

Details

- **Tunneling** probability through the horizon of a **Schwarzschild spacetime** for *one particle* with energy E:

$$\Gamma_E = |\langle \psi_{Ex} | \psi_{Ey} \rangle|^2$$

ψ_{Ex}, ψ_{Ey} localized around $x = (t, r_1, \theta, \varphi)$ and $y = (t, r_2, \theta, \varphi)$ respectively, **separated by the horizon**.

- "Quantization" w.r.to the **Painlevé time** t:

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + 2\sqrt{\frac{2M}{r}}\, dt\, dr + dr^2 + r^2 d\Omega$$

- **WKB method** to approximate Γ_E:

$$\Gamma_E = \lim_{y \to x} |\langle \psi_{Ex} | \psi_{Ey} \rangle|^2 \sim \lim_{y \to x} |e^{i \int_{r_1}^{r_2} p_r^{(E)} dr}|^2$$

The integral diverges $\Rightarrow$ complex plane regularization (Feynman-regularization). An imaginary part arises in the integral:

$$\Gamma_E \sim e^{-2 Im S_{reg}} \sim e^{-E/kT_H}$$

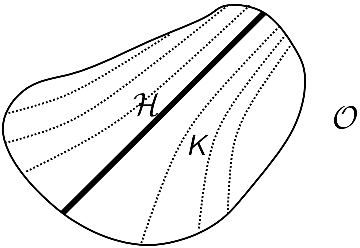
Problems

- The appearance of T_H is related with a particular choice of the (complex-plane) regularization procedure: is it independent form it?
- The notions of **particle**, **time** and **energy** are ambiguously defined in **curved spacetime** (no Poincaré symmetry).
- There is no **Schrödinger equation** to handle by means of the **WKB machinery**.
- It is not completely clear if the results depend on the local BH geometry only.

  It is however difficult to suppose that T_H pops up by chance.
  To investigate the issue, clarifying the physical meaning of this Parikh-Wilczek's interesting result, the whole computation should be performed within the proper framework of QFT in curved spacetime.

The geometry of local Killing horizon

- From now on, we focus on some **neighborhood** $\mathcal{O}$ of a point on the **horizon** $\mathcal{H}$ of a black hole generated by the **Killing field** K,
- Actually, since our computation will not depend on the geometry outside $\mathcal{O}$ the horizon may (smoothly) cease to exist outside $\mathcal{O}$.



Local geometric hypotheses (LGH) for $\mathcal{O}$

Here are the precise hypotheses we assume for $\mathcal{O}$.

Definition

If (M, g) is a time-orientable smooth spacetime, **LGH** hold for an open set $\mathcal{O} \subset M$, if a smooth vector field K exists thereon such that:

(a) K is a Killing field for g in $\mathcal{O}$.

(b) $\mathcal{O}$ contains the **local Killing horizon** $\mathcal{H}$ i.e. a 3-submanifold invariant under the action of K with $K^a K_a = 0$ on $\mathcal{H}$.

(c) The **orbits** of K in $\mathcal{O}$ are diffeomorphic to an **open interval** I and topologically $\mathcal{H} = I \times \mathcal{B}$ ($\mathcal{B}$ being a 2-dimensional cross section).

(d) The **surface gravity** κ is **constant** on $\mathcal{H}$. ($\kappa : \mathcal{H} \to \mathbb{R}$ is defined by $\nabla^a(K_b K^b) = -2\kappa K^a$.)

Comments on our geometric hypotheses

- The requirement $\kappa =$ constant along $\mathcal{H}$ means that the **thermodynamical equilibrium** has been reached on $\mathcal{H}$, since $\kappa = 2\pi T_H$.

- **LGH** are in particular satisfied by neighborhoods of points on the **future horizon** of a **non-extremal** black hole in the **Kerr-Newman family**, including **charged** and **rotating black holes**.

- **LGH** are valid for "realistic" black holes produced by collapsed matter, so that only the future horizon exists, but even for **ethernal black holes** (whose manifolds include **white hole** regions as in the whole Kruskal spacetime)
- Our picture includes also situations where the collapse starts, reaches a sort of local equilibrium and it **stops** after a while, without giving rise to a complete BH structure.

The algebra of quantum fields

- $\mathcal{A}$ is the $*$-**algebra** generated by the linear abstract **field operators** $\phi(f)$ with $f \in C_0^\infty(M)$ such that:
  - $\phi(f)^* = \phi(\overline{f})$
  - $[\phi(f), \phi(f')] = 0$ for causally disjoint $supp(f)$, $supp(g)$.

  No field equation is necessary.
- We intend to compute the **correlation function** $\omega(\phi(f)\phi(f'))$ when $supp(f), supp(f') \subset \mathcal{O}$ are "very close" to $\mathcal{H}$.
- When $supp(f), supp(f')$ are separated by the horizon, $|\omega(\phi(f)\phi(f'))|^2$ is a **tunneling probability** if ω is Gaussian (quasifree)
- Supposing that $\mathcal{O}$ is **geodesically convex** we assume that:

$$\omega_2(x, y) := \frac{U(x, y)}{\sigma(x, y)} + \text{less singular bi-distribution}$$

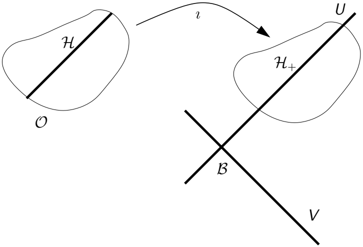
$\sigma(x, y)$ **geodesical distance** of x and y, $U = \Delta_{VVM}^{1/2}/4\pi^2$.
Hadamard states fulfil the requirement in particular.

Adding the bifurcation surface $\mathcal{B}$

An important technical step is the following result.

Theorem

(Racz and Wald [1992 1996])
Let $\mathcal{O} \subset M$ satisfy the LGH. There is a time-orientable spacetime (M', g') with a **bifurcate Killing horizon** *generated by a Killing field K' and an isometric imbedding $i : \mathcal{O} \to M'$ such that $i_*(K) = K'$.*



$\mathcal{H}$ becomes a part of a **future Killing horizon** $\mathcal{H}_+$ in the future of the **bifurcation surface** $\mathcal{B}$, where $K' = 0$.

Coordinate patch adapted to $\mathcal{H}_+$, using $\mathcal{B}$

- In a neighborhood of $\mathcal{H}_+$ we can define coordinates (U, V, x^3, x^4)
  - x^3,x^4 coordinates on $\mathcal{B}$.
  - U affine parameter along nul geodesics forming $\mathcal{H}_+$ ($U = 0$ on $\mathcal{B}$)
  - V affine parameter along null geodesics crossing $\mathcal{H}_+$ ($V = 0$ on $\mathcal{H}_+$)
- Exploiting that coordinate frame with $g(\partial_V, \partial_U) = -1$ on $\mathcal{H}_+$, the metric on $\mathcal{H}$ in $\mathcal{O}$ takes the form:

$$g \restriction_{\mathcal{H}} = -\frac{1}{2} dU \otimes dV - \frac{1}{2} dV \otimes dU + \sum_{i,j=3}^{4} h_{ij}(x^3, x^4) dx^i \otimes dx^j \ ,$$

- h – independent from V, U – is the **Euclidean** metric on the bifurcation surface $\mathcal{B}$: a **spacelike** 2-**surface**.
- In $\mathcal{O}$ the Killing field K takes the form:

$$K^V(p) = -\kappa V + V^2 R_1(p) \ , \quad K^U(p) = \kappa U + V^2 R_2(p) \ ,$$

$$K^i(p) = V R_i(p) \ , \ i = 3, 4 \ ,$$

R_1, R_2, R_i bounded smooth functions.

Shrinking procedure for smearing functions

Re-intepreting the limit "$x \to y$" in Parikh-Wilczek picture, we are interested in computing:

$$\lim_{\lambda \to 0^+} \omega(\phi(f_\lambda), \phi(f'_\lambda))$$

where the limit $\lambda \to 0^+$ **shrinks the supports of f and f' on $\mathcal{H}$**. Explicitly, making use of our coordinates adapted to $\mathcal{H}$:

$$f_\lambda(V, U, x) = \frac{1}{\lambda} f\left(\frac{V}{\lambda}, U, x\right)$$

We restrict the choice of f, f' to

$$f = \partial_V F \ , \qquad f' = \partial_V F' \ , \quad F, F' \in C_0^\infty(\mathcal{O}),$$

to remove an infrared divergence arising in the computation of $\omega(\phi(f_\lambda)\phi(f'_\lambda))$ as $\lambda \to 0^+$.

Main Theorem

Theorem

(V.M., N. Pinamonti [2010])
Assuming that LGH hold in $\mathcal{O}$ and that

$$\omega_2(x,y) = \frac{U(x,y)}{\sigma(x,y)} + \textit{less singular terms}\,,$$

for f_λ and f'_λ defined as above it holds:

$$\lim_{\lambda\to 0^+} \omega\left(\phi(f_\lambda)\phi(f'_\lambda)\right) =$$

$$= -\lim_{\epsilon\to 0^+} \int\limits_{\mathbb{R}^4\times\mathcal{B}} \frac{F(V,U,s)F'(V',U',s)}{16\pi(V-V'-i\epsilon)^2}\, dUdVdU'dV'd\mu(s)\,.$$

μ being the measure induced by h on the bifurcation.

Some technical comments

- Result similar to that by Fredenhagen and Haag but now it holds for a generic local Killing horizon and no spherical symmetry is required.
- The proof is mainly based on the following expression for $\sigma(p,q)$ appearing in $\omega_2(p,q)$. Let $s : \mathcal{O} \to \mathcal{B}$ be the natural projection onto $\mathcal{B}$, if $p, q \in \mathcal{O}$:

$$\sigma(p,q) = \ell(s(p), s(q)) - (U_p - U_q)(V_p - V_q) + R(p, V_q, U_q)$$

ℓ squared **geodesic distance** on $(\mathcal{B}, h)$,
$R(p, V_q, U_q) = AV_p^2 + BV_q^2 + CV_pV_q$,
A, B, C bounded smooth functions of p, V_q, U_q.
It generalizes a similar technical result found and used by Kay and Wald.

Physical interpretation

$$\omega\left(\phi(f_\lambda)\phi(f'_\lambda)\right)|_{\lambda=0} = -\int_{\mathbb{R}^4\times\mathcal{B}} \frac{F(V,U,s)F'(V',U',s)}{16\pi(V-V'-i0_+)^2}\, dUdVdU'dV'd\mu(s)\,.$$

- To go on we need a notion of time for the external region at least. Natural choice: **the parameter** τ **of the Killing field** K.
  In $\mathcal{O}$, close to $\mathcal{H}$ in view of the LGH one has:

$$V(\tau) \simeq -e^{-\kappa\tau} \text{ “internal region”},\ K \text{ spacelike and } V < 0$$

$$V(\tau) \simeq e^{-\kappa\tau}\ , \text{“external region”}\ ,\ K \text{ timelike and } V > 0$$

- The Fourier transform with respect to τ individuates the **energy spectrum** with respect to the notion of energy E associated with τ.

Tunneling Probability

Case of f_λ and f'_λ with supports separated by the horizon $\mathcal{H}$.

Tunneling probability:

$$\omega(\phi(f_\lambda)\phi(f'_\lambda))|_{\lambda=0} = \lim_{\epsilon\to 0^+} \frac{\kappa^2}{64\pi} \int \frac{F(\tau,U,x)F'(\tau',U',x')}{\cosh(\frac{\kappa}{2}(\tau-\tau')+i\epsilon)^2} d\tau dU d\tau' dU' d\mu(x)$$

Passing to the τ-**Fourier transform** we have, if $\beta_H = 2\pi/\kappa = 1/T_H$

$$\omega(\phi(f_\lambda)\phi(f'_\lambda))|_{\lambda=0} = \int_{\mathbb{R}^2\times\mathcal{B}} \int_{-\infty}^{\infty} \frac{\overline{\widehat{F}(E,U,x)}\widehat{F'}(E,U',x)}{16\sinh(\beta_H E/2)} E dE \, dU dU' d\mu(x) \, .$$

For **wave packets** sharply centered around E_0 it produces:

$$\lim_{\lambda\to 0} |\omega(\phi(f_\lambda)\phi(f'_\lambda))|^2 \sim \text{const.}\ E_0^2 \, e^{-\beta_H E_0} \, ,$$

in agreement with the result of Parikh and Wilczek.

Source of the Hawking radiation

Case of f_λ and f'_λ with both supports in the external region.

$$\omega(\phi(f_\lambda)\phi(f'_\lambda))|_{\lambda=0} = -\lim_{\epsilon\to 0^+}\int \frac{\kappa^2 F(\tau,U,x)F'(\tau',U',x)}{64\pi(\sinh(\frac{\kappa}{2}(\tau-\tau'))+i\epsilon)^2} d\tau dU d\tau' dU' d\mu$$

The τ-**Fourier transfation** produces the **Bose spectrum** at the Hawking temperature:

$$\omega(\Phi(f_\lambda)\Phi(f'_\lambda))|_{\lambda=0} = \int_{\mathbb{R}^2\times\mathcal{B}} \int_{-\infty}^{\infty} \frac{\overline{\widehat{F}(E,U,x)}\widehat{F'}(E,U',x)}{1-e^{-\beta_H E}} EdE \frac{dUdU'd\mu(x)}{32},$$

That result generalises the core of the Fredenhagen-Haag's explanation of the Hawking radiation for $\mathcal{H}$ complete in the future (in the spherically symmetric case and expoiting Klein-Gordon equation).

Comments, further developments, open issues I

- We have proved that Parikh and Wilczek's result has a **precise** and **rigorous** meaning if adopting the viepoint of algebraic QFT in curved spacetime.
- Our computation of the tunneling probability is **local in space and time**, and it strongly supports the idea that the **Hawking radiation is (also) a local phenomenon**, independent from the existence of a whole black hole. However our results work for the full Kerr-Newman class of non-extreme black holes, including the **charged rotating** one, ethernal or produced by a collapse.
- The results are **independent from the state** of the quantum field, provided it belongs to a large class including **Hadamard states**.
- The results are **independent from any particular field equation satisfied by the quantum field**.

Comments, further developments, open issues II

- We expect that the results may be true even considering **interacting quantum fields**, at least **perturbatively** and taking the **renormalization procedure in curved spacetime** into account (Hollands - Wald, Brunetti - Fredenhagen).
  We are investigating the case of the $\lambda\phi^3$ **interaction** for Hadamard states. At one-loop (i.e. λ^2 order) our results survives in Rindler spacetime, since the renormalized two-point functions show the short distance behaviour $\sim \Delta^{1/2}\sigma^{-1} + \cdots$.
  We expect that it is valid at least in conformally flat spacetimes.

Thanks a lot for your attention!