Quantum space-time and the horizon problem

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Motivations

- At short distance the spacetime should be **non-commutative**.
- This feature should be encoded in the **"Quantum Gravity"**

No satisfactory description.

- We can get information about such a theory analyzing **some particular regimes** *[Hawking]*.
- Gravity classically Matter by quantum theory.

$$G_{ab} = 8\pi \langle T_{ab} \rangle_{\omega}$$

- *Doplicher, Fredenhagen and Roberts 95* use this to obtain **uncertainty relations** for the coordinates on a **flat** quantum space.

Questions

Does it work also on curved spacetimes?
If yes, which are the implication of sp-non-commutativity on GR?
And at the early universe?

Plan of the talk

- Formation of trapped surfaces out of measurements
  - A measuring process: model of the quantum detector
  - The influence on the curvature and appearance of trapped surfaces
- An application in cosmological
  - Energy density on Quantum Minkowski Spacetime
  - Backreaction on Quantum (FRW) Spacetime

Biblipgraphy

- S. Doplicher, K. Fredenhagen, J. Roberts CMP **172**, 187 (1995)
- D. Bahn, S. Doplicher, K. Fredenhagen, G. Piacitelli CMP **237**, 221 (2003)
- S. Doplicher, G. Morsella, NP, [arXiv:1201.2519] (2012)

- In *[DFR 95]* the authors find the commutation rules among the coordinates

$$[q^\mu, q^\nu] = iQ^{\mu\nu}$$

compatible with the following uncertainty relations

$$\Delta x_0 \left(\Delta x_1 + \Delta x_2 + \Delta x_3\right) \geq \lambda_P^2,$$

$$\Delta x_1 \Delta x_2 + \Delta x_2 \Delta x_3 + \Delta x_3 \Delta x_1 \geq \lambda_P^2$$

which are obtained using the following:

Minimal Principle (P0):

We cannot create a singularity just observing a system.

- Together with the **Heisenberg principle (HP)** (valid in Minkowski).
- The uncertainties are tailored to the flat spacetime.
- On a curved spacetime we have to **replace** it with something else.
- We use **QFT on CST** and their comm. rel. in combination with **P0**.
- We shall perform such analysis on a spherically symmetric space.

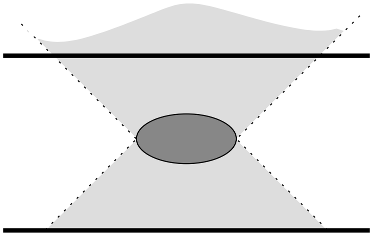
Formation of trapped surfaces out of measurements

- A measuring process: model of the quantum detector
- Influence on curvature and appearance of trapped surfaces

A measuring process:
model of a spherically symmetric quantum detector

Idealized measurement:
light scattered by a target.

- two steps:
  1. preparation: **Incoming focused light**
  2. detection: **scattered light** by an object localized in $\mathcal{O}$



- **Problem when:**
  incoming light is too focused.
- Let's look at this.

Simple model for light

Quantum field theory over a curved space-time M. Consider

$$-\Box\phi = 0$$

Quantize using **algebraic methods**:

- Construct $\mathcal{A}(M)$ the $*-$algebra of **observables** generated by $\phi(f)$ and its *Wick products* $\phi^2(f), \dots$
- It encompasses the **commutation relations**.

$$[\phi(f), \phi(g)] = \Delta(f, g)$$

- **States** are positive linear functionals over $\mathcal{A}$ (With further nice properties: **Hadamard condition**).

Example: $\phi(f)$, $T_{\mu\nu}(f)$ are contained in $\mathcal{A}(M)$.
Their **expectation values** are

$$\langle\phi(f)\rangle := \omega(\phi(f)) \,, \qquad \langle T_{\mu\nu}(f)\rangle := \omega(T_{\mu\nu}(f))$$

- Assume that on a state ω it holds

$$G_{\mu\nu} = 8\pi\, \omega(T_{\mu\nu})$$

- **Prepare the incoming light** by applying $\phi(f)$ on ω.

Proposition

We call ω_f **prepared state**

$$\omega_f(A) = \frac{\omega\left(\phi(f)\; A\; \phi(f)\right)}{\omega(\phi(f)\phi(f))} \qquad \forall A \in \mathcal{A}(M)$$

Thus the expectation values of the observables are changed. For example

$$\langle T_{\mu\nu}\rangle_{f,0} := \omega_f(T_{\mu\nu}) - \omega(T_{\mu\nu})\,.$$

Evaluation of the change in $\langle T_{\mu\mu} \rangle_{f,0}$

Proposition

Consider $\mathcal{A}(M)$ generated by ϕ real and a quasi free Hadamard state ω

$$\langle \partial_\mu \phi \partial_\mu \phi(x) \rangle_{f,0} \geq \frac{1}{2} \frac{|\partial_\mu \Delta(f)(x)|^2}{\omega\left(\phi(f)\phi(f)\right)}, \qquad f \in C_0^\infty(M)$$

x is a point of M and Δ is the causal propagator.
Whenever $\mu-$direction is light-like $T_{\mu\mu}(x) = \partial_\mu \phi \partial_\mu \phi(x)$

▸ Proof

Influence on curvature and appearance of trapped surfaces

We want to solve

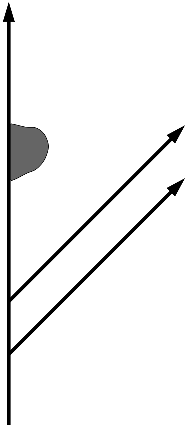
$$G_{\mu\nu} = 8\pi\, \omega_f(T_{\mu\nu}).$$

It is **very** difficult. Assume **spherical symmetry**.

- Spacetime is $\mathbb{R}^2 \times \mathbb{S}^2$, **retarded coordinates**:
- spanned by **future null geodesic** emanated from the center of the sphere γ
  - u proper time on the worldline line γ
  - s **affine parameter along** the null geodesics with $s(0) = 0$ and $\dot{s}(0) = 1$
  - $r(u, s)$ is the retarded distance.
- The most **generic metric** is

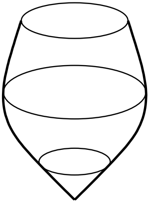
$$ds^2 := -A(u,s)du^2 - 2dsdu + r(u,s)^2 d\mathbb{S}^2$$

- Fix u, the family of null geodesics form a cone $\mathcal{C}_u$



- For every $\mathcal{C}_u$ consider the **expansion parameter** θ of that family

θ measures the **rate of change** of $4\pi r^2$ along $\mathcal{C}_u$



  - $\theta > 0$ expansion
  - $\theta = 0$ trapped surface
  - $\theta < 0$ contraction

- Its evolution along $\mathcal{C}_u$ is governed by the **Raychaudhuri** equation

$$\dot{\theta} = -\frac{\theta^2}{2} - R_{ss}; \qquad \lim_{s\to 0^+} s\theta = 2$$

- We solve this equation **semiclassically** namely:

$$R_{ss} = 8\pi\, \omega_f(T_{ss}) = 8\pi\, \omega(T_{ss}) + 8\pi\langle T_{ss}\rangle_{f,0} = R^{(0)}_{ss} + 8\pi\langle T_{ss}\rangle_{f,0}$$

$R^{(0)}_{\mu\nu}$ is the *"curvature"* without the influence of the measurement.

Theorem about the formation of a trapped surfaces

M spherically symmetric. $\mathcal{A}(M)$ and ω as before. Assume:

1. Semiclassical Einstein equations are satisfied by ω and M;
2. $R_{ss}^{(0)} = 8\pi\omega(T_{ss})$ is positive on $\mathcal{C}_0$;
3. For every f supported in $J^+(\mathcal{C}_0)$

$$|\omega_2(f,f)| \leq C\|s\psi_f\|_2\|\partial_s(s\psi_f)\|_2,$$

$\psi_f = \Delta(f) \restriction \mathcal{C}_0$. $\|\cdot\|_2$ is the L^2 norm on $\mathcal{C}_0$ w.r.t $ds \wedge d\mathbb{S}^2$.

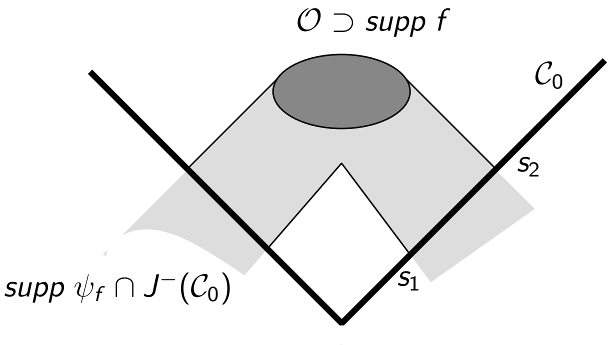
Consider ω_f a state **perturbed** by a symmetric $\phi(f)$ such that:

$supp\ f \subset J^+(\mathcal{C}_0)$ $\qquad$ $supp\ \psi_f \subset \{(s,\Omega) \in \mathcal{C}_0 | s_1 < s < s_2\}$ where

$$s_1 < s_2 < \frac{3}{2}s_1, \qquad (s_2)^2 < \bar{s}^2\ , \qquad \bar{s}^2 := \frac{1}{6\,C}.$$

Hence **semiclassically** θ vanishes in $\mathcal{C}_0$ and thus $J^+(\mathcal{C}_0)$ contains a **trapped surface**.

Hypotheses about $\mathcal{C}_0$ and about f w.r.t. $\mathcal{O}$



$$s_1 < s_2 < \frac{3}{2}s_1, \qquad (s_2)^2 < \bar{s}^2\ , \qquad \bar{s}^2 := \frac{1}{6\,C}.$$

▸ Proof

Comments on the obtained results

Question

Are the hypotheses 1. 2. 3. too strong?

- Solutions of the semiclassical Einstein equation **do exists**.
  (At least in cosmology is a **well posed** system of equations *[NP 2011]*).
- $R_{ss}^{(0)} \geq 0$ is realized in every **reasonable** cosmological model.
  (So it is physically acceptable).
- The asked continuity for the state **occurs** in many concrete examples.
  (Minkowski vacuum, many other Hadamard states of interest *[DPP 2011]*).

Comparison with classical results by Christodoulou:

Our hypotheses imply the Christodoulou one.

Further comments

Fact

There is a minimal lenghtscale λ under which a trapped surface occurs. (θ becomes negative and remains negative).

principle of gravitational stability under localization of events (P0)

$\Longrightarrow$ we cannot detect object smaller then that lengthscale λ.

- The obtained minimal length scale does not permit to obtain a full set of commutation relations among coordinates.
- Non spherical symmetric situations are not addressed.

An application in cosmology

Question:

Which role has the lengthscale λ when back reaction of matter on curvature is considered?

- We don't have a full set of commutation relation.
- Solving semiclassical Einstein equations is a difficult task.

Thus let us consider a cosmological model

$$ds^2 = -dt^2 + a(t)^2 \left[dx^2 + dy^2 + dz^2\right]$$

- a is called **scale factor** it is the single degree of freedom in the model.
- This model **fits the observation** of our universe at large scales:

$$a(t) = exp(Ht)$$

with H (Hubble constant) small but strictly positive.

To understand the role of λ on the dynamics, we would like to solve

$$G_{\mu\nu} = 8\pi \langle T^{NC}_{\mu\nu} \rangle$$

in a *"cosmological non-commutative spacetime"*.

- **Big big problem:** We **don't have** a complete theory in curved space-time,
- We have info on the **minimal length scale** λ which should appear in the products of fields
- **Poor man strategy:** **Learn** how to implement the minimal length scale from the theory in flat space.
- In flat space $\langle T^{NC}_{\mu\nu} \rangle$ are constructed using **states of maximal localization** compatible with λ *[BDFP 2003]*.
- We implement the same effect on a curved background in order to **estimate** the influence of λ on the curvature.

Energy density on Quantum Minkowski Spacetime

Let's have a look at **non commutative** Minkowski:

$\mathcal{E}$ is the C^*-algebra of **quantum coordinates** generated by $q^\mu, Q^{\mu\nu}$ with

$$[q^\mu, q^\nu] = i\lambda^2 Q^{\mu\nu}, \qquad [q^\rho, Q^{\mu\nu}] = 0,$$

$$Q_{\mu\nu}Q^{\mu\nu} = 0, \qquad \left(\frac{1}{4}Q^{\mu\nu}(*Q)_{\mu\nu}\right)^2 = 1,$$

any function f on M can be used to define a function on $\mathcal{E}$, that is:

$$f(q) := \int \check{f}(k)\, e^{ikq}\, d^4k \ ,$$

$T_{\mu\nu}$ is defined as a product of fields **at the same point**.

Remember: on a nc space-time we cannot localize precisely.
The **best approx.** of points is furnished by states of optimal localization!
[Bahns, Doplicher, Fredenhagen, Piacitelli 2003]

$$\mathcal{E}^{(2)} := \mathcal{E} \otimes \mathcal{E} \ , \qquad q_1^\mu := q^\mu \otimes 1, \qquad q_2^\mu := 1 \otimes q^\mu,$$

introduce center of mass and relative coordinates (which commutes)

$$\bar{q}^\mu := \frac{1}{2}(q_1^\mu + q_2^\mu), \qquad \xi^\mu := \frac{1}{\lambda}(q_1^\mu - q_2^\mu),$$

Evaluating ξ^μ on optimally localized states (partial trace) yields the **quantum diagonal map**

$$E^{(2)} : \mathcal{E}^{(2)} \to \bar{\mathcal{E}} := C^*(\{e^{ik\bar{q}}\}) \ .$$

On $f(q_1, q_2) \in \mathcal{E}^{(2)}$, using $|k|^2$ the Euclidean length of $k \in \mathbb{R}^4$

$$E^{(2)}(f(q_1, q_2)) = \int d^4k_1 d^4k_2 \, \check{f}(k_1, k_2) \; e^{-\frac{\lambda^2}{4}|k_1 - k_2|^2} e^{i(k_1+k_2)\bar{q}}.$$

The lenghtscale λ represents the maximal localization.

Wick square and Energy density

Free (scalar) field on QST can be formally defined as

$$\phi(q) = \int d^4k \check{\phi}(k) \otimes e^{ikq}$$

which is an element of $\mathcal{A} \otimes \mathcal{E}$.

Using $E^{(2)}$ we can define the **Quantum Wick Square**

$$\phi_Q^2(\bar{q}) := E^{(2)}(\phi(q_1)\phi(q_2)).$$

and the **Energy Density**: the 00 component of the stress tensor:

$$\rho_Q(\bar{q}) := E^{(2)}\left(\partial_0\phi(q_1)\partial_0\phi(q_2) - \frac{1}{2}\eta_{\mu\nu}\partial_\mu\phi(q_1)\partial^\nu\phi(q_2)\right) .$$

Evaluation in a Thermal state

- Our cosmologist friend: "*our universe was very hot and dense in the past*".
- A relic of this thermal matter is present in the **CMB** (cosmic microwave background).
- For this reason we evaluate those observables on a **KMS state** ω_β at fixed inverse temperature β

$$\omega_\beta(\check{\phi}(k_1)\check{\phi}(k_2)) = \delta(k_1 + k_2)\delta(k_1^2)\frac{\varepsilon(k_1^0)}{1 - e^{-\beta k_1^0}}.$$

- It is **"translationally invariant"** $\implies$ the expectation value of $\overline{q}$ does not appear

Expectation values

The **energy density** is

$$\omega_\beta(:\rho:_Q) := (\omega_\beta - \omega_0)(\rho_Q) = 4\pi \int_0^{+\infty} k^3 \frac{e^{-\lambda^2 k^2}}{e^{\beta k} - 1}\, dk\,.$$

There are **two characteristic lengths** λ and β.

All the information about QST is in λ. It alters the **spectrum** of the state. Effectively it is a change in the smearing of the two-point function

$$\omega_\beta(:\phi^2:) := (\omega_\beta^2 - \omega_0^2)(\delta) \qquad \Longrightarrow \qquad \omega_\beta(:\phi^2:_Q) := (\omega_\beta^2 - \omega_0^2)(g).$$

where $\delta = \delta(x-y)$ is replaced by $g = Ne^{-(x-y)^2/\lambda^2}$

Let's study the **asymptotic form** of ρ for small and large λ/β:

$$\omega_\beta(:\rho:_Q) \simeq C_1 \frac{1 - \frac{\lambda^2}{\beta^2} C_2}{\beta^4}\,, \qquad \omega_\beta(:\rho:_Q) \simeq C_2 \frac{1}{\beta\lambda^3}\,,$$

Backreaction on Quantum (FRW) Spacetime

$$G_{\mu\nu} = 8\pi\omega(:\, T_{\mu\nu} :_Q).$$

In a flat FRW space-time it reduces to the Friedmann equation

$$H^2(t) = \omega(:\, \rho :_Q).$$

- Choose **conformal matter**.
- ω_β^M is a **conform KMS state** for the commutative theory.
- Temperature **scales** with a, $(\beta = \beta_0 a(t))$.
- The length scale λ its **scale invariant**.
- A *"reasonable expression"* for the expectation value of the energy density is

$$\rho_\beta(t) := \omega_\beta^M(:\, \rho :_Q) = 4\pi \int_0^{+\infty} dk\, k^3 \frac{e^{-\lambda^2 k^2}}{e^{\beta(t)k} - 1}.$$

Asymptotic form of the energy density

In an **ethereally expanding** universe with a Big Bang we have

$$\beta(t) = a(t)\beta_0$$

- In the **future** it holds

$$\frac{\lambda}{\beta} << 1 \,, \qquad \rho_\beta(t) \simeq C_1 \frac{1}{\beta_0^4 a(t)^4}$$

The effect due to spacetime noncommutativity can be neglected

- When the **universe was very small** (close to the Big Bang)

$$\frac{\lambda}{\beta} >> 1 \,, \qquad \rho_\beta(t) \simeq \frac{C_2}{\beta_0 a(t) \lambda^3}.$$

Which is **less divergent** for $a \to 0$, than classical matter.

Form of the Big Bang singularity

Consider

$$H^2(t) = \frac{C}{a(t)}.$$

$M = (a, b) \times \mathbb{R}^3$ is conf. flat $\Longrightarrow$ **imbed** M in $\mathbb{R}^4$

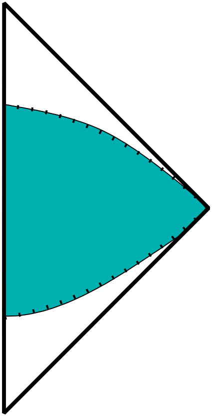
At which **conformal time** occurs the singularity?

$$\tau_0 - \tau = \int_t^{t_0} \frac{1}{a(t')} dt' = \int_a^{a_0} \frac{1}{a'^2 H(a')} da'$$

$$= \frac{2}{\sqrt{C}} \left(\frac{1}{\sqrt{a}} - \frac{1}{\sqrt{a_0}} \right)$$

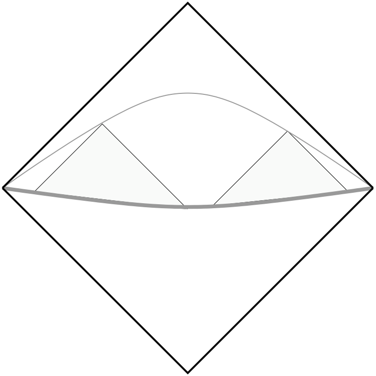
Thus

$$\lim_{a \to 0} \tau = -\infty$$



What does it mean? Horizon problem

$$ds^2 = a^2\left(-d\tau^2 + d\mathbf{x}^2\right).$$



- **Classical solution**
  Radiation dominated:
  $\tau =\rightarrow \tau_0$ for $a \rightarrow a_0$
  Horizon problem.

- **Quantum NC Corrections**
  $\rho = 1/a(t)$:
  $\tau \rightarrow -\infty$ for $a \rightarrow 0$
  Singularity is light like,
  No Horizon Problem

  Power law inflation with
  Null Big Bang $\Im^- \cup i^-$

Summary

- Analysis of the measure process.
- Semiclassical Backreaction can be used to constraint the **non commutativity**.
- Using the obtained minimal length we can **estimate** the role of the non commutativity on the curvature.
- In a cosmological model the **Horizon problem disappears**.

Open Questions

- Can we say something for the generic case?
- Can we construct a full fledged non commutative quantum theory on curved space-time?

Thanks a lot for your attention!

Proof

$\{\xi_n\} \subset C_0^\infty(M)$ converging weakly to $\partial_\mu \delta(x)$. Notice that

$$\lim_{n\to\infty} \langle \phi(\xi_n)\phi(\xi_n)\rangle_{f,0} = \langle T_{\mu\mu}(x)\rangle_{f,0}$$

$$\langle \phi(\xi_n)\phi(\xi_n)\rangle_{f,0} = 2\frac{|\omega_2(f,\xi_n)|^2}{\omega_2(f,f)} = 2\frac{|\omega_{2,A}(f,\xi_n)|^2 + |\omega_{2,S}(f,\xi_n)|^2}{\omega_2(f,f)}$$

Hence

$$\langle \phi(\xi_n)\phi(\xi_n)\rangle_{f,0} \geq \frac{1}{2}\frac{|\langle \xi_n, \Delta(f)\rangle|^2}{\omega_2(f,f)} .$$

back

Proof

Raychauduri equation in integral form

$$\theta(s_2) = \theta(s_1) - \int_{s_1}^{s_2} \frac{\theta^2}{2} ds - \int_{s_1}^{s_2} R_{ss} ds \,.$$

Use **hypothesis 1** and **hypothesis 2** to get

$$s_2\ \theta(s_2) \leq s_2\ \theta(s_1) - 8\pi s_2 \int_{s_1}^{s_2} \langle T_{ss}\rangle_{f,0} ds \,.$$

In the past of s_1 the space-time is unperturbed and from **hypothesis 3**

$$s_2\theta(s_2) \leq 2\frac{s_2}{s_1} - \frac{s_2}{C} \frac{\|\partial_s\psi_f\|_2^2}{\|s\psi_f\|_2 \|\partial_s(s\psi_f)\|_2}$$

thanks to the support properties of ψ_f, using standard properties of L^2 norms

$$\|\psi_f\|_2 \leq s_2 \|\partial_s\psi_f\|_2 \,, \qquad \|\partial_s(s\psi_f)\|_2 \leq 2s_2\|\partial_s\psi_f\|_2$$

Using the constraints on s_1 and s_2

$$s_2\theta(s_2) \leq 3 - \frac{1}{2C\,(s_2)^2} \,.$$

Thus $\theta(s_2)$ is surely negative because $(s_2)^2 < 1/(6C)$ per hypothesis.