Noncommutative Gravity on Moyal plane: Second order correction

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Outline

Introduction

1. M. Dimitrijević, V. Radovanović and H. Štefančić, NC gravity, work in progress

Introduction

Quantum field theory encounters problems at high energy/small distances.
Some modifications are needed. One possibility is noncommutativity among space time coordinates. It is given by

$$[x^\mu, x^\nu] = i\theta^{\mu\nu}(x) \ .$$

Canonical noncommutativity

$$[x^\mu, x^\nu] = i\theta^{\mu\nu} = \text{const.}$$

Different modes are constructed on canonical NC space time: ϕ^4,QED, standard model, SUSY models;
renormalizability, unitarity,...

Introduction

Generalization general relativity or some other gravity theory (like Poincare gauge theory) on NC spacacetime is a difficult task.
Many attempts:
• Twist approach
1. P.Aschieri, C.Blohmann, M.Dimitrijević, F. Meyer, P.Schupp and J. Wess, CQG **2**2, 3511-3522 (2005),
2. P.Aschieri, M. Dimitrijević, F.Meyer and J.Wess, CQG **2**3, 1883-1912 (2006), [hep-th/0510059].

Introduction

- Sieberg-Witten approach

1. Ali Chamseeddine, PLB. **5**04 (2001) 33;PRD. **6**9 (2004) 024015
2 M. A. Cardella and D. Zanon, CQG **2**0 (2003) L95
3. P. Aschieri and L. Castellani, JHEP (0906) (2009) 086
4. P. Aschieri and L. Castellani, arXiv:1111.4822, ArXiv: 12051911
5. R. Banerjee, P. Mukherjee and S. Samanta, PRD **7**5, (2007) 125020
6. Yang-Gang Miao, Zhao Xue and Shao-Jun Zhang, PRD **8**3, (2011) 024023

AdS gauge theory on commutative spacetime

Consider a gauge theory with $SO(2,3)$ as a gauge group in $4D$ Minkowski spacetime.
$SO(2,3)$ is the isometry group anti de Sitter space.
Anti de Sitter space is a maximally symmetric space with a negative constant curvature.
M_{AB}-generators of $SO(2,3)$ group
$A, B, \dots$ take values $0, 1, 2, 3, 5$.
Commutation relations:

$$[M_{AB}, M_{CD}] = i(\eta_{AD}M_{BC} + \eta_{BC}M_{AD} - \eta_{AC}M_{BD} - \eta_{BD}M_{AC}), \quad (1)$$

$\eta_{AB} = \mathrm{diag}(+,-,-,-,+)$ is $5D$ metric.

AdS gauge theory on commutative spacetime

Clifford generators Γ_A in $5D$ Minkowski space satisfy

$$\{\Gamma_A, \Gamma_B\} = 2\eta_{AB} \ . \quad (2)$$

M_{AB} are

$$M_{AB} = \frac{i}{2}[\Gamma_A, \Gamma_B] \ . \quad (3)$$

$\gamma_a, \ (a = 0, 1, 2, 3)$ are the gamma matrices in $4D$ Minkovski spacetime
The gamma matrices in $5D$ are

$$\Gamma_A = (i\gamma_a\gamma_5, \gamma_5) \ .$$

γ_5 is defined by $\gamma^5 = \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$.

AdS gauge theory on commutative spacetime

It is easy to show that

$$\begin{aligned} M_{ab} &= \frac{i}{4}[\gamma_a,\gamma_b] = \frac{1}{2}\sigma_{ab} \ , \\ M_{5a} &= \frac{i}{2}\gamma_a \ . \end{aligned} \tag{4}$$

If we introduce momenta $P_a = \frac{1}{l}M_{a5}$,where l is a constant with dimensions of length AdS algebra (1) becomes

$$\begin{aligned} [M_{ab},M_{cd}] &= i(\eta_{ad}M_{bc}+\eta_{bc}M_{ad}-\eta_{ac}M_{bd}-\eta_{bd}M_{ac}) \\ [M_{ab},P_c] &= i(\eta_{bc}P_a-\eta_{ac}P_b) \\ [P_a,P_b] &= -i\frac{1}{l^2}M_{ab} \ . \end{aligned} \tag{5}$$

In the limit $l\to\infty$ AdS algebra reduces usual Poincare algebra in $4D$ spacetime. (Wigner-Inonu contraction)

AdS gauge theory on commutative spacetime

ψ spinor matter field in the fundamental representation
Under the infinitesimal $SO(2,3)$ gauge transformations it transforms as

$$\delta_\epsilon \psi = i\epsilon\psi = \frac{i}{2}\epsilon^{AB} M_{AB}\psi \tag{6}$$

The covariant derivative in the fundamental representation

$$D_\mu\psi = \partial_\mu\psi - i\omega_\mu\psi \ , \tag{7}$$

$$\omega_\mu = \frac{1}{2}\omega_\mu^{AB} M_{AB} = \frac{1}{4}\omega_\mu^{ab}\sigma^{ab} - \frac{1}{2}\omega_\mu^{a5}\gamma_a \tag{8}$$

is the $SO(2,3)$ gauge potential. Decomposition: ω_μ^{AB} *to* ω_μ^{ab}, ω_μ^{a5},
ω_μ^{ab} is a spin connection
$\omega_\mu^{a5} = \frac{1}{l} e_\mu^a$ are vielbeins (tetrads).

AdS gauge theory on commutative spacetime

The transformation law of the $SO(2,3)$ potential is given by

$$\delta_\epsilon \omega_\mu^{AB} = \partial_\mu \epsilon^{AB} - \epsilon^A{}_C \omega_\mu^{CB} + \epsilon^B{}_C \omega_\mu^{CA} \ . \tag{9}$$

If $\epsilon^{a5} = 0$ we obtain the transformation laws for spin connection and vielbeins under the $SO(1,3)$ gauge transformation:

$$\delta_\epsilon \omega_\mu^{ab} = \partial_\mu \epsilon^{ab} - \epsilon^a{}_c \omega_\mu^{cb} + \epsilon^b{}_a \omega_\mu^{ca}, \tag{10}$$

$$\delta_\epsilon e_\mu^a = -\epsilon^{ad} e_\mu^d \tag{11}$$

We reduce the local anti de Sitter symmetry down to the local Lorentz symmetry:

$$SO(2,3) \to SO(1,3)$$

AdS gauge theory on commutative spacetime

The field strength

$$F_{\mu\nu} = \partial_\mu \omega_\nu - \partial_\nu \omega_\mu - i[\omega_\mu, \omega_\nu] = \frac{1}{2} F_{\mu\nu}^{AB} M_{AB} \ . \tag{12}$$

Decomposition of $F_{\mu\nu}^{AB}$

$$F_{\mu\nu}^{AB} \ \rightarrow \ F_{\mu\nu}^{ab}, \ F_{\mu\nu}^{a5}$$

$$F_{\mu\nu} = \left[R_{\mu\nu}^{ab} - \frac{1}{l^2}(e_\mu^a e_\nu^b - e_\mu^b e_\nu^a) \right] \frac{\sigma_{ab}}{4} + F_{\mu\nu}^{a5} \frac{\gamma_a}{2} \ , \tag{13}$$

where the Reiman curvature tensor and torsion are

$$R_{\mu\nu}^{ab} = \partial_\mu \omega_\nu^{ab} - \partial_\nu \omega_\mu^{ab} + \omega_\mu^{ac} \omega_\nu^{cb} - \omega_\mu^{bc} \omega_\nu^{ca} \tag{14}$$

$$l F_{\mu\nu}^{a5} = D_\mu e_\nu^a - D_\nu e_\mu^a = T_{\mu\nu}^a \tag{15}$$

AdS gauge theory on commutative spacetime

Under the local anti de Sitter transformation field strength transforms as

$$\delta_\epsilon F_{\mu\nu} = i[\epsilon, F_{\mu\nu}] \tag{16}$$

or more explicitly

$$\begin{aligned}
\delta_\epsilon F^{ab}_{\mu\nu} &= -\epsilon^{ac} F_{\mu\nu c}{}^{b} + \epsilon^{bc} F_{\mu\nu c}{}^{a} - \epsilon^{a5} F_{\mu\nu 5}{}^{b} + \epsilon^{b5} F_{\mu\nu 5}{}^{a} \\
\delta_\epsilon T^{a}_{\mu\nu} &= -\epsilon^{ac} T_{\mu\nu c} + \epsilon^{5c} F_{\mu\nu c}{}^{a} \ .
\end{aligned} \tag{17}$$

If we reduce the initial anti de Sitter gauge symmetry down to its Lorentz subgroup by taking $\epsilon^{a5} = 0$ we obtain the correct transformation laws:

$$\begin{aligned}
\delta_\epsilon F^{ab}_{\mu\nu} &= -\epsilon^{a}{}_{c} F^{cb}_{\mu\nu} + \epsilon^{b}{}_{c} F^{ca}_{\mu\nu} \\
\delta_\epsilon T^{a}_{\mu\nu} &= -\epsilon^{a}{}_{c} T^{c}_{\mu\nu} \ .
\end{aligned} \tag{18}$$

AdS gauge theory on commutative spacetime

Action:

$$S = -\frac{il}{16\pi G_N}\mathrm{Tr}\int F\wedge F\phi + \lambda\int d^4x\Big(\frac{1}{4}\mathrm{Tr}\,\phi^2 - l^2\Big) \qquad (19)$$

is invariant under the $SO(2,3)$ gauge transformations
G_N is the Newton gravitational constant.
λ is the Lagrange multiplier.
$\phi = \phi^A\Gamma_A$ is a vector field

$$\delta\phi = i[\epsilon, \phi]\ , \qquad (20)$$

Constraint $\phi_A\phi^A = l^2$.

AdS gauge theory on commutative spacetime

Taking $\phi^a = 0,\ \phi^5 = l$ the $SO(2,3)$ symmetry is broken spontaneously down to $SO(1,3)$

The action after SSB:

$$
\begin{aligned}
S &= -\frac{il^2}{16\pi G_N}\mathrm{Tr}\int (F\wedge F\gamma_5) \\
&= -\frac{il^2}{64\pi G_N}\epsilon^{\mu\nu\rho\sigma}\int d^4x\,\mathrm{Tr}\left(F_{\mu\nu}F_{\rho\sigma}\gamma_5\right) \\
&= -\frac{il^2}{16\pi G_N}\int\Big[\mathrm{Tr}\left(R\wedge R\gamma_5\right) - \frac{2}{l^2}\mathrm{Tr}\left(R\wedge e\wedge e\gamma_5\right) \\
&\quad + \frac{1}{l^4}\mathrm{Tr}\left(e\wedge e\wedge e\wedge e\gamma_5\right)\Big] \\
&= \frac{1}{16\pi G_N}\int d^4x\Big[\frac{l^2}{16}\epsilon^{\mu\nu\rho\sigma}\epsilon_{abcd}R_{\mu\nu}^{ab}R_{\rho\sigma}^{cd} + \sqrt{-g}R - 2\sqrt{-g}\Lambda\Big] \qquad (21)
\end{aligned}
$$

where $\Lambda = -3/l^2$ and $\sqrt{-g} = \det e^a_\mu$.

AdS gauge theory on commutative spacetime

In this action (see the third line) the vielbeins and spin connection are independent variables. Varying the action with respect to the spin connection we obtain an equation which relates connection and vielbeins. In this way we can express the spin connection in terms of vielbeins. Since there is no fermionic matter in the action (21) this equation gives the vanishing torsion. In that case the first term in (21) is the Gauss-Bonnet term; it is a topological term and does not contribute to the equations of motion. The second term is the Einstein-Hilbert action, while the last term is the cosmological constant.

AdS gauge theory on commutative spacetime

From vielbeins we can construct the metric tensor

$$g_{\mu\nu} = \eta_{ab} e^a_\mu e^b_\nu \ . \tag{22}$$

The action (21) is invariant under the Lorentz gauge transformations by construction. In addition this action posses invariance under general coordinate transformations. This action will be our starting point for the construction of a noncommutative gravity theory.

References:

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Stelle,West,PRD, 1980
P. Townsend, PRD, 1977

Noncommutative $SO(1,3)_\star$ symmetry

Replace the usual product by the Moyal-Weyl $\star$ product

$$f(x) \star g(x) = e^{\frac{i}{2}\theta^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial y^\nu}} f(x)g(y)|_{y\to x} \ , \qquad (23)$$

where $\theta^{\mu\nu}$ is a constant antisymmetric matrix,
Replace the commutative fields by their noncommutative counterparts.
Noncommutative gauge potential $\hat{\omega}_\mu$
Noncommutative vielbeins: $\hat{E}_\mu$
Noncommutative curvature tensor $\hat{R}_{\mu\nu}$

$$\hat{R}_{\mu\nu} = \partial_\mu \hat{\omega}_\nu - \partial_\nu \hat{\omega}_\mu - i[\hat{\omega}_\mu \overset{\star}{,} \hat{\omega}_\nu] \ . \qquad (24)$$

Noncommutative $SO(1,3)_\star$ symmetry

Under the deformed gauge transformations the gauge potential, the field strength and vielbeins transform as

$$\begin{aligned} \delta^\star_\epsilon \hat{\omega}_\mu &= \partial_\mu \hat{\Lambda}_\epsilon - i[\hat{\omega}_\mu \stackrel{\star}{,} \hat{\Lambda}_\epsilon] \\ \delta^\star_\epsilon \hat{R}_{\mu\nu} &= i[\hat{\Lambda}_\epsilon \stackrel{\star}{,} \hat{R}_{\mu\nu}] \ , \end{aligned} \tag{25}$$

$$\delta^\star_\epsilon \hat{E}_\mu = i[\hat{\Lambda}_\epsilon \stackrel{\star}{,} \hat{E}_\mu] \ , \tag{26}$$

where $\hat{\Lambda}_\epsilon$ is the noncommutative gauge parameter.
The noncommutative fields belong to the enveloping algebra of $so(1,3)$. For example, the $\star$-commutator in (26) does not close in the Lie algebra.

Noncommutative $SO(1,3)_\star$ symmetry

NC action

$$
\begin{aligned}
S &= -\frac{il^2}{16\pi G_N}\int \Big[\mathrm{Tr}\,(\hat{R}\wedge_\star \hat{R}\gamma_5) \\
&\quad -\frac{2}{l^2}\mathrm{Tr}\,(\hat{R}\wedge_\star \hat{E}\wedge_\star \hat{E}\gamma_5)+\frac{1}{l^4}\mathrm{Tr}\,(\hat{E}\wedge_\star \hat{E}\wedge_\star \hat{E}\wedge_\star \hat{E}\gamma_5)\Big] \\
&= -\frac{il^2}{64\pi G_N}\epsilon^{\mu\nu\rho\sigma}\int d^4x\Big[\mathrm{Tr}\,(\hat{R}_{\mu\nu}\star \hat{R}_{\rho\sigma}\gamma_5) \\
&\quad -\frac{2i}{l^2}\mathrm{Tr}\,(\hat{R}_{\mu\nu}\star \hat{E}_\rho\star \hat{E}_\sigma\gamma_5)+\frac{1}{l^4}\mathrm{Tr}\,(\hat{E}_\mu\star \hat{E}_\nu\star \hat{E}_\rho\star \hat{E}_\sigma\gamma_5)\Big]\ .
\end{aligned}
$$

The noncommutative action is invariant under the deformed $SO(1,3)_\star$ gauge symmetry.

Noncommutative $SO(1,3)_\star$ symmetry

Commutative and noncommutative symmetries which correspond to the same gauge group can be related by the Seiberg-Witten map: the map enables one to express the noncommutative variables in terms of the commutative variables. In that way no new degrees of freedom are introduced. SW map can also be seen as an expansion in $\theta^{\mu\nu}$, so the SW approach is known as a $\theta-$expanded theory.
K. Ulker, B. Yapiskan, Phys. Rev. D77 (2008) 065006

Noncommutative $SO(1,3)_*$ symmetry

The noncommutative quantities $\hat{\Lambda}_\epsilon, \hat{\omega}_\mu, \hat{E}_\mu$ are power series in the noncommutative parameter $\theta^{\mu\nu}$:

$$\begin{aligned}
\hat{\Lambda}_\epsilon &= \epsilon + \hat{\Lambda}^{(1)} + \hat{\Lambda}^{(2)} + \dots \ , \\
\hat{\omega}_\mu &= \omega_\mu + \hat{\omega}_\mu^{(1)} + \hat{\omega}_\mu^{(2)} + \dots \\
\hat{E}_\mu &= e_\mu + \hat{E}_\mu^{(1)} + \hat{E}_\mu^{(2)} + \dots ,
\end{aligned} \tag{27}$$

where the higher order corrections are functions of the commutative variables $\epsilon, \omega_m, e_\mu$ and their derivatives. The requirement that the commutator of two deformed gauge transformations is a deformed transformation again:

$$[\delta^\star_\alpha \stackrel{\star}{,} \delta^\star_\beta] = \delta^\star_{-i[\alpha,\beta]} \tag{28}$$

gives the solution for $\Lambda^{(1)}_\epsilon, \Lambda^{(2)}_\epsilon, \dots$.

Noncommutative $SO(1,3)_\star$ symmetry

The solution, up to second order in $\theta^{\mu\nu}$ is given by

$$\hat{\Lambda}^{(1)} = -\frac{1}{4}\theta^{\kappa\lambda}\{\omega_\kappa, \partial_\lambda\epsilon\} \tag{29}$$

and

$$\begin{aligned}\hat{\Lambda}^{(2)} &= \frac{1}{32}\theta^{\kappa\lambda}\theta^{\rho\sigma}\Big(\{\omega_\rho, \{\partial_\sigma\omega_\kappa, \partial_\lambda\epsilon\}\} + \{\omega_\rho, \{\omega_\kappa, \partial_\sigma\partial_\lambda\epsilon\}\} \\ &+ \{\{\omega_\rho, \partial_\sigma\omega_\kappa\}, \partial_\lambda\epsilon\} - \{\{F_{\rho\kappa}, \omega_\sigma\}, \partial_\lambda\epsilon\} \\ &- 2i[\partial_\rho\omega_\kappa, \partial_\sigma\partial_\lambda\epsilon]\Big)\ .\end{aligned} \tag{30}$$

Since ϵ and ω_μ contain σ_{ab} matrices then the noncommutative gauge parameter has the following structure

$$\hat{\Lambda}_\epsilon = \frac{1}{4}\Lambda^{ab}\sigma_{ab} + \tilde{\Lambda} + \tilde{\Lambda}^5\gamma_5\ . \tag{31}$$

This means $[\hat{\Lambda}_\epsilon, \gamma_5] = 0$.

Noncommutative $SO(1,3)_\star$ symmetry

Solving the equation

$$\hat\omega_\mu(\omega)+\delta^\star_\epsilon\hat\omega_\mu(\omega)=\hat\omega_\mu(\omega+\delta_\epsilon\omega) \tag{32}$$

order by order in the noncommutative parameter we can express noncommutative gauge potential $\hat\omega_\mu$ in terms of the commutative one. The first order solution is

$$\begin{aligned}\hat\omega^{(1)}_\mu &= -\frac{1}{4}\theta^{\kappa\lambda}\{\omega_\kappa,\partial_\lambda\omega_\mu+R_{\lambda\mu}\}\\ &= \omega^{(1)}_\mu I+\omega^{(1)}_{\mu5}\gamma_5\ ,\end{aligned} \tag{33}$$

where

$$\begin{aligned}\omega^{(1)}_\mu &= -\frac{1}{16}\theta^{\kappa\lambda}\omega^{ab}_\kappa(\partial_\lambda\omega^{ab}_\mu+R^{ab}_{\lambda\mu})\\ \omega^{(1)}_{\mu5} &= -\frac{i}{32}\theta^{\kappa\lambda}\epsilon_{abcd}\omega^{ab}_\kappa(\partial_\lambda\omega^{cd}_\mu+R^{cd}_{\lambda\mu})\ .\end{aligned} \tag{34}$$

Noncommutative $SO(1,3)_\star$ symmetry

In the second order we obtain

$$\begin{aligned}\hat{\omega}_\mu^{(2)} &= \hat{\omega}_\mu^{ab}\frac{\sigma_{ab}}{4} \\ &= -\frac{1}{8}\theta^{\kappa\lambda}\{\hat{\omega}_\kappa^{(1)}, \partial_\lambda\omega_\mu + R_{\lambda\mu}\} + \{\omega_\kappa, \partial_\lambda\hat{\omega}_\mu^{(1)} + \hat{R}_{\lambda\mu}^{(1)}\} \\ &- \frac{i}{16}\theta^{\kappa\lambda}\theta^{\alpha\beta}[\partial_\alpha\omega_\kappa, \partial_\nu(\partial_\lambda\omega_\mu + R_{\lambda\mu})] \; , \end{aligned} \quad (35)$$

where $\hat{R}_{\mu\nu}^{(1)}$ is the first order corrections to the field strength (see below).

Noncommutative $SO(1,3)_\star$ symmetry

The curvature

$$\hat{R}_{\mu\nu} = R_{\mu\nu} + \hat{R}^{(1)}_{\mu\nu} + \hat{R}^{(2)}_{\mu\nu} + \dots \tag{36}$$

can be obtained from definition (24). The first order is

$$\hat{R}^{(1)}_{\mu\nu} = R^{(1)}_{\mu\nu} I + R^{(1)}_{\mu\nu 5}\gamma^5 \ , \tag{37}$$

where

$$R^{(1)}_{\mu\nu} = \frac{1}{8}\theta^{\rho\sigma}\Big(R^{ab}_{\mu\rho}R_{\nu\sigma ab} - \omega^{ab}_{\rho}\partial_\sigma R^{ab}_{\mu\nu} - \frac{1}{2}\omega^{ab}_{\rho}(\omega^{ae}_{\sigma}R^{eb}_{\mu\nu} - \omega^{be}_{\sigma}R^{ea}_{\mu\nu})\Big) \tag{38}$$

and

$$R^{(1)}_{\mu\nu 5} = \frac{i}{16}\theta^{\rho\sigma}\epsilon_{abcd}\Big(R^{ab}_{\mu\rho}R^{cd}_{\nu\sigma} - \omega^{ab}_{\rho}\partial_\sigma R^{cd}_{\mu\nu} - \omega^{ab}_{\rho}\omega^{ce}_{\sigma}R^{ed}_{\mu\nu}\Big) \ . \tag{39}$$

In the second order we obtain

Noncommutative $SO(1,3)_\star$ symmetry

$$
\begin{aligned}
\hat{R}^{(2)}_{\mu\nu} &= -\frac{1}{8}\theta^{\kappa\lambda}\Big(\{\omega_\kappa, \partial_\lambda \hat{R}^{(1)}_{\mu\nu} + (D_\lambda R_{\mu\nu})^1\} + \{\hat{\omega}^{(1)}_\kappa, \partial_\lambda R_{\mu\nu} + (D_\lambda R_{\mu\nu})\} \\
&\quad -2\{R_{\mu\kappa}, \hat{R}^{(1)}_{\nu\lambda}\} - 2\{\hat{R}^{(1)}_{\mu\kappa}, R_{\nu\lambda}\}\Big) \\
&\quad - \frac{i}{16}\theta^{\kappa\lambda}\theta^{\rho\sigma}\Big([\partial_\rho\omega_\kappa, \partial_\sigma(\partial_\lambda R_{\mu\nu} + D_\lambda R_{\mu\nu})] - 2[\partial_\rho R_{\mu\kappa}, \partial_\sigma R_{\nu\lambda}]\Big)\ ,
\end{aligned}
$$

where ω_μ and $R_{\mu\nu}$ are the gauge potential and the field strength for an arbitrary gauge group.

Noncommutative $SO(1,3)_\star$ symmetry

For the $SO(1,3)$ gauge group we find

$$\hat{R}^{(2)}_{\mu\nu} = R^{(2)ab}_{\mu\nu}\frac{\sigma_{ab}}{4}, \tag{40}$$

$$\begin{aligned}
R^{(2)ab}_{\mu\nu} &= -\frac{1}{8}\theta^{\kappa\lambda}\omega^{ab}_{\kappa}(4\partial_\lambda R^{(1)}_{\mu\nu} + \frac{1}{4}\theta^{\alpha\beta}\partial_\alpha\omega^{cd}_{\lambda}\partial_\beta R^{cd}_{\mu\nu}) \\
&- \frac{i}{8}\epsilon^{abpq}\omega^{pq}_{\kappa}(2\partial_\lambda R^{(1)}_{\mu\nu 5} + \frac{i}{16}\theta^{\alpha\beta}\epsilon_{cdef}\partial_\alpha\omega^{cd}_{\lambda}\partial_\beta R^{ef}_{\mu\nu}) \\
&- \frac{1}{4}\theta^{\kappa\lambda}\omega^{(1)}_{\kappa}(\partial_\lambda R^{ab}_{\mu\nu} + (D_\lambda R_{\mu\nu})^{ab}) \\
&- \frac{i}{8}\theta^{\kappa\lambda}\epsilon_{abcd}\omega^{(1)}_{\kappa 5}(\partial_\lambda R^{cd}_{\mu\nu} + (D_\lambda R_{\mu\nu})^{cd}) \\
&+ \frac{1}{2}\theta^{\kappa\lambda}(2R^{ab}_{\mu\kappa}R^{(1)}_{\nu\lambda} + i\epsilon_{abcd}R^{cd}_{\mu\kappa}R^{(1)}_{\nu\lambda 5}) \\
&+ \frac{1}{16}\theta^{\kappa\lambda}\theta^{\rho\sigma}\Big(\partial_\rho\omega^{ea}_{\kappa}\partial_\sigma(\partial_\lambda R^{be}_{\mu\nu} + (D_\lambda R_{\mu\nu})^{be})
\end{aligned}$$

Noncommutative $SO(1,3)_\star$ symmetry

The noncommutative vielbeins is

$$\hat{E}_\mu = e_\mu + \hat{E}^{(1)}_\mu + \hat{E}^{(2)}_\mu + \dots, \quad (42)$$

where

$$\hat{E}^{(1)}_\mu = -\frac{1}{4}\theta^{\kappa\lambda}\{\omega_\kappa, \partial_\lambda e_\mu + D_\lambda e_\mu\} , \quad (43)$$

$$\begin{aligned} \hat{E}^{(2)}_\mu &= -\frac{1}{8}\Big[\{\hat{\omega}^{(1)}_\kappa, \partial_\lambda e_\mu + D_\lambda e_\mu\} + \{\omega_\kappa, \partial_\lambda \hat{E}^{(1)}_\mu + (D_\lambda E_\mu)^{(1)}\}\Big] \\ &- \frac{i}{16}\theta^{\kappa\lambda}\theta^{\rho\sigma}[\partial_\rho\omega_\kappa, \partial_\sigma(\partial_\lambda e_\mu + D_\lambda e_\mu)] , \end{aligned} \quad (44)$$

where $D_\lambda e_\mu = \partial_\lambda e_\mu - i[\omega_\lambda, e_\mu]$ and

$$(D_\lambda E_\mu)^{(1)} = \partial_\lambda \hat{E}^{(1)}_\mu - i[\omega_\lambda, \hat{E}^{(1)}_\mu] - i[\hat{\omega}^{(1)}_\lambda, e_\mu] + \frac{1}{2}\theta^{\alpha\beta}\{\partial_\alpha\omega_\lambda, \partial_\beta e_\mu\} . \quad (45)$$

Noncommutative $SO(1,3)_\star$ symmetry

The expressions found above are valid for an arbitrary gauge group. In the special case of $SO(1,3)$ we obtain

$$\begin{aligned}\hat{E}_\mu^{(1)} &= E_{\mu d5}^{(1)}\gamma_5\gamma^d \\ &= -\frac{1}{8}\theta^{\kappa\lambda}\epsilon_{abcd}\omega_\kappa^{ab}(2\partial_\lambda e_\mu^c + \omega_\lambda^{ce} v_{\mu e}) \qquad (46)\end{aligned}$$

and

Noncommutative $SO(1,3)_\star$ symmetry

$$\begin{aligned}
\hat{E}^{(2)}_\mu &= E^{(2)}_{\mu a}\gamma^a \\
&= -\frac{1}{4}\theta^{\kappa\lambda}\hat{\omega}^{(1)}_\kappa(2\partial_\lambda e^a_\mu + \omega^{ab}_\lambda e_{\mu b})\gamma_a \\
&- \frac{1}{16}\epsilon_{cdfa}\omega^{cd}_\kappa(2\partial_\lambda E^{(1)f}_{\mu 5} + \omega^{fe}_\lambda E^{(1)}_{\mu e5} - 2i\omega^{(1)}_{\lambda 5} e^f_\mu \\
&\quad + \frac{1}{4}\theta^{\alpha\beta}\epsilon_{mnef}\partial_\alpha\omega^{mn}_\lambda\partial_\beta e^e_\mu)\gamma^a \\
&+ \frac{1}{16}\theta^{\kappa\lambda}\theta^{\rho\sigma}\partial_\rho\omega^{ab}_\kappa\partial_\sigma(\partial_\lambda e^b_\mu + \omega^{bd}_\lambda e_{\mu d})\gamma_a. \qquad (47)
\end{aligned}$$

Action: SW expansion

$$\hat{S} = S + S^{(1)} + S^{(2)} + \dots, \tag{48}$$

The first order we have

$$\begin{aligned} S^{(1)} &= -\frac{il^2}{64\pi G_N}\epsilon^{\mu\nu\rho\sigma}\int d^4x\Big[2\mathrm{Tr}\,(\hat{R}^{(1)}_{\mu\nu}R_{\rho\sigma}\gamma_5) - \frac{2i}{l^2}\mathrm{Tr}\,(\hat{R}^{(1)}_{\mu\nu}e_\rho e_\sigma\gamma_5) \\ &- \frac{4i}{l^2}\mathrm{Tr}\,(R_{\mu\nu}\hat{E}^{(1)}_\rho e_\sigma\gamma_5) + \frac{2}{l^4}\mathrm{Tr}\,[e_\mu e_\nu(e_\rho\hat{E}^{(1)}_\sigma + \hat{E}^{(1)}_\rho e_\sigma \\ &+ \frac{i}{2}\theta^{\alpha\beta}\partial_\alpha e_\rho\partial_\beta e_\sigma)\gamma_5]\Big] \end{aligned} \tag{49}$$

$$S^{(1)} = 0 \tag{50}$$

SW freedom does not change the result.

Action: SW expansion

The second order term

$$S^{(2)} = S^{(2)}_{GB} + S^{(2)}_{EH} + S^{(2)}_{\Lambda} \ . \tag{51}$$

The first term is a deformation of Gauss-Bonnet topological term;
the second term is deformation of Einstein-Hilbert action
the last term is a deformation of the cosmological constant term.
To simplify the calculation we take $\omega_\mu = \mathrm{const.}$

$$\begin{aligned} \omega_\mu^{ac}\omega_\nu^{cb} - \omega_\mu^{bc}\omega_\nu^{ca} &\rightarrow R_{\mu\nu}^{ab} \\ \omega_\mu^{ac}R_{\nu\rho}^{cb} - \omega_\mu^{bc}R_{\nu\rho}^{ca} &\rightarrow (D_\mu R_{\nu\rho})^{ac} \ . \end{aligned} \tag{52}$$

Action:SW expansion

The deformation of the Gauss-Bonet term is

$$\begin{aligned} S^{(2)}_{GB} &= -\frac{il^2}{64\pi G_N}\epsilon^{\mu\nu\rho\sigma}\int d^4x\Big[\mathrm{Tr}\,(\hat{R}_{\mu\nu}\hat{R}_{\rho\sigma}\gamma_5)\Big|_{\theta^2} \\ &= -\frac{il^2}{64\pi G_N}\epsilon^{\mu\nu\rho\sigma}\int d^4x\Big[\frac{i}{2}\epsilon_{abcd}R^{ab}_{\rho\sigma}R^{(2)cd}_{\mu\nu} + 8R^{(1)}_{\mu\nu}R^{(1)}_{\mu\nu 5}\Big] . \end{aligned} \tag{53}$$

$$\begin{aligned} S^{(2)}_{GB} &= \frac{l^2}{1024\pi G_N}\theta^{\kappa\lambda}\theta^{\rho\sigma}\epsilon^{\mu\nu\alpha\beta}\epsilon_{abcd}\int d^4x\Big[R^{cd}_{\alpha\beta}R^{ab}_{\mu\kappa}R^{mn}_{\nu\rho}R_{\lambda\sigma mn} \\ &-\frac{1}{2}R^{cd}_{\alpha\beta}R^{ab}_{\mu\kappa}R^{mn}_{\nu\lambda}R_{\rho\sigma mn} + R^{mn}_{\alpha\beta}R^{mn}_{\mu\kappa}R^{ab}_{\nu\rho}R^{cd}_{\lambda\sigma} \\ &+R^{mn}_{\mu\rho}R_{\nu\sigma mn}R^{ab}_{\alpha\kappa}R^{cd}_{\beta\lambda} - \frac{1}{2}R^{ab}_{\alpha\kappa}R^{cd}_{\beta\lambda}R^{mn}_{\rho\sigma}R_{\mu\nu mn}\Big] + X \,, \end{aligned} \tag{54}$$

X- the terms that are not written in an explicitly covariant way

Action:SW expansion

$$X = -\frac{l^2}{1024\pi G_N}\theta^{\kappa\lambda}\theta^{\rho\sigma}\epsilon^{\mu\nu\alpha\beta}\epsilon_{mnpq}\int d^4x\Big[-R^{ab}_{\alpha\beta}R^{ab}_{\mu\kappa}\omega_\rho^{mn}(D_\sigma R_{\nu\lambda})^{pq} -\frac{1}{2}R^{ab}_{\mu\rho}R^{ab}_{\nu\sigma}\omega_\kappa^{mn}(D_\lambda R_{\alpha\beta})^{pq} - \frac{1}{4}R^{be}_{\rho\sigma}R^{eb}_{\mu\nu}\omega_\kappa^{mn}(D_\lambda R_{\alpha\beta})^{pq}\Big]\ . \tag{55}$$

This term can be rewritten in the following form

$$\frac{1}{2}\theta^{\kappa\lambda}\epsilon^{\mu\nu\alpha\beta}\epsilon_{mnpq}\omega_\kappa^{mn}(D_\lambda R_{\alpha\beta})^{pq}R^{ab}_{\mu\rho}R^{ab}_{\nu\sigma}\Big[\theta^{\rho\nu}\epsilon^{\mu\sigma\alpha\beta} +\theta^{\rho\mu}\epsilon^{\sigma\alpha\beta\nu} + \theta^{\rho\sigma}\epsilon^{\alpha\beta\nu\mu} + \theta^{\rho\alpha}\epsilon^{\beta\nu\mu\sigma} + \theta^{\rho\beta}\epsilon^{\nu\mu\sigma\alpha}\Big].$$

Action:SW expansion

$$X = 0$$

$$\begin{aligned} S^{(2)}_{GB} &= \frac{l^2}{1024\pi G_N}\theta^{\kappa\lambda}\theta^{\rho\sigma}\epsilon^{\mu\nu\alpha\beta}\epsilon_{abcd}\int d^4x\Big[R^{cd}_{\alpha\beta}R^{ab}_{\mu\kappa}R^{mn}_{\nu\rho}R_{\lambda\sigma mn} \\ &\quad -\frac{1}{2}R^{cd}_{\alpha\beta}R^{ab}_{\mu\kappa}R^{mn}_{\nu\lambda}R_{\rho\sigma mn} + R^{mn}_{\alpha\beta}R^{mn}_{\mu\kappa}R^{ab}_{\nu\rho}R^{cd}_{\lambda\sigma} \\ &\quad +R^{mn}_{\mu\rho}R_{\nu\sigma mn}R^{ab}_{\alpha\kappa}R^{cd}_{\beta\lambda} - \frac{1}{2}R^{ab}_{\alpha\kappa}R^{cd}_{\beta\lambda}R^{mn}_{\rho\sigma}R_{\mu\nu mn}\Big] . \end{aligned} \tag{56}$$

Action:SW expansion

$$S_{EH}^{(2)} = -\frac{1}{32\pi G_N}\epsilon^{\mu\nu\rho\sigma}\int d^4x \mathrm{Tr}\left(\hat{E}_\mu \star \hat{E}_\nu \star \hat{R}_{\rho\sigma}\gamma_5\right)\Big|_{\theta^2} . \qquad (57)$$

Expanding the noncommutative fields and $\star$ product up to second order in θ we obtain

$$\begin{aligned} S_{EH}^{(2)} &= -\frac{1}{32\pi G_N}\epsilon^{\mu\nu\rho\sigma}\int d^4x \mathrm{Tr}\Big[\frac{1}{2}[e_\mu, \epsilon_\nu]\hat{R}^{(2)}_{\rho\sigma}\gamma_5 + [\hat{E}^{(1)}_\mu, e_\nu]\hat{R}^{(1)}_{\rho\sigma}\gamma_5 \\ &+ \frac{i}{2}\theta^{\alpha\beta}\{\partial_\alpha e_\mu, \partial_\beta v_\nu\}\hat{R}^{(1)}_{\rho\sigma}\gamma_5 + [\hat{E}^{(2)}_\mu, e_\nu]R_{\rho\sigma}\gamma_5 + \frac{1}{2}[\hat{E}^{(1)}_\mu, \hat{E}^{(1)}_\nu]R_{\rho\sigma}\gamma_5 \\ &+ \frac{i}{2}\theta^{\alpha\beta}\{\partial_\alpha e_\mu, \partial_\beta \hat{E}^{(1)}_\nu\}R_{\rho\sigma}\gamma_5 - \frac{1}{16}\theta^{\alpha\beta}\theta^{\gamma\delta}[\partial_\alpha\partial_\gamma e_\mu, \partial_\beta\partial_\delta e_\nu]R_{\rho\sigma}\gamma_5\Big] \end{aligned}$$

Action:SW expansion

Taking both ω_μ and e_μ are constant:

$$S^{(2)}_{EH} = -\frac{1}{32\pi G_N}\epsilon^{\mu\nu\rho\sigma}\theta^{\kappa\lambda}\theta^{\alpha\beta}\int d^4x\Big[\epsilon_{abcd}e^a_\mu\epsilon^b_\nu R^{(2)cd}_{\rho\sigma} + 8E^{(1)d}_{\mu 5}e_{\nu d}R^{(1)}_{\rho\sigma} + 2\epsilon_{abcd}e^b_\nu R^{(cd)}_{\rho\sigma}E^{(2)a}_\mu - \epsilon_{abcd}R^{ab}_{\rho\sigma}E^{(1)c}_{\mu 5}E^{(1)d}_{\nu 5}\Big] . \tag{58}$$

Action:SW expansion

$$\begin{aligned} S_{EH}^{(2)} &= -\frac{1}{32\pi G_N}\epsilon^{\mu\nu\rho\sigma}\theta^{\kappa\lambda}\theta^{\alpha\beta}\epsilon_{abcd}\int d^4x\Big[e_\mu^a e_\nu^b\big(\frac{1}{8}R_{\rho\kappa}^{cd}R_{\sigma\alpha}^{mn}R_{\lambda\beta mn} \\ &- \frac{1}{16}R_{\rho\kappa}^{cd}R_{\alpha\beta}^{mn}R_{\sigma\lambda mn}\big) \\ &+ \frac{1}{8}e_\mu^m\epsilon_\nu^n R_{\rho\kappa}^{mn}R_{\sigma\alpha}^{ab}R_{\lambda\beta}^{cd} - \frac{1}{16}e_\mu^m\epsilon_\nu^n R_{\rho\kappa}^{mn}R_{\alpha\beta}^{cd}R_{\sigma\lambda}^{ab} + Y\Big], \end{aligned} \tag{59}$$

where the gauge covariant and noncovariant parts are separated.

Action:SW expansion

Y- term can be transformed in explicitly covariant form. The result reads

$$
\begin{aligned}
- & \frac{1}{32\pi G_N}\epsilon^{\mu\nu\alpha\beta}\theta^{\kappa\lambda}\theta^{\rho\sigma}\int d^4x Y \\
& = \frac{1}{128\pi G_N}\theta^{\kappa\lambda}\theta^{\alpha\beta}\int d^4x \det e\Big[- R_{\kappa\lambda}^{\rho\alpha}(R_{\rho\sigma}^{\gamma\delta}R_{\sigma\beta\gamma\delta} - \frac{1}{2}R_{\alpha\beta}{}^{\gamma\delta}R_{\rho\sigma\gamma\delta}) \\
- & \frac{1}{2}R_{\rho\sigma\gamma\delta}(R_{\alpha\gamma}{}^{\gamma\delta}R_{\beta\lambda}{}^{\rho\sigma} + R_{\alpha\beta}{}^{\rho\sigma}R_{\kappa\lambda}{}^{\gamma\delta} + 2R_{\alpha\kappa}^{\gamma\sigma}R_{\beta\lambda}{}^{\delta\rho}) \\
+ & 2R_{\rho\sigma ab}(R_{\alpha\lambda}^{ac}(D_\beta e^\rho)_c(D_\lambda e^\sigma)^b + R_{\kappa\lambda}^{bc}(D_\alpha e^\rho)^a(D_\beta e^\sigma)_c \\
- & (D_\alpha D_\kappa e^\rho)^a(D_\beta D_\lambda e^\sigma)^b) \\
- & \frac{1}{2}e_a^\rho(D_\alpha e^\sigma)^a R_{\beta\lambda}^{cd}(D_\kappa R_{\rho\sigma})_{cd} + e_a^\rho(D_\lambda e_\sigma)^b R_{\rho\sigma bc}(D_\kappa R_{\alpha\beta})_{ac})\Big] \ . \quad (60)
\end{aligned}
$$

Action:SW expansion

We use vielbeins to convert $SO(1,3)$ indexes into world indexes:
$e^a_\mu X_a = X_\mu$.
The local index $a, b, \dots$ are raised by the metric η^{ab}.
The inverse vielbeines e^μ_a are defined by $e^a_\mu e^\mu_b = \delta^a_b$.

Action:SW expansion

$$\begin{aligned}
S^{(2)}_{EH} &= -\frac{1}{32\pi G_N}\epsilon^{\mu\nu\rho\sigma}\theta^{\kappa\lambda}\theta^{\alpha\beta}\epsilon_{abcd}\int d^4x\Big[\frac{1}{8}R^{cd}_{\rho\kappa}R^{mn}_{\sigma\alpha}R_{\lambda\beta mn}e^a_\mu e^b_\nu \\
&- \frac{1}{16}R^{cd}_{\rho\kappa}R^{mn}_{\alpha\beta}R_{\sigma\lambda mn}e^a_\mu e^b_\nu \\
&+ \frac{1}{8}e^m_\mu\epsilon^n_\nu R^{mn}_{\rho\kappa}R^{ab}_{\sigma\alpha}R^{cd}_{\lambda\beta} - \frac{1}{16}e^m_\mu\epsilon^n_\nu R^{mn}_{\rho\kappa}R^{cd}_{\alpha\beta}R^{ab}_{\sigma\lambda} \\
&+ \frac{1}{148\pi G_N}\theta^{\kappa\lambda}\theta^{\alpha\beta}\int d^4x\det e\Big[-R^{\rho\sigma}_{\kappa\lambda}(R^{\gamma\delta}_{\rho\alpha}R_{\sigma\beta\gamma\delta} - \frac{1}{2}R^{\gamma\delta}_{\alpha\beta}R_{\rho\sigma\gamma\delta}) \\
&- \frac{1}{2}R^{\gamma\delta}_{\rho\sigma}(R^{\gamma\delta}_{\alpha\gamma}R^{\rho\sigma}_{\beta\lambda} + R^{\rho\sigma}_{\alpha\beta}R^{\gamma\delta}_{\kappa\lambda} + 2R^{\gamma\sigma}_{\alpha\kappa}R^{\delta\rho}_{\beta\lambda}) \\
&+ 2R_{\rho\sigma ab}(R^{ac}_{\alpha\lambda}(D_\beta e^\rho)_c(D_\lambda e^\sigma)^b \\
&+ R^{bc}_{\kappa\lambda}(D_\alpha e^\rho)^a(D_\beta e^\sigma)_c - (D_\alpha D_\kappa e^\rho)^a(D_\beta D_\lambda e^\sigma)^b) \\
&- \frac{1}{2}e^\rho_a(D_\alpha e^\sigma)^a R^{cd}_{\beta\lambda}(D_\kappa R_{\rho\sigma})_{cd} + e^\rho_a(D_\lambda e_\sigma)^b R_{\rho\sigma bc}(D_\kappa R_{\alpha\beta})_{ac})\Big] \qquad (61)
\end{aligned}$$

Correction to the cosmological term

$$S_\Lambda^{(2)} = ?$$

Conclusion

We consider a deformation of gravity with local $SO(1,3)_\star$ symmetry; We start from $SO(1,3)$ commutative action obtained by SSB from $SO(2,3)$ action

The second order corrections to the classical action are found in the covariant form.

Future investigations:

NC corrections to the black hole/ cosmological solutions

NC terms are important in early evolution of universe (inflationary expansion)

renormalizability of NC gravity

Conclusion

Remark:
NC action

$$\begin{aligned} S &= -\frac{il}{16\pi G_N}\mathrm{Tr}\int \hat{F}\wedge_\star \hat{F}\gamma_5 \\ &= -\frac{il^2}{64\pi G_N}\epsilon^{\mu\nu\rho\sigma}\int d^4x \mathrm{Tr}\,(\hat{F}_{\mu\nu}\star\hat{F}_{\rho\sigma}\gamma_5) \end{aligned} \tag{62}$$

is not invariant under deformed symmetry with NC gauge parameter $\hat{\Lambda}_\epsilon^{SO(2,3)_\star}\Big|_{\epsilon_{a5}=0}$.