Bayrischzell 24/5/2013

THE MULTIPLICATIVE INTEGRABILITY of the MODULAR FUNCTION

F. BONECHI
(INFN, Firenze)

* joint work with N. Ciccoli
J. Qiu
M. Tarlini

PLAN of THE TALK

- motivations
- Poisson manifolds and symplectic groupoids
- Groupoid C^*-algebras
- Geometric quantization of the symplectic groupoid
- Multiplicative integrability of the modular function
- A bihamiltonian system on $\mathbb{C}P_n$

SOME MOTIVATIONS

- A.J. Sheu ('90s) showed that C^*-algebras of quantum groups and quantum homogeneous spaces are groupoid C^*-algebras: $C^*(\mathbb{C}P_{n,q,0})$, $C^*(S_q^{2n+1})$, $C^*(SU_q(n))$, $C^*(\mathbb{C}P_{q\,n,t})$ only subalgebras

<u>ex.</u> $C^*(\mathbb{C}P_{1,q,0}) = C^*(\mathcal{G}_s)$

$$\mathcal{G}_s = \{ (m,n) \mid m, m+n \geq 0,\ m=\infty \Rightarrow n=0 \} \subset \bar{\mathbb{Z}} \times \mathbb{Z} \Big|_{\mathbb{N}}$$

- can we understand these results?

Sheu groupoids as Bohr-Sommerfeld leaves of some polarization of the symplectic groupoid

this is true in the examples I'm going to present provided we use

very singular polarizations

integrable systems

POISSON MANIFOLD

M smooth manifold

$\{ , \}$ Poisson bracket on $C^\infty(M)$

$$\{f,g\} = \pi^{ij} \partial_i f \, \partial_j g \qquad \pi = \pi^{ij} \partial_i \wedge \partial_j$$ Poisson tensor

Jacobi identity $\Longleftrightarrow [\pi,\pi] = 0$

↳ Schouten bracket

symplectic foliation

$\mathrm{Im}\,\pi \subset TM$ is a generalized integrable distribution

each leaf inherits a symplectic structure

Lichnerowicz-Poisson cohomology

$$d_{LP} = [\pi, -] : \mathcal{V}^\bullet(M) \longrightarrow \mathcal{V}^{\bullet+1}(M)$$

multivector fields

$$d_{LP}^2 = 0$$

$$H_{LP}(M,\pi)$$

the Poisson tensor itself defines a class

$$[\pi] \in H^2_{LP}(M,\pi)$$

the modular class

(M,π) choose a volume form Ω on M

$$X_\Omega = \mathrm{div}_\Omega \pi \quad \text{modular vector field}$$

satisfies $d_{LP} X_\Omega = [\pi, X_\Omega] = 0$

$$[X_\Omega] = H^1_{LP}(M) \text{ is independent on } \Omega$$

The MODULAR CLASS is the obstruction to find an invariant volume form, i.e.

$$\mathcal{L}_{X_f}\Omega = 0 \qquad f \in C^\infty(M), \quad X_f = \{f, -\}$$

EXAMPLES

1) $M = \mathfrak{g}^*$ is unimodular iff $\mathfrak{g}$ is unimodular as a Lie algebra

2) M symplectic is unimodular

3) Standard Poisson Lie group and related hom. spaces are not unimodular.

THE SYMPLECTIC GROUPOID (Weinstein, Karasev 80's)

DEFINITION: A symplectic groupoid is a Lie groupoid $\mathcal{G}$ equipped with a symplectic form $\Omega_{\mathcal{G}}$ s.t.

$$\mathrm{graph}(m_{\mathcal{G}}) \subset \bar{\mathcal{G}} \times \bar{\mathcal{G}} \times \mathcal{G} \text{ is Lagrangian}$$

multiplication

- integration of Poisson manifolds

$$\begin{array}{c} \mathcal{G} \\ s \downarrow\downarrow t \\ M \end{array}$$

there is a unique Poisson structure on M s.t. s and t are Poisson and anti Poisson maps.

" $\mathcal{G}$ integrates (M,π) "

" (M,π) is integrable "

- If you assume that, given (M,π), such a $\mathcal{G}$ exists, then there exists only one which is source simply connected (ssc)

- integration of cocycles

$$H^1_{LP}(M,\pi) = H^1(\mathcal{G})$$

↳ groupoid cohomology

$$\chi \in \mathfrak{X}(M) \quad [\pi,\chi]=0 \longrightarrow c_\chi \in C^\infty(\mathcal{G})$$

$$c_\chi(\gamma_1\gamma_2) = c_\chi(\gamma_1) + c_\chi(\gamma_2)$$

for each volume form Ω on M

χ_Ω modular vector field $\longrightarrow$ c_Ω modular function

SYMPLECTIC GROUPOID VS QUANTIZATION

Lagrangian submanifolds $\xrightarrow[\text{quantization}]{\text{semiclassical}}$ vectors in the Hilbert space

means

$$\mathrm{graph}(m_g) \xrightarrow[\text{quant.}]{} \psi_{m_g} \in \mathcal{H}^* \otimes \mathcal{H}^* \otimes \mathcal{H}$$

multiplication of g

$$\Updownarrow$$

$$*_{m_g} : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$$

Output of the quantization procedure must be an algebra on the space of states

We have to think to $\mathcal{H}$ as the algebra of operators rather than the Hilbert space of states.

- example M symplectic $\mathcal{G}(M) = M \times \bar{M}$

$M \xrightarrow[\text{quant}]{} \mathcal{h}$ (Hilbert space of states) $\mathcal{G}(M) \longrightarrow \mathcal{h} \otimes \mathcal{h}^* = \mathrm{End}\, \mathcal{h}$ algebra of quantum observables

HOW TO PRODUCE AN ALGEBRA: GROUPOID C^*-ALGEBRAS (Renault)

$$\mathcal{G} \underset{s}{\overset{t}{\rightrightarrows}} \mathcal{G}_0 \qquad \text{topological groupoid}$$

(left) Haar system $\equiv$ family of measures supported on the source fibre $\lambda = \{ \lambda^x,\ x \in \mathcal{G}_0 \}$

$\lambda^{-1} \equiv \lambda \circ i \quad i(\gamma) = \gamma^{-1}$ right Haar system

- quasi invariant measures

$\mu \equiv$ measure on $\mathcal{G}_0 \Rightarrow \nu_\mu, \nu_\mu^{-1} =$ measure on $\mathcal{G}$

$$f \in C_c(\mathcal{G}) \qquad \int_{\mathcal{G}} f \nu_\mu \equiv \int_{\mathcal{G}_0} \lambda(f) \mu$$

μ is quasi invariant if $\nu_\mu \sim \nu_\mu^{-1}$

$$D = \frac{d\nu}{d\nu^{-1}} \quad \text{modular function}$$

$c_\mu = \log D \in Z^1(\mathcal{G}, \mathbb{R})$ groupoid 1-cocycle

- Convolution algebra $C_c(\mathcal{G})$

$f, g \in C_c(\mathcal{G})$

$$(f * g)(\gamma) = \int f(\gamma\gamma')\, g(\gamma'^{-1})\, \xi(\gamma, \gamma')\, d\lambda^{r(\gamma)}$$

$$f^*(\gamma) = \overline{f(\gamma^{-1})}\ \overline{\xi(\gamma, \gamma^{-1})}$$

$\xi \in Z^2(\mathcal{G}, \mathbb{T})$ groupoid 2-cocycle

$C_c(\mathcal{G}, \xi)$ is a $*$-algebra $\Longrightarrow$ $C^*(\mathcal{G}, \xi)$ groupoid C^*-algebra

Let $c \in Z^1(\mathcal{G}, \mathbb{R})$ groupoid 1-cocycle

defines an algebra automorphism

$$A_c : \mathbb{R} \to C_c(\mathcal{G})$$

$$\left(A_c(t)(f)\right)(\gamma) = e^{itc(\gamma)} f(\gamma)$$

Let μ a measure on $\mathcal{G}_0$

$$\phi_\mu(f) = \int_{\mathcal{G}_0} f\, d\mu$$ weight on $C^*(\mathcal{G}, \xi)$

ϕ_μ satisfies the KMS condition for A_c $\Longleftrightarrow$ μ is a quasi-invariant measure with modular function c

- modular operator

GNS representation defined by ϕ_μ:

$C_c(\mathcal{G},\xi)$ acts by convolution on $L^2(\mathcal{G},\nu_\mu^{-1})$

$L^2(\mathcal{G},\nu_\mu^{-1})$ is a left Hilbert algebra

(*-algebra + scalar product)

$S(f)=f^*$

polar decomposition of S

$S=J\Delta^{1/2}$ $\quad \Delta=S^\dagger S$ modular operator

$(\Delta f)(\gamma)=e^{-c(\gamma)}f(\gamma)$

We look for

- a topological groupoid with a Haar system
- a groupoid 1-cocycle (the modular cocycle)

from the symplectic groupoid? No

from the geometric quantization of the symplectic groupoid? YES

GEOMETRIC QUANTIZATION of THE SYMPLECTIC GROUPOID

geometric quantization of $\mathcal{G}$ as symplectic manifold + compatibility with the groupoid structure

- prequantization cocycle φ

Let $\left(\begin{smallmatrix} L \\ \downarrow \\ \mathcal{G} \end{smallmatrix}, \nabla\right)$ be a prequantization of $\mathcal{G}$ as symplectic mfld

THEOREM (Weinstein, Xu '91)

If $\mathcal{G}$ is prequantizable then there exists a unique prequantization $\left(\begin{smallmatrix} L \\ \downarrow \\ \mathcal{G} \end{smallmatrix}, \nabla\right)$ with $\varphi \in \Gamma(\partial^* L^*)$ s.t.

1) $|\varphi| = 1$

2) $\varphi(\gamma_1, \gamma_2\gamma_3)\varphi(\gamma_2, \gamma_3) = \varphi(\gamma_1, \gamma_2)\varphi(\gamma_1\gamma_2, \gamma_3) \quad \gamma_i \in \mathcal{G}_3$

3) $\nabla\varphi = d\varphi + \partial^* \Theta_{\mathcal{G}}\varphi = 0 \qquad (d\Theta_{\mathcal{G}} = \Omega_{\mathcal{G}})$

By a result of Crainic ('04)

$$\varphi = 1 \iff 0 = [\pi] \in H^2_{LP}(M, \pi)$$

- polarization (according E. Hawkins, 2006)

Let $F \subset T_{\mathbb{C}}\mathcal{G}$ be a polarization of $\mathcal{G}$

Denote $F_{(2)} = (F \oplus F) \cap T\mathcal{G}_{(2)} \subset T\mathcal{G} \oplus T\mathcal{G}$

DEFINITION A polarization $F \subset T\mathcal{G}$ is

i) multiplicative if $m_{\mathcal{G}*}(F_{(2)}) = F$

ii) hermitian if $i_{\mathcal{G}*}(F) = \overline{F}$

Main task of Hawkins construction is to define a convolution algebra $C(\mathcal{G}, F)$ space of polarized section

BASIC EXAMPLE: Real polarization $F = \overline{F}$

$p: \mathcal{G} \to \mathcal{G}_F = \mathcal{G}/F$ fibration of Lie groupoids

$A = C_c(\mathcal{G}, F) = C(\mathcal{G}_F)$

the convolution algebra coincides with groupoid convolution algebra of $\mathcal{G}_F$

A QUANTIZATION SCHEME FROM SINGULAR POLARIZATIONS

We look for

a) $\pi_F : \mathcal{G} \longrightarrow \mathcal{G}_F$ topological groupoid quotient

b) $\pi_F^{-1}(e) \subset \mathcal{G}$ is isotropic $\forall\, \ell \in \mathcal{G}_F$ and lagrangian almost everywhere

$\mathcal{G}/F \supset \mathcal{G}_F^{bs}$ is the set of Bohr-Sommerfeld leaves

$$\ell \in \mathcal{G}_F^{bs} \text{ if } [\Theta|_\ell] \in H^1(\ell; \mathbb{Z})$$

$$d\Theta = \Omega_{\mathcal{G}}$$

$\mathcal{G}_F^{bs}$ inherits a groupoid structure but can be quite singular. We require only that

c) $\mathcal{G}_F^{bs}$ admits a Haar system

Apply Renault theory of groupoid C^*-algebras

d) the modular and the prequantization cocycles descend to $\mathcal{G}_F$

multiplicative integrability of the modular function

$c_\Omega \in C^\infty(\mathcal{G})$ modular function

We can ask if c_Ω is integrable i.e.

$F = \{f_i\}_{i=1}^{\dim M}$ $f_i \in C^\infty(\mathcal{G})$ in involution and independent

$\{f_\Omega, f_i\} = 0$

$\mathcal{G}_F \equiv$ contour levels of f_i

a) multiplicativity: $\mathcal{G}_F$ inherits a topological groupoid structure

b) $\mathcal{G}_F^{bs}$ admits a Haar system (for instance it is ètale)

A BIHAMILTONIAN SYSTEM on $\mathbb{CP}_n$

(Koroshkin, Radul, Rubtsov '93)

$X \equiv$ flag mfld

- X as a coadjoint orbit

$$X = \mathcal{O}_\lambda = \{ g^{-1}\lambda g, \ g \in su(n+1) \} \quad \lambda \in \mathfrak{t}_n \subset su(n+1)$$

$\Pi_\lambda \equiv$ (inverse of) Kirillov-Kostant symplectic form

- X as a Poisson-homogeneous space

On $SU(n+1)$ choose the standard Poisson-Lie structure: H_λ is a Poisson subgroup

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\ \ \ ||| $ stability group of λ

On $X \simeq H_\lambda \backslash G$

$\Pi_0 \equiv$ Bruhat-Poisson structure

$$[\Pi_\lambda, \Pi_0] = 0 \quad \text{(KRR, '93)}$$

- Poisson pencil

$\pi_t = \pi_0 + t\pi_1 \quad t \in \mathbb{R}$ is a family of covariant Poisson structures

- Bihamiltonian system (Magri '80)

$J_\lambda = \pi_0 \circ \omega_\lambda : TX \to TX$

has vanishing Nijenhuis torsion

$$I_k = \frac{1}{k!} \operatorname{Tr} J_\lambda^k$$

- $\{I_k, I_{k'}\}_{\pi_t} = 0 \qquad \forall t$

- $X_{\omega_\lambda^n} = \pi_\lambda \, d I_1$ is the modular vector field of π_0 with respect to ω_λ^n

(Damianou-Fernandes) '08

- diagonalization of J_λ

$$J_\lambda \sigma_i = c_i \sigma_i \qquad \begin{cases} c_i \in C^\infty(X) \\ \sigma_i \in \mathrm{Vect}(X) \end{cases}$$

$$\sigma_i = \Pi_\lambda d c_i$$

$[\Pi_0, \sigma_i] = 0 \equiv$ Poisson vector field for Π_0 ($\Rightarrow \Pi_t$)

$h_i \equiv$ lift of σ_i to $\mathcal{G}(X, \Pi_t)$

- $h = \sum_i h_i$ is the modular function

- $F = \{s^*(c_i), h_i\}$ are in involution

- $\mathcal{G}_F(X, \Pi_t) \equiv$ contour level set is a topological gpd

$$\begin{cases} l(c,h) = c \\ r(c,h) = e^{-h}(c+t) - t \\ (c,h)(c',h') = (c, h+h') \end{cases}$$

for $X = \mathbb{C}P_n$

- the eigenvalues of J_λ are the components of the momentum map (with respect to ω_λ)

$$c : \mathbb{C}P_n \longrightarrow \mathfrak{t}_n = \text{Cartan}$$

$$c_k = \operatorname{Tr}\left(g^{-1} \lambda g H_k\right)$$

with respect to the basis $\langle H_k \rangle$ of $\mathfrak{t}_n$

$$H_k = i \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & 0 & \\ & & & & \ddots \\ & & & & & 0 \end{pmatrix} - ik \mathbb{1}$$

The lift of c is the momentum map

$$h : \mathcal{G}(\mathbb{C}P_n, \pi_H) \longrightarrow \mathfrak{t}_n^*$$

of the (hamiltonian) action of the Cartan on the symplectic groupoid

• symplectic foliation of π_t $t \in [0,1]$

$c_k \neq 1-t$ is symplectic of max dim

$P_k = \{ x \in \mathbb{CP}_n, \quad c_k(x) = 1-t \} \subset \mathbb{CP}_n$

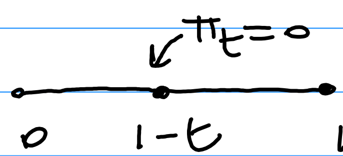
is a Poisson submfld

• $P_1 \sim P_n \sim S^{2n-1}$

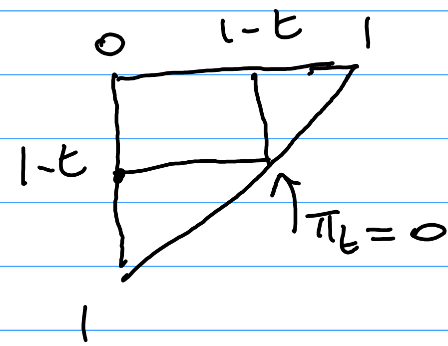
• $O_n \bigcap_k P_k \simeq S^1$ $\qquad \pi_t = 0$

$\mathrm{Im}\, c \simeq \Delta_n$ the standard simplex

$n=1$



$n=2$



- Bohr - Sommerfeld conditions

our main result is the following theorem

THEOREM

- The BS groupoid is the following subgroupoid of the groupoid of leaves

$$\mathcal{G}^{bs}_{F}(\mathbb{C}P_n, \pi_t) = \left\{ (c,h) \;\middle|\; \begin{array}{l} \log |c_i - 1 + t| \in \hbar\mathbb{Z} \\ h_i \in \hbar\mathbb{Z} \end{array} \right\}$$

- It is an étale gpd and admits a unique Haar system
- The modular function is quantized to the following cocycle

$$f_{FS}(c,h) = \hbar \sum_{i=1}^{n} h_i$$

the space of units is contained in the standard simplex $\Delta_n = \{ c,\ 1 \leq \cdots c_i \leq c_{i+1} \cdots \leq 1 \}$

$$\mathcal{G}_F(\mathbb{C}P_n, \Pi_t)_0 = \{ c \in \Delta_n,\ \log |c_i - 1 + t| \in t_0\mathbb{Z} \}$$

The modular cocycle determines the following quasi invariant measure on $\mathcal{G}_F(\mathbb{C}P_n, \Pi_t)_0$

$$\mu_{FS}(c) = \det\left(J_\lambda(c) + t\right) = \prod_k |c_k - 1 + t|$$

$$= \exp -t_0 \sum_{k=1}^{n} n_k$$

$$c_k - 1 + t = \pm e^{-t_0 n_k}$$

More explicit description of components

(r,s) $0\leq r,s$ $r+s\leq n$ $\overline{\mathbb{Z}}=\mathbb{Z}\cup\{\infty\}$

$$\Delta^{\mathbb{Z}}_{(r,s)}(t)=\Big\{ (m_1 \dots m_r \infty \dots \infty n_1 \dots n_s)\in\overline{\mathbb{Z}}^n,$$
$$-\tfrac{1}{\hbar}\log(1-t)\leq \dots m_i\leq m_{i+1}\leq\dots$$
$$\dots n_i\geq n_{i+1}\dots\geq -\tfrac{1}{\hbar}\log t \Big\}$$

$t\in(0,1)$

$$\mathcal{J}^{bs}_F(\mathbb{CP}_n,\Pi_t)^{(r,s)}=\Big\{(q,p)\in \underbrace{\overline{\mathbb{Z}}^n\ltimes\mathbb{Z}^n}_{\Delta^{\mathbb{Z}}_{(rs)}(t)},\ q_i=q_{i+1}=\infty\Rightarrow p_i=p_{i+1}\Big\}$$

$t=0$

$$\mathcal{J}^{bs}_F(\mathbb{CP}_n,\Pi_t)^{(r,0)}=\Big\{(q,p)\in \underbrace{\overline{\mathbb{Z}}^n\ltimes\mathbb{Z}^n}_{\Delta^{\mathbb{Z}}_{(r0)}(0)},\ q_i=\infty\Rightarrow p_i=0\Big\}$$

CONCLUSIONS

- comparison with Shen's results

we recover Shen's results for $t=0$ and for odd spheres. For $t \neq 0,1$ and other Poisson submanifolds Shen's results are looser. We conjecture that these quantum spaces are groupoid C^*-algebras.

- extend to general flag mflds.

the missing ingredient is the diagonalization of the Nijenhuis operator

- Poisson morphisms

complete the description of the quantization of Poisson morphisms.