HSGRA: off-shell formulation and gauge algebra

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Plan

1. Introduction
2. Brief intro to Vasiliev's 4D equations
3. Classical off-shell unfolding
4. Results and discussions
5. AKSZ quantization
6. Action in non-commutative case

Higher-spin gauge theories : some motivations

- Gauge Principle : HS theories contain gravity ; ∞-dim gauge algebra ;
- Vasiliev's unfolding : a geometric approach to field theory ;
- AdS/CFT dualities between Vasiliev's theory and free CFT's [Sezgin-Sundell, Klebanov-Polyakov] for AdS_4/CFT_3 and [Gaberdiel-Gopakumar] for AdS_3/CFT_2 . Relations with statistical physics, integrable models, strings etc.

THE GAUGE PRINCIPLE [H. WEYL, 1929]

In Classical Field Theory : remarkable achievement by M. A. Vasiliev with formulation of *fully nonlinear field equations* for higher-spin gauge fields in 4D [Vasiliev, 1990 – 1992] and in D space-time dimensions `[hep-th/0304049]`. Some salient features are

- Manifest diffeomorphism invariance, no explicit reference to a metric ;
- Manifest Cartan integrability $\Rightarrow$ *gauge invariance* under infinite-dimensional HS algebra ;
- Formulation in terms of two infinite-dimensional modules of $\mathfrak{so}(2, D-1)$: The *adjoint* and *twisted-adjoint* representations $\rightsquigarrow$ master 1-form and master zero-form. Uses unfolding in terms of FDA.

Unfolded equations and FDA

A free (graded commutative, associative) differential algebra $\mathfrak{R}$ is set $\{X^\alpha\}$ of *a priori* independent variables, locally-defined differential forms obeying first-order equations of motion

$$\mathscr{R}^\alpha = \mathrm{d}X^\alpha + Q^\alpha(X) \approx 0\,, \quad Q^\alpha(X) = \sum_n f^\alpha_{\beta_1\ldots\beta_n} X^{\beta_1}\cdots X^{\beta_n}\ .$$

Nilpotency of d and integrability condition $\mathrm{d}\mathscr{R}^\alpha \approx 0$ require

$$Q^\beta\,\frac{\partial^L Q^\alpha}{\partial X^\beta} \equiv 0 \quad .$$

For $X^\alpha_{[p_\alpha]}$ with $p_\alpha > 0$, gauge transformation preserving $\mathscr{R}^\alpha \approx 0$:

$$\delta_\epsilon X^\alpha = \mathrm{d}\epsilon^\alpha - \epsilon^\beta \frac{\partial^L}{\partial X^\beta} Q^\alpha \quad .$$

- The concepts of spacetime, dynamics and observables are *derived* from infinite-dimensional FDA's.
- Unfolded dynamics is an inclusion of local d.o.f. into field theories described *on-shell* by flatness conditions on generalized curvatures.
- Spin-2 couplings arise in the limit in which the $\mathfrak{so}(2, D-1)$-valued part of the higher-spin connection one-form is treated exactly while its remaining spin $s > 2$ components become weak fields together with all curvature (Weyl) zero-forms.

  $\hookrightarrow$ Lorentz-covariant derivative, minimal coupling.

Action principle with QP-structure

Want an action principle reproducing non-linear and background-independent Vasiliev equations in four spacetime dimensions. These equation possess

- an algebraic structure that enables one to construct a *Hamiltonian action* with nontrivial QP-structures in a manifold with boundary ;
- a geometric structure which allows to construct additional boundary deformations.

Manifold : bulk with non-empty boundary

- Like for the nonlinear Poisson sigma-model [yesterday's talk by Th. Strobl], introduce bulk with non-empty boundary, and add extra momentum-like variables.

- Impose boundary conditions compatible with a *globally* well-defined action principle
  $\hookrightarrow$ the action $S = \int_B L$ should be gauge invariant, and $\delta_\varepsilon L = \mathrm{d}K_\varepsilon$;
  $\hookrightarrow$ compatibility between gauge transformations of field configurations and transition functions between charts.

- The action has two pieces : a bulk part plus various classically marginal deformations on boundary $\rightarrow$ amplitudes.

Relation with Fronsdal's programme

Unlike the original Fronsdal programme [formulate higher-spin gauge theory off shell in a perturbative expansion around constantly curved spacetime], *background-independent* formulation in terms of master fields living in the correspondence space, *i.e.* the local product of a non-commutative phase-spacetime containing the commutative spacetime as a Lagrangian submanifold and a non-commutative twistor space.

Vasiliev's system has a huge classical solution space that admits many different perturbative expansions of which *only some* reduce to Fronsdal systems (with Λ).

Brief review of Vasiliev's 4D equations (1)

The master fields are locally-defined (chart index ξ) **operators**

$$O_\xi(X^M_\xi, \mathrm{d}X^M_\xi; Z^{\underline{\alpha}}, \mathrm{d}Z^{\underline{\alpha}}; Y^{\underline{\alpha}}; K) ,$$

where

$$[Y^{\underline{\alpha}}, Y^{\underline{\beta}}] \;=\; 2iC^{\underline{\alpha\beta}} , \quad [Z^{\underline{\alpha}}, Z^{\underline{\beta}}] \;=\; -2iC^{\underline{\alpha\beta}} , \qquad \underline{\alpha}, \underline{\beta} = 1,2,3,4,$$

with charge conjugation matrix $C^{\alpha\beta} = \epsilon^{\alpha\beta}$, $C^{\dot\alpha\dot\beta} = \epsilon^{\dot\alpha\dot\beta}$, $\underline{\alpha} = (\alpha, \dot\alpha)$, and where $K = (k, \bar{k})$, are two outer Kleinian operators.

The operators are represented by **symbols** $f[O_\xi]$ obtained by going to **specific bases** for the operator algebra $\rightsquigarrow$ **ordering prescriptions**.

Brief review of Vasiliev's 4D equations (2)

One may think of the symbols as functions $f(X,Z;\mathrm{d}X,\mathrm{d}Z;Y)$ on a **correspondence space** $\mathfrak{C}$

$$\mathfrak{C} \;=\; \bigcup_{\xi}\mathfrak{C}_{\xi}\;, \qquad \mathfrak{C}_{\xi} \;=\; \mathfrak{B}_{\xi}\times\mathfrak{Y}\;, \qquad \mathfrak{B}_{\xi} \;=\; \mathfrak{M}_{\xi}\times\mathfrak{Z}$$

equipped with a suitable `associative` star-product operation $\star$ which reproduces, in the space of symbols, the composition rule for operators.

$\looparrowright$ The exterior derivative on $\mathfrak{B}$ is given by

$$\mathrm{d} \;=\; \mathrm{d}X^{M}\partial_{M} + \mathrm{d}Z^{\underline{\alpha}}\,\partial_{\underline{\alpha}} \quad .$$

Brief review of Vasiliev's 4D equations (3)

The master fields of the *minimal bosonic model* are an adjoint one-form

$$A = W + V \ ,$$

$$W = \mathrm{d}X^M\, W_M(X,Z;Y) \ , \qquad V = \mathrm{d}Z^{\underline{\alpha}}\, V_{\underline{\alpha}}(X,Z;Y) \ ,$$

and a twisted-adjoint zero-form

$$\Phi = \Phi(X,Z;Y) \ .$$

Generically, start with locally-defined differential forms of *total degree* p

$$f = \sum_{p=0}^{\infty} f_{[p]}(X^M, \mathrm{d}X^M; Z^{\underline{\alpha}}, \mathrm{d}Z^{\underline{\alpha}}; Y^{\underline{\alpha}}; k, \bar{k}) \ ,$$

$$f_{[p]}(\lambda\, \mathrm{d}X^M; \lambda\, \mathrm{d}Z^{\underline{\alpha}}) = \lambda^p\, f_{[p]}(\mathrm{d}X^M; dZ^{\underline{\alpha}}) \ , \quad \lambda \in \mathbb{C} \ .$$

Brief review of Vasiliev's 4D equations (4)

The X^M's are commuting coordinates, while $(Y^{\underline{\alpha}}, Z^{\underline{\alpha}}) = (y^\alpha, \bar{y}^{\dot{\alpha}}; z^\alpha, \bar{z}^{\dot{\alpha}})$ are non-commutative , k , $\bar{k}$ are outer Kleinians :

$$k \star f \;=\; \pi(f) \star k \;, \quad \bar{k} \star f \;=\; \bar{\pi}(f) \star \bar{k} \;, \quad k \star k \;=\; 1 \;=\; \bar{k} \star \bar{k} \;,$$

with automorphisms π and $\bar{\pi}$ defined by $\pi \mathrm{d} = \mathrm{d}\pi \,, \bar{\pi}\mathrm{d} = \mathrm{d}\bar{\pi}$ and

$$\pi[f(z^\alpha, \bar{z}^{\dot{\alpha}}; y^\alpha, \bar{y}^{\dot{\alpha}})] \;=\; f(-z^\alpha, \bar{z}^{\dot{\alpha}}; -y^\alpha, \bar{y}^{\dot{\alpha}}) \;,$$

$$\bar{\pi}[f(z^\alpha, \bar{z}^{\dot{\alpha}}; y^\alpha, \bar{y}^{\dot{\alpha}})] \;=\; f(z^\alpha, -\bar{z}^{\dot{\alpha}}; y^\alpha, -\bar{y}^{\dot{\alpha}}) \;.$$

Bosonic and irreducibility projections : $\pi\bar{\pi}(f) = f = P_+ \star f$, $P_+ = \frac{1}{2}(1 + k \star \bar{k})$,

$$\hookrightarrow \; f = \left[f^{(+)}(X, \mathrm{d}X; Z, \mathrm{d}Z; Y) + f^{(-)}(X, \mathrm{d}X; Z, \mathrm{d}Z; Y) \star \frac{(k + \bar{k})}{2} \right] \star P_+ \;.$$

Brief review of Vasiliev's 4D equations (5)

- **Bosonic projection** : removes component fields $\rightsquigarrow$ spacetime spinors.
- **Irreducible *minimal* bosonic models : by imposing** reality conditions and discrete symmetries that remove all **odd** spins.

$\hookrightarrow$ † and anti-automorphism τ defined by $\mathrm{d}[(\cdot)^\dagger] = [\mathrm{d}(\cdot)]^\dagger\,,\quad \mathrm{d}\,\tau = \tau\,\mathrm{d}\,,$

$$
\begin{aligned}
[f(z^\alpha,\bar z^{\dot\alpha};y^\alpha,\bar y^{\dot\alpha};k,\bar k)]^\dagger &= \bar f(\bar z^{\dot\alpha},z^\alpha;\bar y^{\dot\alpha},y^\alpha;\bar k,k)\ ,\\
\tau[f(z^\alpha,\bar z^{\dot\alpha};y^\alpha,\bar y^{\dot\alpha};k,\bar k)] &= f(-iz^\alpha,-i\bar z^{\dot\alpha};iy^\alpha,i\bar y^{\dot\alpha};k,\bar k)\ ,\\
[f_{[p]}\star f'_{[p']}]^\dagger &= (-1)^{pp'}(f'_{[p']})^\dagger\star(f_{[p]})^\dagger\ ,\\
\tau(f_{[p]}\star f'_{[p']}) &= (-1)^{pp'}\tau(f'_{[p']})\star\tau(f_{[p]})\ .
\end{aligned}
$$

Brief review of Vasiliev's 4D equations (6)

Back to Vasiliev's A and Φ, the minimal models are imposed by the following projection and reality conditions :

$$\tau(A,\Phi) \;=\; (-A,\pi(\Phi)) \;, \qquad (A,\Phi)^\dagger \;=\; (-A,\pi(\Phi)) \;.$$

Full equations of motion of the minimal bosonic model with fixed interaction ambiguity : $F+\Phi\star J=0$, with two-form J defined globally on correspondence space, obeying $\tau(J)=-J=J^\dagger$ and

$$\mathrm{d}J \;=\; 0 \;, \quad [f,J]^\pi_\star \;:=\; f\star J - J\star\pi(f) \;=\; 0 \quad \forall\, f \quad \text{s.t.} \quad \pi\bar\pi(f)=f \;. \qquad (1)$$

In the minimal model,

$$J \;=\; -\frac{i}{4}(b\,dz^2\;\kappa + \bar b\,d\bar z^2\;\bar\kappa) \;,$$

... where the chiral inner Kleinians

$$\kappa \;=\; \exp(iy^{\alpha}z_{\alpha})\;, \qquad \bar{\kappa} \;=\; \kappa^{\dagger} = \exp(-i\bar{y}^{\dot{\alpha}}\bar{z}_{\dot{\alpha}})\;.$$

By making use of field redefinitions $\Phi \to \lambda\Phi$ with $\lambda \in \mathbb{R}\,,\ \lambda \neq 0\,,$ the complex parameter b in J can be taken to obey

$$|b| \;=\; 1\;, \qquad \arg(b) \;\in\; [0,\pi]\;.$$

The phase breaks parity P $\quad [P\,\mathrm{d} = \mathrm{d}\,P]$

$$P\left[f(X^M; z^{\alpha}, \bar{z}^{\dot{\alpha}}; y^{\alpha}, \bar{y}^{\dot{\alpha}}; k, \bar{k})\right] \;=\; (Pf)(X^M; -\bar{z}^{\dot{\alpha}}, -z^{\alpha}; \bar{y}^{\dot{\alpha}}, y^{\alpha}; \bar{k}, k)\;,$$

except in the following two cases :

$$\text{Type-A model (parity-even physical scalar)} \quad : \quad b = 1\;,$$

$$\text{Type-B model (parity-odd physical scalar)} \quad : \quad b = i\;.$$

Brief review of Vasiliev's 4D equations (8)

[The integrability of $F + \Phi \star J = 0$ implies that $D\Phi \star J = 0$, that is, $D\Phi = 0$, where the twisted-adjoint covariant derivative $D\Phi = d\Phi + A \star \Phi - \Phi \star \pi(A)$. This constraints is integrable since $D^2\Phi \;=\; F \star \Phi - \Phi \star \pi(F) \;=\; -\Phi \star J \star \Phi + \Phi \star \pi(\Phi) \star J$ gives zero, using the constraint on F and (1).]

$\hookrightarrow$ Summary : minimal higher-spin gravity given by

$$F + \Phi \star J \;=\; 0 \;, \qquad D\,\Phi \;=\; 0 \;, \qquad \mathrm{d}J = 0 \;,$$

$$F \;:=\; \mathrm{d}A + A \star A \;, \qquad D\Phi \;:=\; \mathrm{d}\Phi + [A, \Phi]_\pi \;,$$

$$\tau(A, \Phi) \;=\; (-A, \pi(\Phi)) \;, \qquad (A, \Phi)^\dagger \;=\; (-A, \pi(\Phi)) \;,$$

$$\hookrightarrow [A, J]_\pi = 0 = [\Phi, J]_\pi \;.$$

Brief review of Vasiliev's 4D equations (9)

$\hookrightarrow$ Integrability implies invariance under Cartan gauge transformations

$$\delta_\epsilon A \ = D\epsilon \ , \qquad \delta_\epsilon \Phi \ = \ -[\epsilon, \Phi]_\pi \ ,$$

for zero-form gauge parameters $\epsilon(X, Z; Y)$ obeying the same kinematic constraints as the master one-form, *i.e.* $\tau(\epsilon) = -\epsilon$ and $(\epsilon)^\dagger = -\epsilon$.

$\hookrightarrow$ The closure of the gauge transformations reads

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}] \ = \ \delta_{\epsilon_{12}} \ , \qquad \epsilon_{12} \ = \ [\epsilon_1, \epsilon_2]_\star \ ,$$

defining the algebra $\mathfrak{hs}(4)$.

Classical action principle (1)

Starting from $\{X^\alpha\}$ defined locally on B_ξ (base manifold $B = \cup_\xi B_\xi$ of dim. $\hat{p}+1$) satisfying some unfolded constraints with given Q-structure,
$\hookrightarrow$ off-shell extensions based on sigma models with maps

$$\phi_\xi \ : \ T[1]B_\xi \ \to \ M \ ,$$

between two $\mathbb{N}$-graded manifolds, from the parity-shifted tangent bundle $T[1]B$ to a target space M that is a differential $\mathbb{N}$-graded symplectic manifold with two-form $\mathscr{O}$, Q-structure $\mathscr{Q}$ and Hamiltonian $\mathscr{H}$ with the following degrees :

$$\deg(\mathscr{O}) \ = \ \hat{p}+2 \ , \quad \deg(\mathscr{Q}) \ = \ 1 \ , \quad \deg(\mathscr{H}) \ = \ \hat{p}+1 \ .$$

Hamiltonian bulk action

$$S^{\rm cl}_{\rm bulk}[\phi|B] \;=\; \sum_\xi \int_{B_\xi} \mathscr{L}^{\rm cl}_\xi \;=\; \sum_\xi \int_{B_\xi} \pi\, \phi^*_\xi(\vartheta - \mathscr{H})\ ,$$

where $\phi_\xi \equiv \phi|_{B_\xi}$ and $\pi : \Omega(T[1]B) \to \Omega(B)$ degree-preserving canonical homomorphism that takes k-forms on $T[1]B$ of degree p to p-forms on B, *viz.*

$$\pi \;:\; \Omega^{[k|p]}(T[1]B) \to \Omega^{[p]}(B)\ ,$$

and that intertwines the actions of the exterior derivative d in $\Omega(B)$ and the Lie derivative $\mathscr{L}_q = i_q \circ {\rm d} - {\rm d} \circ i_q$ in $\Omega(T[1]B)$ along the canonical Q-structure on $T[1]B$ as follows :

$${\rm d} \circ \pi \;=\; \pi \circ {\rm d} \;=\; \pi \circ \mathscr{L}_q\ , \quad q \;:=\; \theta^\mu \partial_\mu\ .$$

Equipping $T[1]B$ with coordinates

$$(x^\mu, \theta^\mu) \ , \qquad \deg(x^\mu, \theta^\mu) \ = \ (0, 1) \ ,$$

one has

$$\pi(f(x^\mu, \theta^\mu; dx^\mu, d\theta^\mu)) \ = \ f(x^\mu, dx^\mu; dx^\mu, 0) \ .$$

Thus the exterior differential d , which has form-degree one, has degree one, *i.e.*

$$\deg(\mathrm{d}) \ = \ \deg(q) \ = \ 1 \ .$$

The assumption that the sigma-model maps ϕ have vanishing intrinsic degree implies

$$\Omega^{[k|p]}(M) \stackrel{\phi^*}{\to} \Omega^{[k|p]}(T[1]B) \stackrel{\pi}{\to} \Omega^{[p]}(B) \ ,$$

that is, the pull-back ϕ^* of a k-form of $\mathbb{N}$-degree p on M is a ditto on $T[1]B$, in its turn sent by π to a p-form on B ; the condition that M is $\mathbb{N}$-graded (instead of $\mathbb{Z}$-graded) and $\deg(\mathrm{d}) = 1$ implies that $p \geqslant k$. Thus, since

$$\mathscr{O} \ = \ \mathrm{d}\vartheta \in \Omega^{[2|\hat{p}+2]}(M) \ , \quad \vartheta \in \Omega^{[1|\hat{p}+1]}(M) \ , \quad \mathscr{H} \in \Omega^{[0|\hat{p}+1]}(M) \ ,$$

it follows that

$$\pi\phi^*(\vartheta - \mathscr{H}) \ \in \ \Omega^{[\hat{p}+1]}(B) \ ,$$

which can then be integrated by decomposing B into charts B_ξ.

Classical action principle (2)

$\hookrightarrow$ Classical action principle of Hamiltonian type :

$$S^{\mathrm{cl}}_{\mathrm{bulk}}[\phi|B] \;=\; \sum_{\xi}\int_{B_\xi}\mathscr{L}^{\mathrm{cl}}_{\xi} \;=\; \sum_{\xi}\int_{B_\xi}\pi\,\phi^*_\xi(\vartheta - \mathscr{H})\ ,$$

where ϑ is a pre-symplectic form.

$\hookrightarrow$ Writing $\vartheta = \mathrm{d}Z^i\vartheta_i\,,\ \mathscr{O} = \frac{1}{2}\,\mathrm{d}Z^i\mathrm{d}Z^j\,\widetilde{\mathscr{O}}_{ij} = \frac{1}{2}\,\mathrm{d}Z^i\,\mathscr{O}_{ij}\,\mathrm{d}Z^j$ and defining

$$\{A,B\}^{[-\hat{p}]} = (-1)^{\hat{p}+(\hat{p}+i+1)A}\ \partial_i A\ \mathscr{P}^{ik}\ \partial_j B$$

where $\mathscr{P}^{ik}\mathscr{O}_{kj} = (-1)^{\hat{p}}\delta^i_j$, then ...

Classical action principle (3)

- ... the variation of the Lagrangian :

$$\delta \mathscr{L}^{\text{cl}}_{\text{bulk}} \;=\; \delta Z^i \mathscr{R}^j \widetilde{\mathscr{O}}_{ij} + \mathrm{d}\left(\delta Z^i \vartheta_i\right) \;,$$

where generalized curvatures and Hamiltonian vector field

$$\begin{aligned} \mathscr{R}^i \;&=\; \mathrm{d}Z^i + \mathscr{Q}^i \;, \qquad \mathscr{Q}^i \;=\; (-1)^{\hat{p}+1} \mathscr{P}^{ij} \partial_j \mathscr{H} \;, \\ \overrightarrow{\mathscr{Q}} \;&=\; \mathscr{Q}^i \vec{\partial}_i \;, \qquad \deg(\overrightarrow{\mathscr{Q}}) \;=\; 1 \;. \end{aligned}$$

- Variational principle $\;\Longrightarrow\;$ $\mathscr{R}^i \approx 0$, whose Cartan integrability on shell requires $\overrightarrow{\mathscr{Q}}$ to be a Hamiltonian Q-structure

$$\mathscr{L}_{\overrightarrow{\mathscr{Q}}} \overrightarrow{\mathscr{Q}} \equiv 0 \quad \Leftrightarrow \quad \mathscr{Q}^j \partial_j \mathscr{Q}^i \equiv 0 \quad \Leftrightarrow \quad \partial_i \{\mathscr{H}, \mathscr{H}\}^{[-\hat{p}]} \equiv 0 \;.$$

Classical action principle (4)

Nilpotency of $\overrightarrow{\mathscr{Q}}$ with suitable boundary conditions on the fields and gauge parameters ensure invariance of the action under

$$\begin{aligned} \delta_\epsilon Z^i &= \mathrm{d}\epsilon^i - \epsilon^j \partial_j \mathscr{Q}^i + \tfrac{1}{2}\,\epsilon^k \mathscr{R}^l\,\partial_l \widetilde{\mathscr{O}}_{kj}\,\mathscr{P}^{ji}\ , \\ \delta_\epsilon \mathscr{L}^{\mathrm{cl}}_{\mathrm{bulk}} &= \mathrm{d}K_\epsilon\ , \qquad K_\epsilon = \epsilon^i \mathscr{R}^j \widetilde{\mathscr{O}}_{ij} + \delta_\epsilon Z^i \vartheta_i\ , \end{aligned}$$

Closure of gauge transformations :

$$[\delta_{\varepsilon_1}, \delta_{\varepsilon_2}] Z^i = \delta_{\varepsilon_{12}} Z^i - \overrightarrow{\mathscr{R}} \varepsilon^i_{12}\ ,$$

where $\overrightarrow{\mathscr{R}} = \mathscr{R}^i \partial_i$ and

$$\varepsilon^i_{12} = -\tfrac{1}{2}\,[\overrightarrow{\varepsilon}_1, \overrightarrow{\varepsilon}_2]\,\mathscr{Q}^i\ .$$

Classical action principle (5)

- *Under certain extra assumptions* on ϑ and $\mathscr{H}$, the action can be defined globally by gluing together the locally defined fields and gauge parameters along chart boundaries using gauge transitions $\delta_t Z^i$ and $\delta_t \epsilon^i$ with parameters $\{t^i\} = t^{\xi}_{\xi'}$ defined on overlaps.

  Assumptions :

$$(i) \quad \delta_t K_\epsilon \; = \; 0\,, \quad (ii) \quad \partial_j \partial_k \overrightarrow{t}\,\mathscr{Q}^i \; = \; 0\,, \quad (iii) \quad K_t \; \equiv \; 0\,.$$

- Assumption $(i) \Longrightarrow$ cancellation of contributions to $\delta_\epsilon S^{\mathrm{cl}}_{\mathrm{bulk}}$ from chart boundaries in the interior of B, s.t. the variational principle implies the BC on fields and gauge parameters

$$K_\epsilon|_{\partial B} \; \equiv \; 0\,.$$

Classical action principle (6)

- Assumptions (ii) and (iii) ensure compatibility between gauge transformations and gauge transitions in the sense that performing a transition transformation on fields and gauge parameters between two adjacent charts and moving along the gauge orbit are two operations that commute. Give access to $\delta_{\varepsilon_\xi}\, t^{\xi}_{\xi'}$ and $\delta_{t^{\xi}_{\xi'}}\, \varepsilon_\xi$.
- The $\{t^{\xi}_{\xi'}\}'s \rightsquigarrow$ subalgebra of Cartan transformations that preserve the Lagrangian density, *i.e.* selects the transitions.
- Assuming there are no constants of total degree $\hat{p}+2$ on M , the condition $\partial_i\{\mathscr{H},\mathscr{H}\}^{[-\hat{p}]}\equiv 0$ is equivalent to the *structure equation*

$$\{\mathscr{H},\mathscr{H}\}^{[-\hat{p}]} \;\equiv\; 0 \quad\Leftrightarrow\quad (-1)^{i(\hat{p}+1)}\,\partial_i\mathscr{H}\,\mathscr{P}^{ij}\partial_j\mathscr{H} \;\equiv\; 0\,.$$

Correct amplitudes for unbroken HS

- In the case of Vasiliev's model : PSM action with bulk + boundary deformations.

$$S^{Tot.} \quad = \quad S^{bulk}[X,P] + S^{bound.}[X]\ .$$

  Reproduces the full nonlinear equations, same content perturbatively.

- With the addition of suitable boundary deformations built from the zero-forms of X , $Z[\mu] = \int DX DP \exp[\frac{\mu i}{\hbar}\, S^T]$ reproduces, to lowest order in $\hbar$, the correct N-point functions of the free $O(N)$ model on boundary [Colombo,Sundell], $(N = 2, 3)$ then [Didenko,Skvortsov] $N \geqslant 4$.

On-shell equivalence to Fronsdal approach

Concerning the correspondence with the free $O(N)$ vector model and Gross–Neveu model [Sezgin-Sundell] :

- for any $\mathscr{H}(U,V;B)$ and applying perturbation theory in which $\int_{\mathscr{M}} \mathrm{Tr}'[dX^\alpha \star P_\alpha]$ is treated as the kinetic term, it follows from the fact that the vertices in $\mathscr{H}(U,V;B)$ are built from exterior ($\star$-) products that boundary correlation functions that involve only zero-forms and one-forms are given by their semi-classical limits (as vacuum bubbles cancel), *viz.*

$$\langle B_{[0]}(p_1)\cdots B_{[0]}(p_n)A_{[1]}(p_{n+1})\cdots A_{[1]}(p_{n+m})\rangle|_{p_i\in\partial\mathscr{M}}$$

$$= \langle B_{[0]}(p_1)\rangle\cdots\langle B_{[0]}(p_n)\rangle\langle A_{[1]}(p_{n+1})\rangle\cdots\langle A_{[1]}(p_{n+m})\rangle \ ;$$

- assuming the existence of a perturbative completion $\int_{\partial\mathscr{M}} \mathscr{V}_{\mathrm{FV}}(B_{[0]}, dB_{[0]}; A_{[1]}, dA_{[1]})$ of the Fradkin – Vasiliev action [1], it can be added as a topological vertex operator and treated as an interaction (including its kinetic terms) ;
- it follows that the expectation value of the Fradkin–Vasiliev action is tree-level exact, *i.e.*

$$Z(\mu) \ := \ \left\langle \exp(\frac{i\mu}{\hbar}\int_{\partial\mathscr{M}} \mathscr{V}_{\mathrm{FV}}) \right\rangle \ = \ \exp(\frac{i\mu}{\hbar}\int_{\partial\mathscr{M}} \mathscr{V}_{\mathrm{FV}})\Big|_{B_{[0]}=\langle B_{[0]}\rangle ; A_{[1]}=\langle A_{[1]}\rangle} \ ,$$

with expectation values $\langle B_{[0]}\rangle$ and $\langle A_{[1]}\rangle$ obeying the Vasiliev equations of motion subject to boundary conditions at the three-dimensional boundary of $\partial\mathscr{M}$;

1. Whether the completion is given in the standard Fronsdal formulation or in the frame-like formulation is immaterial as in both cases the dynamical field content can be obtained by applying projections to the Vasiliev master fields.

- thus, assuming a suitable topology for $\partial\mathscr{M}$ and that $\langle B_{[0]}\rangle$ and $\langle A_{[1]}\rangle$ are asymptotic to AdS_4 , hence built from the boundary data using *boundary-to-bulk propagators*, we expect that $Z(\mu)$ with $\mu N = \hbar$ is equal to the generating functional of the free $O(N)$ model in the case of the Type A model with scalar field obeying $\Delta = 1$ boundary conditions, and to the generating functional of the free Gross–Neveu model (with N free fermions) in the case of the Type B model with scalar field obeying $\Delta = 2$ boundary conditions.

- We wish to stress the fact that both of the latter higher-spin gravity models are manifestly tree-level unitary : by the very nature of the perturbative treatment of the Poisson sigma models (with kinetic PdX-terms), *the partition function* $Z(\mu)$ *is completely free from loop-corrections in the Fradkin–Vasiliev sector*, in perfect agreement with free three-dimensional CFTs. In other words, $Z(\mu)$ is given by the sum of tree Witten-diagrams in AdS_4 with external boundary-to-bulk and internal bulk-to-bulk Green's functions arising as the result of solving classical equations of motion subject to boundary sources.

- In the case of the strongly-coupled fixed points of the $O(N)$ vector model [Klebanov-Polyakov] and the Gross–Neveu model [Sezgin-Sundell], reached by suitable double-trace deformations, the Fradkin-Vasiliev action needs to be modified with a Gibbons-Hawking term

$$\int_{\partial^2 \mathscr{M}} \mathscr{V}_{\mathrm{GH}} \;=\; \int_{\partial^2 \mathscr{M}} \phi \partial_n \phi + \cdots \;,$$

where the $\cdots$ contain a non-linear completion achieving higher-spin gauge invariance.

AKSZ quantization

Classical coordinates $Z^i \equiv Z^{i\,\langle 0\rangle}_{[p_i]}$ on M is extended into coordinates on $\boldsymbol{M}$:

$$\left\{ Z^{i\,\langle g\rangle}_{[p_i-g]}\ , \qquad Z^{\langle -1-g\rangle}_{i[\hat{p}+1-p_i+g]} \ := \ \left(Z^{i\,\langle g\rangle}_{[p_i-g]} \right)^+ \right\}\ , \qquad g \ = \ 0,\dots,p_i\ ,$$

$O^{\langle g\rangle}_{[p]}$: ghost number g and form degree p.

Total degree and Graßmann parity (for classical theories consisting of only bosonic fields) :

$$|\cdot| \ := \ \deg(\cdot) + \mathrm{gh}(\cdot)\ , \quad \mathrm{Gr}(\cdot) \ = \ |\cdot| \mod 2\ .$$

So,

$$|Z^{i\,\langle g\rangle}_{[p_i-g]}| \ = \ p_i\ , \qquad |Z^{\langle -1-g\rangle}_{i[\hat{p}+1-p_i+g]}| \ = \ \hat{p} - p_i\ .$$

Integrate a total form on a p-chain

Given a differential form $L \in \Omega(\boldsymbol{M})$ of fixed total degree $|L|$, described locally on $\boldsymbol{M}$ by a function $L(Z, Z^+, dZ, dZ^+)$, with pull-back

$$\pi\boldsymbol{\phi}^*(L) \equiv \sum_{p=0}^{\hat{p}+1} [\pi\boldsymbol{\phi}^*(L)]^{\langle |L|-p\rangle}_{[p]} \in \Omega(B)$$

and a p-cycle $C \subseteq B$, the integral

$$I(L|C) \;\equiv\; \sum_\xi \int_{B_\xi \cap C} \pi\, \boldsymbol{\phi}^*{}_\xi(L) \;:=\; \sum_\xi \int_{B_\xi \cap C} [\pi\boldsymbol{\phi}^* L]^{\langle |L|-p\rangle}_{[p]}$$

$$\textit{i.e.}\quad \mathrm{gh}(I(L|C)) \;=\; |L| - p\ .$$

The *canonical coordinates* $Z^i = (X^\alpha, P_\alpha)$ of M induce supercoordinates $\boldsymbol{Z}^i = (\boldsymbol{X}^\alpha, \boldsymbol{P}_\alpha)$ of $\boldsymbol{M}$ of fixed total degree :

$$\begin{aligned}
\boldsymbol{X}^\alpha &= \underbrace{X^{\alpha\,\langle p_\alpha\rangle}_{[0]} + X^{\alpha\,\langle p_\alpha-1\rangle}_{[1]} + \ldots + X^{\alpha\,\langle 0\rangle}_{[p_\alpha]}}_{\text{fields}} + \\
&+ \underbrace{P^{\alpha\,\langle -1\rangle}_{[p_\alpha+1]} + P^{\alpha\,\langle -2\rangle}_{[p_\alpha+2]} + \ldots + P^{\alpha\,\langle p_\alpha-\hat{p}-1\rangle}_{[\hat{p}+1]}}_{\text{anti-fields}} \quad ,
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{P}_\alpha &= \underbrace{P^{\langle \hat{p}-p_\alpha\rangle}_{\alpha\ [0]} + P^{\langle \hat{p}-p_\alpha-1\rangle}_{\alpha\ [1]} + \ldots + P^{\langle 0\rangle}_{\alpha\ [\hat{p}-p_\alpha]}}_{\text{fields}} \\
&+ \underbrace{X^{\langle -1\rangle}_{\alpha\ [\hat{p}-p_\alpha+1]} + X^{\langle -2\rangle}_{\alpha\ [\hat{p}-p_\alpha+2]} + \ldots + X^{\langle -p_\alpha-1\rangle}_{\alpha\ [\hat{p}+1]}}_{\text{anti-fields}} \quad .
\end{aligned}$$

Symplectic and pre-symplectic forms $\boldsymbol{O}$ and $\boldsymbol{\vartheta}$ on $\boldsymbol{M}$:

$$\boldsymbol{O} \;=\; \big[(-1)^{\alpha+1}\mathrm{d}\boldsymbol{X}^{\alpha}\mathrm{d}\boldsymbol{P}_{\alpha}\big]^{\langle 0\rangle}_{[\hat{p}+2]} \;=\; \mathrm{d}\boldsymbol{\vartheta}\;, \qquad \boldsymbol{\vartheta} \;=\; [\mathrm{d}\boldsymbol{X}^{\alpha}\boldsymbol{P}_{\alpha}]^{\langle 0\rangle}_{[\hat{p}+1]}\;,$$

and we denote the corresponding *graded Poisson bracket* on $\boldsymbol{M}$ by

$$\{\cdot,\cdot\} \;\equiv\; \{\cdot,\cdot\}^{\langle 0\rangle}_{[-\hat{p}]}\;,$$

and graded Poisson bracket on $\mathrm{Maps}\,[T[1]B,\boldsymbol{M}]$, referred to as the BV bracket, is denoted by

$$(\cdot,\cdot) \;\equiv\; (\cdot,\cdot)^{\langle 1\rangle}_{[0]}\;,$$

with quantum numbers $\mathrm{gh}\,((\cdot,\cdot)) = 1$ and $\deg\,((\cdot,\cdot)) = 0\,$.

BV bracket induced from Poisson bracket.

As observed by AKSZ, the BV bracket $(\cdot,\cdot)$ on $\mathrm{Maps}\,[T[1]B, \boldsymbol{M}]$ is induced from the graded Poisson bracket $\{\cdot,\cdot\}$ on $\Omega^{[0]}(\boldsymbol{M})$ via the formula

$$(I(F|B) \,,\, \boldsymbol{\phi}^*(F')) \;\equiv\; \boldsymbol{\phi}^*(\{F, F'\}) \;.$$

It follows that the BV-adjoint action of the pre-symplectic form is related to the exterior derivative as follows :

$$(I(\mathrm{d}\boldsymbol{X}^\alpha \boldsymbol{P}_\alpha|B) \,,\, \boldsymbol{\phi}^*(L)) \;\equiv\; \mathrm{d}\,\boldsymbol{\phi}^*(L) \;\equiv\; \boldsymbol{\phi}^*(\mathrm{d}\,L) \;,$$

for $L \in \Omega(\boldsymbol{M})$.

Superfunctionals

Functionals built from ultra-local superfunctionals $\boldsymbol{\phi}^*(\boldsymbol{G})$ where $\boldsymbol{G} \in \Omega(\boldsymbol{M})$ have local representatives of the form $\boldsymbol{G} = G(\boldsymbol{Z}^i, d\boldsymbol{Z}^i)$ where $G \in \Omega(M)$. In particular, if $\boldsymbol{F}, \boldsymbol{F}'$ are superfunctions it follows that

$$\{\boldsymbol{F}, \boldsymbol{F}'\} \;=\; \big(\{F, F'\}_{[-\hat{p}]}(Z^i)\big)\Big|_{Z^i \to \boldsymbol{Z}^i} \;,$$

where $\{F, F'\}_{[-\hat{p}]}$ denotes the Poisson bracket evaluated in the classical target space M .

The AKSZ action

$$\boldsymbol{S}_{\text{bulk}}[\boldsymbol{\phi}|B] \;:=\; I\,(\boldsymbol{L}|B) \;=\; \sum_{\xi}\int_{B_\xi} \pi\boldsymbol{\phi}_\xi^*\,(\boldsymbol{L})\;, \qquad \boldsymbol{L} \;:=\; \mathrm{d}\boldsymbol{X}^\alpha \boldsymbol{P}_\alpha - \mathscr{H}(\boldsymbol{X},\boldsymbol{P})$$

with $\mathscr{H}$ being a solution to the classical structure equation obeying $\mathscr{H}|_{P_\alpha=0} = 0$. Defining

$$\mathrm{s}(\cdot) \quad := \quad (\boldsymbol{S}_{\text{bulk}}, (\cdot))\;,$$

one has

$$\mathrm{s}\boldsymbol{Z}^i \quad = \quad \boldsymbol{R}^i\;,$$

where the generalized supercurvatures

$$\boldsymbol{R}^i \;:=\; \mathrm{d}\boldsymbol{Z}^i + \boldsymbol{Q}^i\;, \qquad \boldsymbol{Q}^i \;:=\; \mathscr{Q}^i(\boldsymbol{Z}^j) \;=\; (-1)^{\hat{p}+1}\mathscr{P}^{ij}\partial_j\mathscr{H}(\boldsymbol{Z}^i)\;.$$

The locally-defined field configurations form equivalence classes modulo gauge transformations

$$\delta_{\boldsymbol{\varepsilon}} \boldsymbol{Z}^i \quad := \quad \mathrm{d}\boldsymbol{\varepsilon}^i - \boldsymbol{\varepsilon}^j \partial_j \boldsymbol{Q}^i \ ,$$

where the parameters have total degree $|\boldsymbol{\varepsilon}^i| = |\boldsymbol{Z}^i| - 1$ and expansions into components with fixed ghost numbers and form degrees given by the suspension of $\boldsymbol{X}^\alpha$ and $\boldsymbol{P}_\alpha$ with one unit of form degree, and zero units of ghost number.

As in the classical case, it follows from

$$\delta_{\boldsymbol{\varepsilon}} \boldsymbol{S}_{\text{bulk}} \ \equiv \ \textstyle\sum_\xi \oint_{\partial B_\xi} \boldsymbol{K}_{\boldsymbol{\varepsilon}} \ ,$$

$$\boldsymbol{K}_{\boldsymbol{\varepsilon}} \ = \ (-1)^{\hat{p}(\alpha+1)} \boldsymbol{\eta}_\alpha \boldsymbol{R}^\alpha + \left((\overrightarrow{\boldsymbol{P}} - 1) \overrightarrow{\boldsymbol{\epsilon}} + \overrightarrow{\boldsymbol{P}} \overrightarrow{\boldsymbol{\eta}} \right) \mathscr{H} \ ,$$

... that the AKSZ action can be defined globally using fiber-bundle type geometries.

(I) the local representatives $\boldsymbol{Z}^i_\xi$ are glued together using transition functions with parameters $\boldsymbol{t}^{i,\xi}_{\xi'} = (\boldsymbol{t}^\alpha, 0)^\xi_{\xi'}$, obeying

$$(\overrightarrow{\boldsymbol{P}} - 1)\overrightarrow{\boldsymbol{t}}\mathscr{H} \equiv 0 \quad i.e. \quad \overrightarrow{\boldsymbol{t}}\Pi_{(n)} \equiv 0 \text{ for } n \neq 1 \,,$$

and

(II) the following Dirichlet conditions are imposed :

$$\boldsymbol{\eta}_\alpha|_{\partial B} = 0 \,, \qquad \boldsymbol{P}_\alpha|_{\partial B} = 0 \,.$$

The AKSZ relation between the BV bracket and the Poisson bracket $\Rightarrow$

$$(\boldsymbol{S}_{\text{bulk}}, \boldsymbol{S}_{\text{bulk}}) \;=\; (-1)^{\hat{p}} \sum_{\xi} \oint_{\partial B_\xi} \pi \boldsymbol{\phi}^{*}_{\xi} \left(\boldsymbol{R}^{\alpha} \boldsymbol{P}_{\alpha} - 2\boldsymbol{L}\right) \;=\; 0 \;,$$

where the latter equality follows from the boundary conditions and the facts that $\delta_{\boldsymbol{t}} \boldsymbol{L} \equiv \boldsymbol{K}_{\boldsymbol{t}} \equiv 0$ and that

$$\delta_{\boldsymbol{t}} \boldsymbol{P}_{\alpha} \;=\; -(-1)^{\alpha} \overrightarrow{\boldsymbol{t}} \partial_{\alpha} \mathscr{H} \;, \qquad \delta_{\boldsymbol{t}} \boldsymbol{R}^{\alpha} \;=\; (-1)^{\hat{p}(\alpha+1)} \overrightarrow{\boldsymbol{R}}_{X} \overrightarrow{\boldsymbol{t}} \partial^{\alpha} \mathscr{H} \;,$$

where $\overrightarrow{\boldsymbol{R}}_{X} := \boldsymbol{R}^{\alpha} \partial_{\alpha}$, implying

$$\delta_{\boldsymbol{t}}(\boldsymbol{R}^{\alpha} \boldsymbol{P}_{\alpha}) \;\equiv\; \overrightarrow{\boldsymbol{R}}_{X} \overrightarrow{\boldsymbol{t}} (\overrightarrow{\boldsymbol{P}} - 1) \mathscr{H} \;\equiv\; 0 \;.$$

↱ The AKSZ action $\boldsymbol{S}_{\text{bulk}}$ solves the classical BV master equation

$$(\boldsymbol{S}_{\text{bulk}}, \boldsymbol{S}_{\text{bulk}}) \;=\; 0 \quad \Leftrightarrow \quad \mathrm{s}^2 \;=\; 0 \;,$$

subject to the functional boundary condition

$$\boldsymbol{S}_{\text{bulk}}[\boldsymbol{\phi}|B]|_{\boldsymbol{\phi}=\phi} \;=\; S^{\text{cl}}_{\text{bulk}}[\phi|B] \;.$$

Hamiltonian action principle ; chiral trace

$\hookrightarrow$ Integration over $\mathfrak{C}$ of a globally-defined $(\hat{p}+1)$-form $\mathscr{L}$:

$$\int_{\mathfrak{C}} \mathscr{L} \;=\; \sum_{\xi} \int_{M_\xi} \mathrm{Tr}\,[f_{\mathscr{L}}] \;,$$

where $f_{\mathscr{L}}$ denotes a symbol of $\mathscr{L}$ and the chiral trace operation is defined by

$$\mathrm{Tr}\,[f] \;=\; \sum_{m} \int_{\mathfrak{Z}\times\mathfrak{Y}} \frac{d^2y d^2\bar{y}}{(2\pi)^2} \frac{f_{[m;2,2]}|_{k=0=\bar{k}}}{(2\pi)^2} \;, \tag{2}$$

using $f_{[p]} = \sum_{\substack{m\,+\,q\,+\,\bar{q}\,=\,p \\ q,\,\bar{q}\,\leqslant\,2}} f_{[m;q,\bar{q}]}$ with

$$f_{[m;q,\bar{q}]}(\lambda\, dX^M; \mu\, dz^\alpha, \bar{\mu}\, d\bar{z}^{\dot{\alpha}}) \;=\; \lambda^m\, \mu^q\, \bar{\mu}^{\bar{q}}\, f_{[m;q,\bar{q}]}(dX^M; dz^\alpha, d\bar{z}^{\dot{\alpha}})\;. \tag{3}$$

One integrates over $\{y^\alpha, z^\alpha\}$ and $\{\bar{y}^{\dot{\alpha}}, \bar{z}^{\dot{\alpha}}\}$ viewed as real, independent variables.

Action principle ; Graded cyclic trace

This choice implies

$$\mathrm{Tr}\left[\pi(f)\right] \;=\; \mathrm{Tr}\left[\bar{\pi}(f)\right] \;=\; \mathrm{Tr}\left[f\right] \;,$$

which in its turn implies graded cyclicity,

$$\mathrm{Tr}\left[f_{[p]} \star f'_{[p']}\right] \;=\; (-1)^{pp'}\,\mathrm{Tr}\left[f'_{[p']} \star f_{[p]}\right] \;,$$

Furthermore

$$(\mathrm{Tr}\left[f\right])^{\dagger} \;=\; \mathrm{Tr}\left[(f)^{\dagger}\right] \;, \quad \mathrm{Tr}\left[P(f)\right] \;=\; \mathrm{Tr}\left[f\right] \;, \quad \mathrm{Tr}\left[\pi_k(f)\right] \;=\; \mathrm{Tr}\left[f\right] \;, \quad \text{where}$$

$$\pi_k \;:\; (k,\bar{k}) \;\mapsto\; (-k,-\bar{k}) \;,$$

$$P[f(X^M;z^\alpha,\bar{z}^{\dot{\alpha}};y^\alpha,\bar{y}^{\dot{\alpha}};k,\bar{k})] \;=\; (Pf)(X^M;-\bar{z}^{\dot{\alpha}},-z^\alpha;\bar{y}^{\dot{\alpha}},y^\alpha;\bar{k},k) \;.$$

[where Pf is expanded in terms of parity-reversed component fields.]

Odd-dimensional bulk $(\hat{p} \in 2\mathbb{N})$

$\hookrightarrow$ Finally, we assume that, off shell : $\mathrm{Tr}\,[\tau(f)] \;=\; \mathrm{Tr}\,[f]$, and that the integration over $\mathfrak{C}$ is non-degenerate : If $\mathrm{Tr}\,[f \star g] = 0$ for all f, then $g = 0$.

In the case of an odd-dimensional base manifold of dimension $\hat{p} + 1 = 2n + 5$ with $n \in \{0, 1, 2, \ldots\}$ such that $\dim(M) = 2n + 1$, we propose the bulk action

$$S^{\mathrm{cl}}_{\mathrm{bulk}}[\{A, B, U, V\}_\xi] \;=\; \sum_\xi \int_{M_\xi} \mathrm{Tr}\left[U \star DB + V \star \left(F + \mathscr{G}(B, U; J^I, J^{\bar{I}}, J^{I\bar{I}})\right)\right]$$

with interaction freedom $\mathscr{G}$ and locally-defined master fields $(m = n + 2)$

$$\begin{aligned} A &= A_{[1]} + A_{[3]} + \cdots + A_{[2m-1]}\,, & B &= B_{[0]} + B_{[2]} + \cdots + B_{[2m-2]}\,, \\ U &= U_{[2]} + U_{[4]} + \cdots + U_{[2m]}\,, & V &= V_{[1]} + V_{[3]} + \cdots + V_{[2m-1]}\,. \end{aligned}$$

Why such an extension ?

- Because we want a P-structure and only wedge products in the Lagrangian, (take $n=2$ here) $U_{[8]}$ and $V_{[7]}$ are not sufficient : $U_{[8]} \star V_{[7]}$ is not of total degree $9 = 4+1+4$.
- $\mathscr{G}$ must be constrained in order for the action to be gauge invariant and in order to avoid systems that are trivial. We take

$$\begin{aligned}
\mathscr{G} &= \mathscr{F}(B; J^I, J^{\bar{I}}, J^{I\bar{I}}) + \widetilde{\mathscr{F}}(U; J^I, J^{\bar{I}}, J^{I\bar{I}}) \quad , \\
\mathscr{F} &= \mathscr{F}_I(B) \star J^I_{[2]} + \mathscr{F}_{\bar{I}}(B) \star J^{\bar{I}}_{[2]} + \mathscr{F}_{I\bar{I}}(B) \star J^{I\bar{I}}_{[4]} \quad , \\
\widetilde{\mathscr{F}} &= \widetilde{\mathscr{F}}_I(U) \star J^I_{[2]} + \widetilde{\mathscr{F}}_{\bar{I}}(U) \star J^{\bar{I}}_{[2]} + \widetilde{\mathscr{F}}_{I\bar{I}}(U) \star J^{I\bar{I}}_{[4]} \quad ,
\end{aligned}$$

where the central and closed elements

$$(J^I_{[2]})_{I=1,2} = -\tfrac{i}{4}(1\,,\, k\kappa) \star P_+ \star d^2 z\ ,\ (J^{\bar{I}}_{[2]})_{\bar{I}=\bar{1},\bar{2}} = -\tfrac{i}{4}(1\,,\, \bar{k}\bar{\kappa}) \star P_+ \star d^2 \bar{z}$$

$$J^{I\bar{I}}_{[4]} = 4\, i\, J^I_{[2]} J^{\bar{I}}_{[2]} \quad ,$$

Interaction freedom

Denoting $Z^i = (A, B, U, V)$, the general variation of the action defines generalized curvatures $\mathscr{R}^i$ as follows :

$$\delta S \;=\; \sum_\xi \int_{M_\xi} \mathrm{Tr}\left[\mathscr{R}^i \star \delta Z^j \mathscr{O}_{ij}\right] + \sum_\xi \int_{\partial M_\xi} \mathrm{Tr}\left[U \star \delta B - V \star \delta A\right] \;,$$

where one thus has

$$\mathscr{R}^A \;=\; F + \mathscr{F} + \widetilde{\mathscr{F}} \;, \qquad \mathscr{R}^B \;=\; DB + (V\partial_U) \star \widetilde{\mathscr{F}} \;,$$
$$\mathscr{R}^U \;=\; DU - (V\partial_B) \star \mathscr{F} \;, \qquad \mathscr{R}^V \;=\; DV + [B, U]_\star \;,$$

with $\mathscr{O}_{ij}$ being a constant non-degenerate matrix (defining a symplectic form of degree $\hat{p} + 2$ on the $\mathbb{N}$-graded target space of the bulk theory).

Obstruction to Cartan integrability ?

Generically there are obstructions to Cartan integrability of the unfolded equations of motion $\mathscr{R}^i \approx 0$. These obstructions vanish identically (without further algebraic constraints on Z^i) in at least the following two cases :

$$\text{bilinear } Q\text{-structure} \quad : \quad \mathscr{F} \;=\; B \star J \;, \qquad J \;=\; J_{[2]} + J_{[4]} \;,$$

$$\text{bilinear } P\text{-structure} \quad : \quad \widetilde{\mathscr{F}} \;=\; U \star J' \;, \qquad J' \;=\; J'_{[2]} + J'_{[4]} \;.$$

where $B \star J_{[2]} = B \star (b_I\, J^I_{[2]} + b_{\bar{I}}\, J^{\bar{I}}_{[2]})\,,\;\; B \star J_{[4]} = B \star (c_{I\bar{I}}\, J^{I\bar{I}}_{[4]})$, idem J'.

Consistency

Recall that if $\mathscr{R}^i = dZ^i + \mathscr{Q}^i(Z^j)$ defines a set of generalized curvatures, then one has the following three equivalent statements :

(i) $\mathscr{R}^i$ obey a set of generalized Bianchi identities $d\mathscr{R}^i - (\mathscr{R}^j \partial_j) \star \mathscr{Q}^i \equiv 0$;

(ii) $\mathscr{R}^i$ transform into each other under Cartan gauge transformations $\delta_\varepsilon Z^i = d\varepsilon^i - (\varepsilon^j \partial_j) \star \mathscr{Q}^i$; and

(iii) the quantity $\overrightarrow{\mathscr{Q}} := \mathscr{Q}^i \partial_i$ is a Q-structure, *i.e.* a nilpotent $\star$-vector field of degree one in target space, *viz.* $\overrightarrow{\mathscr{Q}} \star \mathscr{Q}^i \equiv 0$.

Furthermore, in the case of differential algebras on commutative base manifolds, one can show that if $\mathscr{R}^i$ are defined via a variational principle as above (with constant $\mathscr{O}_{ij}$), then the action S remains invariant under $\delta_\varepsilon Z^i$.

Cartan gauge transformations

In the two Cartan integrable cases at hand, one thus has the on-shell Cartan gauge transformations

$$
\begin{aligned}
\delta_{\epsilon,\eta}A &= D\epsilon^A - (\epsilon^B\partial_B)\star\mathscr{F} - (\eta^U\partial_U)\star\widetilde{\mathscr{F}}\ , \\
\delta_{\epsilon,\eta}B &= D\epsilon^B - [\epsilon^A, B]_\star - (\eta^V\partial_U)\star\widetilde{\mathscr{F}} - (\eta^U\partial_U)\star(V\partial_U)\star\widetilde{\mathscr{F}}\ , \\
\delta_{\epsilon,\eta}U &= D\eta^U - [\epsilon^A, U]_\star + (\eta^V\partial_B)\star\mathscr{F} + (\epsilon^B\partial_B)\star(V\partial_B)\star\mathscr{F}\ , \\
\delta_{\epsilon,\eta}V &= D\eta^V - [\epsilon^A, V]_\star - [\epsilon^B, U]_\star + [\eta^U, B]_\star\ .
\end{aligned}
$$

These transformations remain symmetries off shell, although we are in the context of graded non-commutative (but still associative) base manifold.

Cartan gauge algebra

$\longmapsto$ More precisely, the $(\epsilon^A;\epsilon^B)$-symmetries leave the Lagrangian invariant while the (η^U,η^V)-symmetries transform the Lagrangian into a nontrivial total derivative, *viz.*

$$\delta_{\epsilon,\eta}\mathscr{L} \;\equiv\; \mathrm{d}\left(Tr\left[\eta^U \star \mathscr{K}_U + \eta^V \star \mathscr{K}_V\right]\right) \;,$$

for $(\mathscr{K}_U,\mathscr{K}_V)$ that are not identically zero. It follows that the Cartan gauge algebra $\mathfrak{g}$ is of the form

$$\mathfrak{g} \cong \mathfrak{g}_1 \oplus\!\!\!\!\!\!\supset \mathfrak{g}_2$$

with $\mathfrak{g}_1 \cong \mathrm{span}\{\epsilon^A,\epsilon^B\}$ and $\mathfrak{g}_2 \cong \mathrm{span}\{\eta^U,\eta^V\}$, as one can verify explicitly.

$\longmapsto$ In order for the variational principle to be globally well-defined, one has (like in PSM) to impose the following :

$$(U,V)|_{\partial M} \;=\; 0 \;.$$

Global formulation

Exponentiation of the infinitesimal Cartan gauge transformations leads to locally defined gauge orbits consisting of elements

$$Z^i_{\lambda,\mathrm{d}\lambda;Z_0} \;=\; \mathscr{G}_{\lambda,\mathrm{d}\lambda;Z} \star Z^i|_{Z^i=Z^i_0} \;,$$

$$\mathscr{G}_{\lambda,\mathrm{d}\lambda;Z} \;:=\; \exp_\star \overrightarrow{\mathscr{T}}_{\lambda,\mathrm{d}\lambda;Z} \;, \qquad \overrightarrow{\mathscr{T}}_{\lambda,\mathrm{d}\lambda;Z} \;:=\; \left(\mathrm{d}\,\lambda^i - (\lambda^j\partial_j)\star\mathscr{Q}^i\right)\frac{\partial}{\partial Z^i} \;,$$

where λ^i and Z^i_0, respectively, are gauge functions and representatives of the orbits defined in coordinate charts of the base manifold. On shell, one has

$$\mathrm{d}\,Z^i_0 + \mathscr{Q}^i(Z^j_0) \;\approx\; 0 \quad\Rightarrow\quad \mathrm{d}\,Z^i_{\lambda,\mathrm{d}\lambda;Z_0} + \mathscr{Q}^i(Z^j_{\lambda,\mathrm{d}\lambda;Z_0}) \;\approx\; 0 \;.$$

Unbroken phase, global formulation

From $\delta_{\epsilon,\eta}\mathscr{L} = \mathrm{d}(T(\eta, Z))$ it also follows that $(\eta^U, \eta^V) \in \mathfrak{g}_2$ need to be defined globally on M, that is, $(\eta^U, \eta^V)|_\xi$ and $(\eta^U, \eta^V)|_{\xi'}$ must be related by transition functions across the chart boundary between M_ξ and $M_{\xi'}$ (in practice : take (η^U_ξ, η^V_ξ) to have compact support in M_ξ).

The unbroken phase of the theory [no impurities inserted] thus consists of local representatives $Z^i_\xi = (A, B; U, V)|_\xi$ defined up to gauge transformations with parameters $(\epsilon^A_\xi; \epsilon^B_\xi)$ that are unrestricted on ∂M_ξ and parameters (η^U_ξ, η^V_ξ) with the aforementioned restrictions on ∂M_ξ, with transitions of the form

$$Z^i_\xi \;=\; \mathscr{G}^{\xi'}_\xi \star Z^i_{\xi'} \qquad \text{defined on} \quad M_\xi \cap M_{\xi'}$$

where $\mathscr{G}^{\xi'}_\xi = \exp_\star \vec{\mathscr{T}}_{t,dt;Z}|^{\xi'}_\xi$ with transition functions $t^{\xi'}_\xi \in \mathfrak{g}_1$ on $M_\xi \cap M_{\xi'}$.

Classical solutions, Boundary conditions

The natural boundary conditions compatible with the locally defined gauge symmetries are the Dirichlet conditions $(U,V)|_{\partial M} \;=\; 0$.

In summary, on top of the above BC, a classical solution can thus be specified by fixing

(I) the transition functions $\{t_{\xi}^{\xi'}\} \in \mathfrak{l} \subseteq \mathfrak{g}_1$;

(II) an initial datum for the zero-form $B_{[0]}$, say

$$B_{[0]}|_p \;=\; C(Y;k,\bar{k}) \;, \tag{4}$$

at some given point $p \in \mathfrak{B}$ in the base manifold ;

(III) boundary conditions on the gauge functions associated with the softly-broken gauge symmetries, *viz.*

$$\lambda|_{\partial M} \qquad \text{for} \qquad \lambda \in \; \mathfrak{g}_1/\mathfrak{l} \quad . \tag{5}$$