Higher geometric prequantum theory II
L_∞-Algebras of local observables

Domenico Fiorenza

Sapienza Università di Roma

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Based on d.f., Chris Rogers and Urs Schreiber *L_∞-algebras of local observables from higher prequantum bundles*, `arXiv:1304.6292`

$$
\begin{array}{c} (X,\omega) \\ \text{symplectic manifold} \end{array}
\quad \rightsquigarrow \quad
\begin{tikzcd}
C^\infty(X;\mathbb{R}),\ \{,\} \arrow[d] \\
\mathfrak{X}_{\mathrm{Ham}}(X)
\end{tikzcd}
$$

$$(X,\omega) \text{ pre-}n\text{-plectic manifold} \quad \rightsquigarrow \quad \begin{array}{c} L_\infty(X;\omega) \\ \downarrow \\ \mathfrak{X}_{\mathrm{Ham}}(X) \end{array}$$

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An n-plectic manifold is a pair (X, ω) with X a smooth manifold and ω a degree n+1, nondegenerate closed form on X

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Example. G compact simple simply connected Lie group, ω_G canonical 3-form on G.

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A pre-n-plectic manifold is a pair (X, ω) with X a smooth manifold and ω a degree n+1 closed form on X

The vector space of Hamiltonian pairs on a symplectic manifold (X, ω) is

$$Ham^0(X) = \{(v, H) \in \mathfrak{X}(X) \oplus \Omega^0(X) \;:\; \iota_v \omega + dH = 0\}$$

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$$\iota_{[v,w]}\omega = [\mathcal{L}_v, \iota_w]\omega = \mathcal{L}_v \iota_w \omega = -d\iota_{v\wedge w}\omega$$

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The vector field $[v, w]$ is Hamiltonian with Hamiltonian $n-1$-form $\iota_{v\wedge w}\omega$.

$$\iota\omega : \mathfrak{X}(X) \to \Omega^n(X)$$

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Is $\iota\omega$ a morphism of Lie algebras?

$$\iota_{[v,w]}\omega \stackrel{?}{=} [\iota_v\omega, \iota_w\omega]$$

$$\iota_{[v,w]}\omega \stackrel{?}{=} \underbrace{[\iota_v\omega, \iota_w\omega]}_{0}$$

$$\iota_{[v,w]}\omega = \underbrace{[\iota_v\omega, \iota_w\omega]}_{0} + d\iota_{v\wedge w}\omega$$

$$\mathfrak{X}_{Ham}(X) \xrightarrow{\iota\omega} d\Omega^{n-1}(X)$$

$$\begin{tikzcd}
\mathfrak{X}_{Ham}(X) \arrow[r, "\iota\omega"] \arrow[dr, "\iota_{\bullet\wedge\bullet}\omega"'] & d\Omega^{n-1}(X) \\
& \Omega^{n-1} \arrow[u, "d"']
\end{tikzcd}$$

$$
\begin{tikzcd}
\mathfrak{X}_{Ham}(X) \arrow[rr, "\iota\omega"] \arrow[drr, "\iota_{\bullet\wedge\bullet}\omega"] \arrow[ddrr, "\iota_{\bullet\wedge\bullet\wedge\bullet}\omega"'] & & d\Omega^{n-1}(X) \\
& & \Omega^{n-1} \arrow[u, "d"] \\
& & \Omega^{n-2} \arrow[u, "d"] \\
& & \vdots \arrow[u, "d"]
\end{tikzcd}
$$

Now on the left we have a Lie algebra (a dgla concentrated in degree zero) and on the right we have a chain complex (an abelian dgla)

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So we can ask ourselves: is $\iota_\bullet\omega$ an L_∞-morphism?

An L_∞-morphism between a Lie algebra $\mathfrak{g}$ and a chain complex $\mathfrak{h}$ is a sequence of multilinear maps $\varphi_k : \wedge^k\mathfrak{g} \to \mathfrak{h}[1-k]$ such that

$$d_{\mathfrak{h}}\varphi_k(v_1\wedge\cdots\wedge v_k) = \sum_{i<j} \pm\varphi_{k-1}([v_i, v_j]_{\mathfrak{g}}\wedge v_1\wedge\cdots\wedge\widehat{v_i}\wedge\cdots\wedge\widehat{v_j}\wedge\cdots\wedge v_k).$$

$$d\iota_{v_1\wedge\cdots\wedge v_k}\omega \stackrel{?}{=} \sum_{i<j} \pm\iota_{[v_i,v_j]_{\mathfrak{g}}\wedge v_1\wedge\cdots\wedge\widehat{v_i}\wedge\cdots\wedge\widehat{v_j}\wedge\cdots\wedge v_k}\omega$$

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For $\omega \in \Omega^{n+1}(X)$ and $v_1, \ldots, v_k$ in $\mathfrak{X}(X)$ we have

$$\begin{aligned} d\iota_{v_1\wedge\cdots\wedge v_k}\omega &= \sum_{i<j} \pm\iota_{[v_i,v_j]_{\mathfrak{g}}\wedge v_1\wedge\cdots\wedge\widehat{v_i}\wedge\cdots\wedge\widehat{v_j}\wedge\cdots\wedge v_k}\omega \\ &\quad + \sum_i \pm\iota_{v_1\wedge\cdots\wedge\hat{v}_i\wedge\cdots\wedge v_k}\mathcal{L}_{v_i}\omega \\ &\quad \pm \iota_{v_1\wedge\cdots\wedge v_k} d\omega \end{aligned}$$

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For $\omega \in \Omega^{n+1}_{cl}(X)$ and $v_1, \ldots, v_k$ in $\mathfrak{X}(X)$ we have

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For $\omega \in \Omega^{n+1}_{cl}(X)$ and $v_1, \ldots, v_k$ in $\mathfrak{X}_{Ham}(X)$ we have

$$d\iota_{v_1\wedge\cdots\wedge v_k}\omega = \sum_{i<j} \pm\iota_{[v_i,v_j]_{\mathfrak{g}}\wedge v_1\wedge\cdots\wedge\widehat{v_i}\wedge\cdots\wedge\widehat{v_j}\wedge\cdots\wedge v_k}\omega$$

Now that we have an L_∞-morphism...

$$\mathfrak{X}_{Ham}(X) \xrightarrow{\iota_\bullet \omega} \left(\Omega^0(X) \cdots d\Omega^{n-1}\right)$$

Now that we have an L_∞-morphism we can take its homotopy fiber

$$\begin{tikzcd} & 0 \arrow[d] \\ \mathfrak{X}_{Ham}(X) \arrow[r, "\iota_\bullet\omega"] & \left(\Omega^0(X) \cdots d\Omega^{n-1})\right) \end{tikzcd}$$

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$$\begin{tikzcd}
L_\infty(X,\omega) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & 0 \arrow[d] \\
\mathfrak{X}_{Ham}(X) \arrow[r, "\iota_\bullet\omega"] & \left(\Omega^0(X) \cdots d\Omega^{n-1}(X)\right)
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Finding a “concrete” model for a homotopy fiber is not an easy task, in general.

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Luckily in the above situation one can use the following recognition principle

Let $\mathfrak{g}$ be an L_∞-algebra, A be a dgla and $f_\infty : \mathfrak{g} \to A$ an L_∞ morphism. Let $p_A : B \to A$ be a fibrant replacement of the zero morphism $0 \to A$ in the category of dglas. If

$$\begin{tikzcd} (\mathfrak{g} \times_A B, Q) \arrow[r, "\pi_{B,\infty}"] \arrow[d, "\pi_{\mathfrak{g}}"'] & B \arrow[d, "p_A"] \\ \mathfrak{g} \arrow[r, "f_\infty"] & A \end{tikzcd}$$

is a commutative diagram of L_∞-algebras whose linear part is a pullback diagram of chain complexes, then $(\mathfrak{g} \times_A B, Q)$ is a model for the homotopy fiber of f_∞.

An explicit model for $L_\infty(X,\omega)$ is as follows:

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The underlying chain complex is

$$\Omega^0(X) \xrightarrow{d} \Omega^1(X) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{n-2}(X) \xrightarrow{(0,d)} \mathrm{Ham}^{n-1}(X)$$

with $Ham^{n-1}(X)$ in degree zero;

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with $Ham^{n-1}(X)$ in degree zero; the bilinear bracket is

$$[x_1, x_2] = \begin{cases} [v_1, v_2] + \iota_{v_1 \wedge v_2}\omega & \text{if } \deg x_1, x_2 = 0, \\ 0 & \text{otherwise,} \end{cases}$$

where $x_i = v_i \oplus \eta_i$ if $\deg x_i = 0$;

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where $x_i = v_i \oplus \eta_i$ if $\deg x_i = 0$; and, for $k > 2$:

$$[x_1, \ldots, x_k]_k = \begin{cases} \pm\iota_{v_1 \wedge \cdots \wedge v_k}\omega & \text{if } \deg x_1, \ldots, x_k = 0, \\ 0 & \text{otherwise,} \end{cases}$$

$$\begin{tikzcd}
L_\infty(X,\omega) \arrow[rr] \arrow[d] \arrow[drr, phantom, "\lrcorner", very near start] & & 0 \arrow[d] \\
\mathfrak{X}_{Ham}(X) \arrow[r, "\iota_\bullet\omega"] & \big(\Omega^0(X) \cdots & d\Omega^{n-1}(X)\big)
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$$

If X is n-connected and $x \in X$ then

$ev_x : \left(\Omega^0(X)\cdots d\Omega^{n-1}(X)\right) \to \mathbb{R}[n]$ is a quasi-isomorphism

$$
\begin{tikzcd}
L_\infty(X,\omega) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & 0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & 0 \arrow[d] \\
\mathfrak{X}_{Ham}(X) \arrow[r, "\iota_\bullet \omega"] & \left(\Omega^0(X) \cdots d\Omega^{n-1}(X)\right) \arrow[r, "ev_x", "\sim"'] & \mathbb{R}[n]
\end{tikzcd}
$$

If X is $(n-1)$-connected and $x \in X$ then

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\begin{tikzcd}
L_\infty(X,\omega) \arrow[rr] \arrow[d] \arrow[drr, phantom, very near start, "\lrcorner"] & & 0 \arrow[d] \\
\mathfrak{X}_{Ham}(X) \arrow[rr, "ev_X\, \iota_\bullet \omega"] & & \mathbb{R}[n]
\end{tikzcd}
$$

By the pasting law for homotopy pullbacks, if X is $(n-1)$-connected, $L_\infty(X,\omega)$ is presented as an abelian extension of $\mathfrak{X}_{Ham}(X)$ by a Kostant-Souriau-type cocycle.

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$$L_\infty(X, \omega) = Ham^0(X)$$

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One recovers that $C^\infty(X; \mathbb{R}), \{,\}$ is an abelian extension of $\mathfrak{X}_{Ham}(X)$ by the Kostant-Souriau cocycle.

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 & L_\infty(X,\omega) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & 0 \arrow[d] \\
\mathfrak{g} \arrow[r, "\rho"] & \mathfrak{X}_{Ham}(X) \arrow[r, "ev_x\, \iota_\bullet \omega"] & \mathbb{R}[n]
\end{tikzcd}
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$$\begin{tikzcd}
Heis^{\rho}(\mathfrak{g}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & L_\infty(X,\omega) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & 0 \arrow[d] \\
\mathfrak{g} \arrow[r, "\rho"] & \mathfrak{X}_{Ham}(X) \arrow[r, "ev_x\, \iota_\bullet \omega"] & \mathbb{R}[n]
\end{tikzcd}$$

Assume now $\mathfrak{g}$ is a Lie algebra acting on a $(n-1)$-connected (X, ω) via Hamiltonian vector fields

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\mathfrak{g} \arrow[r, "ev_x\, \iota_\bullet \omega\, \rho"] & \mathbb{R}[n]
\end{tikzcd}
$$

Example. (V, ω) symplectic vector space; $\mathfrak{g} = V$ acting by translations; $x = 0$

$$\begin{tikzcd}
Heis(V,\omega) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & 0 \arrow[d] \\
V \arrow[r, "\omega"] & \mathbb{R}[1]
\end{tikzcd}$$

is the classical Heisenberg algebra of (V, ω).

Example. (G, ω_G) compact simple simply connected Lie group; ω_G standard closed 3-form; $\mathfrak{g}$ the Lie algebra of G; $x = e$

$$\begin{tikzcd} \mathfrak{string}(\mathfrak{g}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & 0 \arrow[d] \\ \mathfrak{g} \arrow[r, "{\langle [-,-],-\rangle}"] & \mathbb{R}[2] \end{tikzcd}$$

is the classical string Lie 2-algebra of $\mathfrak{g}$.

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This exists if and only if ω represents an integral cohomology class.

A geometric prequantization of a pre-n-plectic manifold (X, ω) is the choice of a $U(1)$-n-bundle with connection whose curvature $n+1$-form is ω

$$\begin{tikzcd} & \mathbf{B}^n U(1)_{conn} \arrow[d, "F"] \\ X \arrow[ur, "\nabla"] \arrow[r, "\omega"] & \Omega^{n+1}_{cl} \end{tikzcd}$$

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and its infinitesimal version, the Lie n-algebra

$$Lie(Aut(X)/_{\mathbf{B}^n U(1)_{conn}})$$

Theorem. If (X, ω) admits a prequantization, then the L_∞-algebra $L_\infty(X, \omega)$ is a model for $Lie(Aut(X)/_{\mathbf{B}^n U(1)_{conn}})$

Theorem. If (X, ω) admits a prequantization, then the L_∞-algebra $L_\infty(X, \omega)$ is a model for $Lie(Aut(X)/_{\mathbf{B}^n U(1)_{conn}})$, i.e., there is a quasi-isomorphism of L_∞-algebras

$$L_\infty(X, \omega) \cong Lie(Aut(X)/_{\mathbf{B}^n U(1)_{conn}})$$

Concluding remark. One can forget the top layer or all of the connection and look at a prequantized X as an object over $\mathbf{B}(\mathbf{B}^{n-1}U(1)_{conn})$ or over $\mathbf{B}^n U(1)$ instead.

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Concluding remark. One can forget the top layer or all of the connection and look at a prequantized X as an object over $\mathbf{B}(\mathbf{B}^{n-1}U(1)_{conn})$ or over $\mathbf{B}^n U(1)$ instead.

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For $n = 1$ one obtains the Lie algebra of sections of the Atiyah algebroid of a principal $U(1)$-bundle.

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This leads to the Courant and to the Atiyah Lie n-algebras

For $n = 1$ one obtains the Lie algebra of sections of the Atiyah algebroid of a principal $U(1)$-bundle.

For $n = 2$ one obtains the Lie 2-algebra of sections of the Courant Lie 2-algebroid of a $U(1)$-gerbe with connective structure (Collier).