Higher geometric prequantum theory II L_{∞} -Algebras of local observables

Domenico Fiorenza

Sapienza Università di Roma

May 27, 2013

Based on d.f., Chris Rogers and Urs Schreiber L_{∞} -algebras of local observables from higher prequantum bundles, arXiv:1304.6292

イロト イポト イヨト イヨト

From symplectic to pre-n-plectic

Hamiltonian vector fields The L_{∞} -morphism Heisenberg-type algebras Geometric prequantization

$$\begin{array}{c} (X,\omega) \\ \text{symplectic manifold} \end{array} \xrightarrow{} \begin{array}{c} C^{\infty}(X;\mathbb{R}), \ \{,\} \\ \downarrow \\ \\ \mathfrak{X}_{\mathrm{Ham}}(X) \end{array}$$

◆□ > ◆□ > ◆臣 > ◆臣 > 善臣 の < @

From symplectic to pre-n-plectic

Hamiltonian vector fields The L_{∞} -morphism Heisenberg-type algebras Geometric prequantization

$$\begin{array}{c} (X,\omega) \\ \text{pre-}n\text{-plectic manifold} \end{array} \xrightarrow{L_{\infty}(X;\omega)} \\ & \swarrow \\ \mathfrak{X}_{\text{Ham}}(X) \end{array}$$

◆□ > ◆□ > ◆臣 > ◆臣 > 善臣 の < @

From symplectic to pre-*n*-plectic Hamiltonian vector fields The L_{∞} -morphism

The L_{∞} -morphism Heisenberg-type algebras Geometric prequantization

What is a pre-*n*-plectic manifold?

Domenico Fiorenza L_{∞} -Algebras of local observables - Bayrischzell 2013

イロン イヨン イヨン イヨン

What is a pre-n-plectic manifold?

A symplectic manifold is a pair (X, ω) with X a smooth manifold and ω a degree 2, nondegenerate closed form on X

イロト イポト イヨト イヨト

What is a pre-n-plectic manifold?

A symplectic manifold is a pair (X, ω) with X a smooth manifold and ω a degree 2, nondegenerate closed form on X

Nondegenerate means that at any $x \in X$ the contraction of tangent vectors against the differential form, $\iota \omega : T_x X \to T_x^* X$ is an isomorphism.

What is a pre-n-plectic manifold?

A symplectic manifold is a pair (X, ω) with X a smooth manifold and ω a degree 2, nondegenerate closed form on X

Nondegenerate means that at any $x \in X$ the contraction of tangent vectors against the differential form, $\iota \omega : T_x X \to T_x^* X$ is injective.

・ロト ・回ト ・ヨト

What is a pre-n-plectic manifold?

An *n*-plectic manifold is a pair (X, ω) with X a smooth manifold and ω a degree n+1, nondegenerate closed form on X

Nondegenerate means that at any $x \in X$ the contraction of tangent vectors against the differential form, $\iota \omega : T_x X \to \wedge^n T_x^* X$ is injective.

What is a pre-n-plectic manifold?

An *n*-plectic manifold is a pair (X, ω) with X a smooth manifold and ω a degree n+1, nondegenerate closed form on X

Nondegenerate means that at any $x \in X$ the contraction of tangent vectors against the differential form, $\iota \omega : T_x X \to \wedge^n T_x^* X$ is injective.

Example. G compact simple simply connected Lie group, ω_G canonical 3-form on G.

What is a pre-*n*-plectic manifold?

A symplectic manifold is a pair (X, ω) with X a smooth manifold and ω a degree 2 nondegenerate closed form on X

Nondegenerate means that at any $x \in X$ the contraction of tangent vectors against the differential form, $\iota \omega : T_x X \to T_x^* X$ is injective.

What is a pre-n-plectic manifold?

A presymplectic manifold is a pair (X, ω) with X a smooth manifold and ω a degree 2 closed form on X

イロト イヨト イヨト イヨト

What is a pre-n-plectic manifold?

A pre-*n*-plectic manifold is a pair (X, ω) with X a smooth manifold and ω a degree n+1 closed form on X

<ロ> (日) (日) (日) (日) (日)

The vector space of Hamiltonian pairs on a symplectic manifold (X, ω) is

$$Ham^0(X) = \{(v, H) \in \mathfrak{X}(X) \oplus \Omega^0(X) : \iota_v \omega + dH = 0\}$$

・ロン ・回と ・ヨン ・ヨン

The vector space of Hamiltonian pairs on a symplectic manifold (X, ω) is

$$Ham^0(X) = \{(v, H) \in \mathfrak{X}(X) \oplus \Omega^0(X) : \iota_v \omega + dH = 0\}$$

$$Ham^{0}(X) \longrightarrow \Omega^{0}(X)$$

$$\downarrow$$

$$\mathfrak{X}(X)$$

・ロン ・回 と ・ ヨ と ・ ヨ と

The vector space of Hamiltonian pairs on a symplectic manifold (X, ω) is

$$\begin{aligned} \mathsf{Ham}^0(X) &= \{(v, H) \in \mathfrak{X}(X) \oplus \Omega^0(X) \ : \ \iota_v \omega + dH = 0\} \\ \\ \mathsf{Ham}^0(X) \xrightarrow{\sim} \Omega^0(X) \\ & \downarrow \\ \mathfrak{X}(X) \end{aligned}$$

イロン イヨン イヨン イヨン

The vector space of Hamiltonian pairs on a symplectic manifold (X, ω) is

$$Ham^0(X) = \{(v, H) \in \mathfrak{X}(X) \oplus \Omega^0(X) : \iota_v \omega + dH = 0\}$$

$$Ham^{0}(X) \xrightarrow{\sim} \Omega^{0}(X)$$

$$\downarrow \\ \mathfrak{X}_{Ham}(X)$$

<ロ> (四) (四) (三) (三) (三)

The vector space of Hamiltonian pairs on a pre-*n*-plectic manifold (X, ω) is

$$\mathit{Ham}^{n-1}(X) = \{(v,H) \in \mathfrak{X}(X) \oplus \Omega^{n-1}(X) \, : \, \iota_v \omega + dH = \mathsf{0}\}$$

・ロン ・回と ・ヨン ・ヨン

æ

Hamiltonian vector fields preserve the closed form ω , i.e. $\mathcal{L}_{v}\omega = 0$

イロン イヨン イヨン イヨン

Hamiltonian vector fields preserve the closed form ω , i.e. $\mathcal{L}_v \omega = 0$

$$\mathcal{L}_{\mathbf{v}}\omega = [i_{\mathbf{v}}, d]\omega = i_{\mathbf{v}}d\omega + di_{\mathbf{v}}\omega$$

イロン イヨン イヨン イヨン

Hamiltonian vector fields preserve the closed form ω , i.e. $\mathcal{L}_{v}\omega = 0$

$$\mathcal{L}_{\mathbf{v}}\omega = [\mathbf{i}_{\mathbf{v}}, \mathbf{d}]\omega = \mathbf{d}\mathbf{i}_{\mathbf{v}}\omega$$

イロト イロト イヨト イヨト 二日

Hamiltonian vector fields preserve the closed form ω , i.e. $\mathcal{L}_v \omega = 0$

$$\mathcal{L}_{\mathbf{v}}\omega = [i_{\mathbf{v}}, d]\omega = -d^2H$$

イロン イヨン イヨン イヨン

Hamiltonian vector fields preserve the closed form ω , i.e. $\mathcal{L}_v \omega = 0$

$$\mathcal{L}_{v}\omega = [i_{v}, d]\omega = -d^{2}H = 0$$

イロン イヨン イヨン イヨン

Hamiltonian vector fields are a Lie subalgebra of the Lie algebra of vector fields

・ロン ・回 と ・ヨン ・ヨン

Hamiltonian vector fields are a Lie subalgebra of the Lie algebra of vector fields

$$\iota_{[\mathbf{v},\mathbf{w}]}\omega = ?$$

・ロン ・回 と ・ヨン ・ヨン

Hamiltonian vector fields are a Lie subalgebra of the Lie algebra of vector fields

$$\iota_{[\mathbf{v},\mathbf{w}]}\omega = [\mathcal{L}_{\mathbf{v}},\iota_{\mathbf{w}}]\omega$$

・ロン ・回と ・ヨン ・ヨン

Hamiltonian vector fields are a Lie subalgebra of the Lie algebra of vector fields

$$\iota_{[\mathbf{v},\mathbf{w}]}\omega = [\mathcal{L}_{\mathbf{v}},\iota_{\mathbf{w}}]\omega = \mathcal{L}_{\mathbf{v}}\iota_{\mathbf{w}}\omega - \iota_{\mathbf{w}}\mathcal{L}_{\mathbf{v}}\omega$$

・ロン ・回と ・ヨン ・ヨン

Hamiltonian vector fields are a Lie subalgebra of the Lie algebra of vector fields

$$\iota_{[\mathbf{v},\mathbf{w}]}\omega = [\mathcal{L}_{\mathbf{v}},\iota_{\mathbf{w}}]\omega = \mathcal{L}_{\mathbf{v}}\iota_{\mathbf{w}}\omega$$

・ロン ・回と ・ヨン ・ヨン

Hamiltonian vector fields are a Lie subalgebra of the Lie algebra of vector fields

$$\iota_{[\mathbf{v},\mathbf{w}]}\omega = [\mathcal{L}_{\mathbf{v}},\iota_{\mathbf{w}}]\omega = \mathcal{L}_{\mathbf{v}}\iota_{\mathbf{w}}\omega = -d\iota_{\mathbf{v}\wedge\mathbf{w}}\omega$$

・ロン ・回と ・ヨン ・ヨン

Hamiltonian vector fields are a Lie subalgebra of the Lie algebra of vector fields

 $\iota_{[\mathbf{v},\mathbf{w}]}\omega + d\iota_{\mathbf{v}\wedge\mathbf{w}}\omega = 0$

・ロン ・回と ・ヨン ・ヨン

Hamiltonian vector fields are a Lie subalgebra of the Lie algebra of vector fields

$$\iota_{[\mathbf{v},\mathbf{w}]}\omega + d\iota_{\mathbf{v}\wedge\mathbf{w}}\omega = 0$$

The vector field [v, w] is Hamiltonian with Hamiltonian n - 1-form $\iota_{v \wedge w} \omega$.

・ロン ・回 と ・ ヨ と ・ ヨ と

$\iota\omega:\mathfrak{X}(X)\to\Omega^n(X)$

< □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □

$\iota \omega : \mathfrak{X}_{Ham}(X) \to \Omega^n(X)$

< □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □

$\iota \omega : \mathfrak{X}_{Ham}(X) \to d\Omega^{n-1}(X)$

< □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □

$$\iota \omega : \mathfrak{X}_{Ham}(X) \to d\Omega^{n-1}(X)$$

On the left we have a Lie algebra, on the right a vector space

・ロト ・回 ト ・ヨト ・ヨト

æ

 $\iota \omega : \mathfrak{X}_{Ham}(X) \to d\Omega^{n-1}(X)$

On the left we have a vector space, on the right a vector space

イロト イヨト イヨト イヨト

æ

 $\iota \omega : \mathfrak{X}_{Ham}(X) \to d\Omega^{n-1}(X)$

On the left we have a vector space, on the right a vector space

 $\iota\omega$ is a morphism of vector spaces

イロト イヨト イヨト イヨト

$$\iota\omega:\mathfrak{X}_{Ham}(X)\to d\Omega^{n-1}(X)$$

On the left we have a Lie algebra, on the right an abelian Lie algebra

・ロト ・回ト ・ヨト ・ヨト

æ

$$\iota\omega:\mathfrak{X}_{Ham}(X)\to d\Omega^{n-1}(X)$$

On the left we have a Lie algebra, on the right an abelian Lie algebra

Is $\iota \omega$ a morphism of Lie algebras?

イロン イヨン イヨン イヨン

æ

$$\iota_{[\mathbf{v},\mathbf{w}]}\omega \stackrel{?}{=} [\iota_{\mathbf{v}}\omega,\iota_{\mathbf{w}}\omega]$$

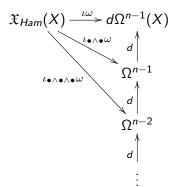
$$\iota_{[v,w]}\omega \stackrel{?}{=} \underbrace{[\iota_v\omega,\iota_w\omega]}_{0}$$

$$\iota_{[\nu,w]}\omega = \underbrace{[\iota_{\nu}\omega,\iota_{w}\omega]}_{0} + d\iota_{\nu\wedge w}\omega$$

 $\mathfrak{X}_{Ham}(X) \xrightarrow{\iota \omega} d\Omega^{n-1}(X)$

◆□ > ◆□ > ◆臣 > ◆臣 > ─ 臣 ─ のへで

$$\mathfrak{X}_{Ham}(X) \xrightarrow{\iota \omega} d\Omega^{n-1}(X)$$



・ロン ・雪 ・ ・ ヨ ・ ・ ヨ ・ ・

æ

Now on the left we have a Lie algebra (a dgla concentrated in degree zero) and on the right we have a chain complex (an abelian dgla)

イロト イヨト イヨト イヨト

Now on the left we have a Lie algebra (a dgla concentrated in degree zero) and on the right we have a chain complex (an abelian dgla)

So we can ask ourselves: is $\iota_{\bullet}\omega$ an L_{∞} -morphism?

・ロト ・回ト ・ヨト

Now on the left we have a Lie algebra (a dgla concentrated in degree zero) and on the right we have a chain complex (an abelian dgla)

So we can ask ourselves: is $\iota_{\bullet}\omega$ an L_{∞} -morphism?

An L_{∞} -morphism between a Lie algebra \mathfrak{g} and a chain complex \mathfrak{h} is a sequence of multilinear maps $\varphi_k : \wedge^k \mathfrak{g} \to \mathfrak{h}[1-k]$ such that

$$d_{\mathfrak{h}}\varphi_{k}(v_{1}\wedge\cdots\wedge v_{k})=\sum_{i< j}\pm\varphi_{k-1}([v_{i},v_{j}]_{\mathfrak{g}}\wedge v_{1}\wedge\cdots\wedge\widehat{v_{i}}\wedge\cdots\wedge\widehat{v_{j}}\wedge\cdots\wedge v_{k}).$$

イロト イポト イヨト イヨト

$$d\iota_{v_1\wedge\cdots\wedge v_k}\omega \stackrel{?}{=} \sum_{i< j} \pm \iota_{[v_i,v_j]_{\mathfrak{g}}\wedge v_1\wedge\cdots\wedge \widehat{v_i}\wedge\cdots\wedge \widehat{v_j}\wedge\cdots\wedge v_k}\omega$$

$$d\iota_{\mathbf{v}_1\wedge\cdots\wedge\mathbf{v}_k}\omega \stackrel{?}{=} \sum_{i< j} \pm \iota_{[\mathbf{v}_i,\mathbf{v}_j]_{\mathfrak{g}}\wedge\mathbf{v}_1\wedge\cdots\wedge\widehat{\mathbf{v}_i}\wedge\cdots\wedge\widehat{\mathbf{v}_j}\wedge\cdots\wedge\mathbf{v}_k}\omega$$

For $\omega \in \Omega^{n+1}(X)$ and v_1, \ldots, v_k in $\mathfrak{X}(X)$ we have

$$d\iota_{\mathbf{v}_{1}\wedge\cdots\wedge\mathbf{v}_{k}}\omega = \sum_{i< j} \pm \iota_{[\mathbf{v}_{i},\mathbf{v}_{j}]_{\mathfrak{g}}\wedge\mathbf{v}_{1}\wedge\cdots\wedge\widehat{\mathbf{v}_{i}}\wedge\cdots\wedge\widehat{\mathbf{v}_{j}}\wedge\cdots\wedge\mathbf{v}_{k}}\omega$$
$$+ \sum_{i} \pm \iota_{\mathbf{v}_{1}\wedge\cdots\wedge\widehat{\mathbf{v}_{i}}\wedge\cdots\wedge\mathbf{v}_{k}}\mathcal{L}_{\mathbf{v}_{i}}\omega$$
$$\pm \iota_{\mathbf{v}_{1}\wedge\cdots\wedge\mathbf{v}_{k}}d\omega$$

イロン イヨン イヨン イヨン

$$d\iota_{v_1\wedge\cdots\wedge v_k}\omega \stackrel{?}{=} \sum_{i< j} \pm \iota_{[v_i,v_j]_{\mathfrak{g}}\wedge v_1\wedge\cdots\wedge \widehat{v_i}\wedge\cdots\wedge \widehat{v_j}\wedge\cdots\wedge v_k}\omega$$

For $\omega \in \Omega^{n+1}_{cl}(X)$ and v_1, \ldots, v_k in $\mathfrak{X}(X)$ we have

$$d\iota_{\mathbf{v}_{1}\wedge\cdots\wedge\mathbf{v}_{k}}\omega=\sum_{i< j}\pm\iota_{[\mathbf{v}_{i},\mathbf{v}_{j}]_{\mathfrak{g}}\wedge\mathbf{v}_{1}\wedge\cdots\wedge\widehat{\mathbf{v}_{i}}\wedge\cdots\wedge\widehat{\mathbf{v}_{j}}\wedge\cdots\wedge\mathbf{v}_{k}}\omega$$
$$+\sum_{i}\pm\iota_{\mathbf{v}_{1}\wedge\cdots\wedge\widehat{\mathbf{v}_{i}}\wedge\cdots\wedge\mathbf{v}_{k}}\mathcal{L}_{\mathbf{v}_{i}}\omega$$

イロン イボン イヨン イヨン 三日

$$d\iota_{v_1\wedge\cdots\wedge v_k}\omega \stackrel{?}{=} \sum_{i< j} \pm \iota_{[v_i,v_j]_{\mathfrak{g}}\wedge v_1\wedge\cdots\wedge \widehat{v_i}\wedge\cdots\wedge \widehat{v_j}\wedge\cdots\wedge v_k}\omega$$

For $\omega \in \Omega^{n+1}_{cl}(X)$ and v_1, \ldots, v_k in $\mathfrak{X}_{Ham}(X)$ we have

$$d\iota_{\mathbf{v}_1\wedge\cdots\wedge\mathbf{v}_k}\omega=\sum_{i< j}\pm\iota_{[\mathbf{v}_i,\mathbf{v}_j]_{\mathfrak{g}}\wedge\mathbf{v}_1\wedge\cdots\wedge\widehat{\mathbf{v}_i}\wedge\cdots\wedge\widehat{\mathbf{v}_j}\wedge\cdots\wedge\mathbf{v}_k}\omega$$

・ロン ・回 と ・ ヨ と ・ ヨ と

Now that we have an L_{∞} -morphism...

$$\mathfrak{X}_{Ham}(X) \xrightarrow{\iota_{\bullet}\omega} \left(\Omega^{0}(X) \cdots d\Omega^{n-1}\right)$$

<ロ> (四) (四) (三) (三) (三)

Now that we have an L_{∞} -morphism we can take its homotopy fiber

$$\begin{array}{c} 0 \\ \downarrow \\ \mathfrak{X}_{Ham}(X) \xrightarrow{\iota_{\bullet}\omega} \left(\Omega^{0}(X) \cdots d\Omega^{n-1} \right) \right) \end{array}$$

・ロン ・回 と ・ヨン ・ヨン

Now that we have an L_{∞} -morphism we can take its homotopy fiber

・ロン ・回 と ・ヨン ・ヨン

Finding a "concrete" model for a homotopy fiber is not an easy task, in general.

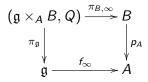
イロン イヨン イヨン イヨン

Finding a "concrete" model for a homotopy fiber is not an easy task, in general.

Luckily in the above situation one can use the following recognition principle

イロト イヨト イヨト イヨト

Let \mathfrak{g} be an L_{∞} -algebra, A be a dgla and $f_{\infty} : \mathfrak{g} \to A$ an L_{∞} morphism. Let $p_A : B \to A$ be a fibrant replacement of the zero morphism $0 \to A$ in the category of dglas. If



is a commutative diagram of L_{∞} -algebras whose linear part is a pullback diagram of chain complexes, then $(\mathfrak{g} \times_A B, Q)$ is a model for the homotopy fiber of f_{∞} .

イロト イヨト イヨト イヨト

An explicit model for $L_{\infty}(X,\omega)$ is as follows:

・ロト ・回ト ・ヨト ・ヨト

æ

An explicit model for $L_{\infty}(X,\omega)$ is as follows:

The underlying chain complex is

$$\Omega^{0}(X) \xrightarrow{d} \Omega^{1}(X) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{n-2}(X) \xrightarrow{(0,d)} \operatorname{Ham}^{n-1}(X)$$

with $Ham^{n-1}(X)$ in degree zero;

イロト イヨト イヨト イヨト

æ

An explicit model for $L_{\infty}(X,\omega)$ is as follows:

The underlying chain complex is

$$\Omega^{0}(X) \xrightarrow{d} \Omega^{1}(X) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{n-2}(X) \xrightarrow{(0,d)} \operatorname{Ham}^{n-1}(X)$$

with $Ham^{n-1}(X)$ in degree zero; the bilinear bracket is

$$[x_1, x_2] = \begin{cases} [v_1, v_2] + \iota_{v_1 \wedge v_2} \omega & \text{if deg } x_1, x_2 = 0, \\ 0 & \text{otherwise}, \end{cases}$$

where $x_i = v_i \oplus \eta_i$ if deg $x_i = 0$;

An explicit model for $L_{\infty}(X,\omega)$ is as follows:

The underlying chain complex is

$$\Omega^0(X) \xrightarrow{d} \Omega^1(X) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{n-2}(X) \xrightarrow{(0,d)} \operatorname{Ham}^{n-1}(X)$$

with $Ham^{n-1}(X)$ in degree zero; the bilinear bracket is

$$[x_1, x_2] = \begin{cases} [v_1, v_2] + \iota_{v_1 \wedge v_2} \omega & \text{if deg } x_1, x_2 = 0, \\ 0 & \text{otherwise}, \end{cases}$$

where $x_i = v_i \oplus \eta_i$ if deg $x_i = 0$; and, for k > 2:

$$[x_1, \dots, x_k]_k = \begin{cases} \pm \iota_{v_1 \wedge \dots \wedge v_k} \omega & \text{if deg } x_1, \dots, x_k = 0, \\ 0 & \text{otherwise}, \end{cases}$$

・ロン ・回と ・ヨン・

$$L_{\infty}(X,\omega) \longrightarrow 0$$

$$\downarrow^{j} \qquad \qquad \downarrow^{u}$$

$$\mathfrak{X}_{Ham}(X) \xrightarrow{\iota_{\bullet}\omega} (\Omega^{0}(X) \cdots d\Omega^{n-1}(X))$$

If X is *n*-connected and $x \in X$ then

 $ev_{\scriptscriptstyle X}: \left(\Omega^0(X)\cdots d\Omega^{n-1}(X)
ight) o \mathbb{R}[n]$ is a quasi-isomorphism

・ロト ・回ト ・ヨト ・ヨト

$$L_{\infty}(X,\omega) \xrightarrow{} 0 \xrightarrow{} 0 \xrightarrow{} 0$$
$$\downarrow^{j} \qquad \qquad \downarrow^{j} \qquad \qquad \downarrow^{j} \qquad \qquad \downarrow^{j}$$
$$\mathfrak{X}_{Ham}(X) \xrightarrow{\iota_{\bullet}\omega} (\Omega^{0}(X) \cdots d\Omega^{n-1}(X)) \xrightarrow{ev_{X}} \mathbb{R}[n]$$

If X is (n-1)-connected and $x \in X$ then

 $ev_{x}: \left(\Omega^{0}(X)\cdots d\Omega^{n-1}(X)
ight)
ightarrow \mathbb{R}[n]$ is a quasi-isomorphism



By the pasting law for homotopy pullbacks, if X is (n-1)-connected, $L_{\infty}(X, \omega)$ is presented as an abelian extension of $\mathfrak{X}_{Ham}(X)$ by a Kostant-Souriau-type cocycle.

イロト イヨト イヨト イヨト

Example. (X, ω) connected symplectic manifold.

$$L_\infty(X,\omega) = Ham^0(X)$$

is a Lie algebra.

イロン 不同と 不同と 不同と

æ

Example. (X, ω) connected symplectic manifold.

$$L_{\infty}(X,\omega) = Ham^0(X)$$

is a Lie algebra. By the linear isomorphism $Ham^0(X) \to \Omega^0(X)$ this Lie algebra structure induces a Lie algebra structire on $C^{\infty}(X; \mathbb{R})$.

イロト イポト イヨト イヨト

Example. (X, ω) connected symplectic manifold.

$$L_{\infty}(X,\omega) = Ham^0(X)$$

is a Lie algebra. By the linear isomorphism $Ham^0(X) \to \Omega^0(X)$ this Lie algebra structure induces a Lie algebra structire on $C^{\infty}(X; \mathbb{R})$. This is the usual Poisson bracket on $C^{\infty}(X; \mathbb{R})$.

イロト イポト イヨト イヨト

Example. (X, ω) connected symplectic manifold.

$$L_{\infty}(X,\omega) = Ham^0(X)$$

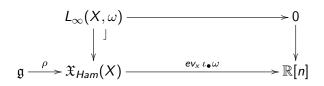
is a Lie algebra. By the linear isomorphism $Ham^0(X) \to \Omega^0(X)$ this Lie algebra structure induces a Lie algebra structire on $C^{\infty}(X; \mathbb{R})$. This is the usual Poisson bracket on $C^{\infty}(X; \mathbb{R})$.

One recovers that $C^{\infty}(X; \mathbb{R}), \{,\}$ is an abelian extension of $\mathfrak{X}_{Ham}(X)$ by the Kostant-Souriau cocycle.

Assume now \mathfrak{g} is a Lie algebra acting on a (n-1)-connected (X, ω) via Hamiltonian vector fields

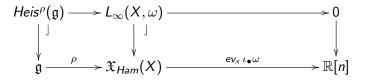
・ロン ・回と ・ヨン ・ヨン

Assume now g is a Lie algebra acting on a (n-1)-connected (X, ω) via Hamiltonian vector fields



イロン イヨン イヨン イヨン

Assume now g is a Lie algebra acting on a (n-1)-connected (X, ω) via Hamiltonian vector fields



イロン イ部ン イヨン イヨン 三日

Assume now g is a Lie algebra acting on a (n-1)-connected (X, ω) via Hamiltonian vector fields



ヘロン 人間 とくほど くほとう

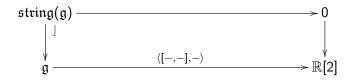
Example. (V, ω) symplectic vector space; $\mathfrak{g} = V$ acting by translations; x = 0



is the classical Heisenberg algebra of (V, ω) .

(ロ) (同) (E) (E) (E)

Example. (G, ω_G) compact simple simply connected Lie group; ω_G standard closed 3-form; \mathfrak{g} the Lie algebra of G; x = e



is the classical string Lie 2-algebra of \mathfrak{g} .

イロト イヨト イヨト イヨト

A geometric prequantization of a presymplectic manifold (X, ω) is the choice of a U(1)-bundle with connection whose curvature 2-form is ω

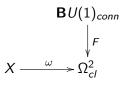
- 4 同 6 4 日 6 4 日 6

A geometric prequantization of a presymplectic manifold (X, ω) is the choice of a U(1)-bundle with connection whose curvature 2-form is ω

$$X \xrightarrow{\omega} \Omega_{cl}^2$$

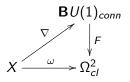
- 4 回 2 4 三 2 4 三 2 4

A geometric prequantization of a presymplectic manifold (X, ω) is the choice of a U(1)-bundle with connection whose curvature 2-form is ω



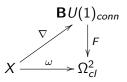
イロン イヨン イヨン イヨン

A geometric prequantization of a presymplectic manifold (X, ω) is the choice of a U(1)-bundle with connection whose curvature 2-form is ω



- 4 同 6 4 日 6 4 日 6

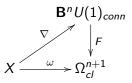
A geometric prequantization of a presymplectic manifold (X, ω) is the choice of a U(1)-bundle with connection whose curvature 2-form is ω



This exists if and only if ω represents an integral cohomology class.

イロト イポト イヨト イヨト

A geometric prequantization of a pre-*n*-plectic manifold (X, ω) is the choice of a U(1)-*n*-bundle with connection whose curvature n + 1-form is ω



This exists if and only if ω represents an integral cohomology class.

- 4 同 6 4 日 6 4 日 6

A prequantization of (X, ω) realizes X as an object over $\mathbf{B}^n U(1)_{conn}$

イロン イヨン イヨン イヨン

A prequantization of (X, ω) realizes X as an object over $\mathbf{B}^n U(1)_{conn}$

We can consider the smooth *n*-group

 $Aut(X)/_{\mathbf{B}^n U(1)_{conn}}$

・ロン ・回と ・ヨン ・ヨン

æ

A prequantization of (X, ω) realizes X as an object over $\mathbf{B}^n U(1)_{conn}$

We can consider the smooth *n*-group

 $Aut(X)/_{\mathbf{B}^n U(1)_{conn}}$

and its infinitesimal version, the Lie n-algebra

 $Lie(Aut(X)/_{\mathbf{B}^n U(1)_{conn}})$

・ロト ・回ト ・ヨト ・ヨト

Theorem. If (X, ω) admits a prequantization, then the L_{∞} -algebra $L_{\infty}(X, \omega)$ is a model for $Lie(Aut(X)/_{\mathbf{B}^n U(1)_{conn}})$

・ロン ・回と ・ヨン ・ヨン

Theorem. If (X, ω) admits a prequantization, then the L_{∞} -algebra $L_{\infty}(X, \omega)$ is a model for $Lie(Aut(X)/_{\mathbf{B}^n U(1)_{conn}})$, i.e., there is a quasi-isomorphism of L_{∞} -algebras

$$L_{\infty}(X,\omega) \cong Lie(Aut(X)/_{\mathbf{B}^n U(1)_{conn}})$$

・ロト ・回ト ・ヨト ・ヨト

Concluding remark. One can forget the top layer or all of the connection and look at a prequantized X as an object over $\mathbf{B}(\mathbf{B}^{n-1}U(1)_{conn})$ or over $\mathbf{B}^{n}U(1)$ instead.

ヘロン 人間 とくほと くほとう

Concluding remark. One can forget the top layer or all of the connection and look at a prequantized X as an object over $\mathbf{B}(\mathbf{B}^{n-1}U(1)_{conn})$ or over $\mathbf{B}^{n}U(1)$ instead.

This leads to the Courant and to the Atiyah Lie *n*-algebras

イロト イポト イヨト イヨト

Concluding remark. One can forget the top layer or all of the connection and look at a prequantized X as an object over $\mathbf{B}(\mathbf{B}^{n-1}U(1)_{conn})$ or over $\mathbf{B}^{n}U(1)$ instead.

This leads to the Courant and to the Atiyah Lie *n*-algebras

For n = 1 one obtains the Lie algebra of sections of the Atiyah algebroid of a principal U(1)-bundle.

イロト イポト イヨト イヨト

Concluding remark. One can forget the top layer or all of the connection and look at a prequantized X as an object over $\mathbf{B}(\mathbf{B}^{n-1}U(1)_{conn})$ or over $\mathbf{B}^{n}U(1)$ instead.

This leads to the Courant and to the Atiyah Lie *n*-algebras

For n = 1 one obtains the Lie algebra of sections of the Atiyah algebroid of a principal U(1)-bundle.

For n = 2 one obtains the Lie 2-algebra of sections of the Courant Lie 2-algebroid of a U(1)-gerbe with connective structure (Collier).

(ロ) (同) (E) (E) (E)