

QUANTIZATION OF NON-GEOMETRIC FLUX BACKGROUNDS

Richard Szabo

Heriot-Watt University, Edinburgh
Maxwell Institute for Mathematical Sciences

with Dionysios Mylonas and Peter Schupp

Quantization, Topological Models and Generalized Geometries
Bayrischzell 2013

String geometry

- ▶ Strings see geometry in different ways than particles do:

T-duality, non-geometric backgrounds, ...

- ▶ Probes of Planck scale quantum geometry:

Spacetime uncertainty $\Delta x \geq \ell_P$ related to noncommutative spacetime structure?

- ▶ Strings and noncommutative geometry:

- ▶ D-branes in B -fields provide realisations of noncommutative spaces
- ▶ Low-energy dynamics described by noncommutative gauge theory, Seiberg–Witten maps, ...

(Chu & Ho '98; Schomerus '99; Seiberg & Witten '99; ...)

Noncommutative string geometry

► Open string noncommutative gauge theory:

- 2-point function on boundary of disk: ordering
- 2-tensor (B -field) deforms to noncommutative 2-bracket
- Quantization produces star-products of fields



► Closed string nonassociative gravity

(Lüst '10; Blumenhagen & Plauschinn '11; Blumenhagen *et al.* '11)

- 3-point function on sphere: orientation
- 3-tensor (R -flux) deforms to nonassociative 3-bracket
- Quantization (Dynamical star-products, categorified Weyl quantization, quasi-Hopf cochain quantization)
- Seiberg–Witten maps



Nonassociativity and gravity in string theory

- ▶ Nonassociative gauge theory:

Open string nonassociative spaces with 3-form $H = dB \neq 0$

(Cornalba & Schiappa '02; Ho '01; Herbst, Kling & Kreuzer '01)

Described locally by associative algebras and noncommutative gerbes (Aschieri *et al.* '10)

- ▶ Noncommutative gravity:

Differential geometry/general relativity on noncommutative spacetime (Aschieri *et al.* '05)

Twisted diffeomorphisms do not arise as physical symmetries of string theory (Álvarez-Gaumé, Meyer & Vázquez-Mozo '06)

Non-geometric flux compactification

T^3 with H -flux gives geometric and non-geometric fluxes via T-duality

(Hull '05; Shelton, Taylor & Wecht '05)

$$H_{abc} \xrightarrow{T_a} f^a{}_{bc} \xrightarrow{T_b} Q^{ab}{}_c \xrightarrow{T_c} R^{abc}$$

► **H -flux:** $H = dB$, gerbes

► **Metric flux:** $de^a = -\frac{1}{2} f^a{}_{bc} e^b \wedge e^c$, twisted torus

► **Q -flux:** T-folds: stringy transition functions

Fibration $T^3 \xrightarrow{T_\theta^2} S^1$ of noncommutative 2-tori

$$[x^i, x^j] = Q^{ij}{}_k x^k \quad (\text{Mathai \& Rosenberg '04; Grange \& Schäfer-Nameki '07})$$

► **R -flux:** Not even locally geometric

(Topological) nonassociative 3-tori $[x^i, x^j, x^k] = R^{ijk}$

(Bouwknegt, Hannabuss & Mathai '06; Ellwood & Hashimoto '06)

Geometrisation of non-geometry

- ▶ **Generalised geometry:** Non-geometric fluxes through T-duality transforms of sections of $C = TM \oplus T^*M$
(Grange & Schäfer-Nameki '07; Graña *et al.* '08; Halmagyi '09)
- ▶ **Doubled geometry/field theory:** $x^i \longrightarrow (x^i, \tilde{x}_i)$, $\partial_i \longrightarrow (\partial_i, \tilde{\partial}^i)$;
implement projection $\tilde{\partial}^i = 0$ (Andriot *et al.* '12)
- ▶ **Membrane σ -model:** Open strings do not decouple from gravity in R -space (Ellwood & Hashimoto '06)

Target space doubling $M \longrightarrow T^*M$, geometric 3-form R -flux

Membrane $\implies$ closed strings with non-geometric flux

+ open string R -twisted Poisson σ -model whose perturbative quantization gives nonassociative dynamical star-product of fields, Jacobiator quantizes 3-bracket (Mylonas, Schupp & RS '12)

Poisson σ -model

- **AKSZ σ -models:** Action functionals in BV formalism for σ -models with target space a symplectic Lie n -algebroid $E \longrightarrow M$

(Alexandrov *et al.* '95)

- **Poisson σ -model:** $n = 1$, $E = T^*M$

Most general 2D TFT from AKSZ construction

$$S_{\text{AKSZ}}^{(1)} = \int_{\Sigma_2} \left(\xi_i \wedge dX^i + \frac{1}{2} \Theta^{ij}(X) \xi_i \wedge \xi_j \right)$$

$$X : \Sigma_2 \longrightarrow M, \quad \xi \in \Omega^1(\Sigma_2, X^* T^* M), \quad \Theta = \frac{1}{2} \Theta^{ij}(x) \partial_i \wedge \partial_j$$

- Perturbative expansion $\implies$ Kontsevich formality maps

(Kontsevich '03; Cattaneo & Felder '00)

On-shell: $[\Theta, \Theta]_S = 0$ Poisson structure

Courant σ -model

- ▶ $n = 2$, $E =$ Courant algebroid with
 fibre metric $h_{IJ} = \langle \psi_I, \psi_J \rangle$, anchor matrix $\rho(\psi_I) = P_I^i(x) \partial_i$,
 3-form $T_{IJK}(x) = [\psi_I, \psi_J, \psi_K]_E$

- ▶ Correspond to TFT on 3D membrane worldvolume Σ_3
 (Hofman & Park '02; Ikeda '03; Roytenberg '07):

$$S_{\text{AKSZ}}^{(2)} = \int_{\Sigma_3} \left(\phi_i \wedge dX^i + \frac{1}{2} h_{IJ} \alpha^I \wedge d\alpha^J - P_I^i(X) \phi_i \wedge \alpha^I \right. \\ \left. + \frac{1}{6} T_{IJK}(X) \alpha^I \wedge \alpha^J \wedge \alpha^K \right)$$

$$X : \Sigma_3 \longrightarrow M, \quad \alpha \in \Omega^1(\Sigma_3, X^*E), \quad \phi \in \Omega^2(\Sigma_3, X^*T^*M)$$

- ▶ Standard Courant algebroid: $E = C = TM \oplus T^*M$ with natural
 frame $(\psi_I) = (\partial_i, dx^i)$, dual pairing $\langle \partial_i, dx^j \rangle = \delta_i^j$

σ-model for geometric fluxes

- ▶ C twisted by H -flux $H = \frac{1}{6} H_{ijk}(x) dx^i \wedge dx^j \wedge dx^k$ with H -twisted Courant–Dorfman bracket:

$$[(Y_1, \alpha_1), (Y_2, \alpha_2)]_H := ([Y_1, Y_2]_{TM}, \mathcal{L}_{Y_1} \alpha_2 - \mathcal{L}_{Y_2} \alpha_1 - \frac{1}{2} d(\alpha_2(Y_1) - \alpha_1(Y_2)) + H(Y_1, Y_2, -))$$

and projection $\rho : C \longrightarrow TM$

- ▶ Brackets: $[\partial_i, \partial_j]_H = H_{ijk} dx^k$, $[\partial_i, \partial_j, \partial_k]_H = H_{ijk}$
- ▶ Topological membrane with $(\alpha') = (\alpha^1, \dots, \alpha^d, \xi_1, \dots, \xi_d)$

(Park '01):

$$S_{\text{AKSZ}}^{(2)} = \int_{\Sigma_3} \left(\phi_i \wedge dX^i + \alpha^i \wedge d\xi_i - \phi_i \wedge \alpha^i + \frac{1}{6} H_{ijk}(X) \alpha^i \wedge \alpha^j \wedge \alpha^k \right)$$

H -twisted Poisson σ -model

- ▶ When $\Sigma_2 := \partial\Sigma_3 \neq \emptyset$, add boundary term $\int_{\Sigma_2} \frac{1}{2} \Theta^{ij}(X) \xi_i \wedge \xi_j$ to get boundary/bulk open topological membrane (Hofman & Park '02)
- ▶ Integrate out ϕ_i :

$$\begin{aligned} \tilde{S}_{\text{AKSZ}}^{(1)} &= \int_{\Sigma_2} \left(\xi_i \wedge dX^i + \frac{1}{2} \Theta^{ij}(X) \xi_i \wedge \xi_j \right) \\ &\quad + \int_{\Sigma_3} \frac{1}{6} H_{ijk}(X) dX^i \wedge dX^j \wedge dX^k \end{aligned}$$

- ▶ On-shell: $[\Theta, \Theta]_S = \wedge^3 \Theta^\sharp(H)$ H -twisted Poisson structure (Jacobi identity for bracket violated)

$$H \longmapsto Q \longmapsto R$$

- Closed strings on Q-space $M = T^3 \xrightarrow{T^2} S^1$ (locally), worldsheet $\mathcal{C} = \mathbb{R} \times S^1 \ni (\sigma^0, \sigma^1)$, winding number $\tilde{p}^3$, twisted boundary conditions at σ'^1

- Closed string noncommutativity (Lüst '10; Blumenhagen *et al.* '11; Condeescu, Florakis & Lüst '12; Andriot *et al.* '12):

$$\{x^i, x^j\}_Q = Q^{ij}{}_k \tilde{p}^k, \quad \{x^i, \tilde{p}^j\}_Q = 0 = \{\tilde{p}^i, \tilde{p}^j\}_Q$$

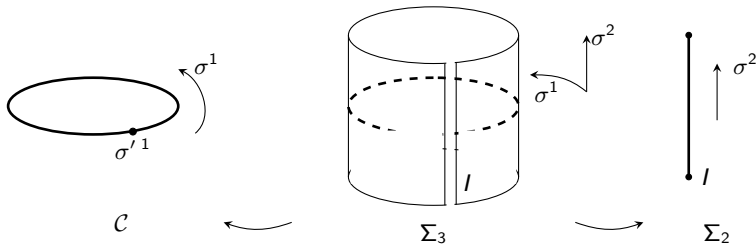
- T-duality sends $Q^{ij}{}_k \longmapsto R^{ijk}$, $\tilde{p}^k \longmapsto p_k$:

$$\{x^i, x^j\}_\Theta = R^{ijk} p_k, \quad \{x^i, p_j\}_\Theta = \delta^i_j, \quad \{p_i, p_j\}_\Theta = 0$$

- Twisted Poisson structure on T^*M : Closed string nonassociativity $\{x^i, x^j, x^k\}_\Theta = R^{ijk}$; transgresses to Poisson structure on loop space of M (Sämann & RS '12)

Closed/open string duality in Q -space

- ▶ CFT correlators: Insert twist field at $\sigma'^1 \in S^1$ creating branch cut
- ▶ Worldsheet $\mathcal{C} = \mathbb{R} \times S^1 \implies$ Worldvolume $\Sigma_3 = \mathbb{R} \times (S^1 \times \mathbb{R})$
- ▶ “Branch surface” $\equiv$ open string worldsheet Σ_2 :



σ-model for non-geometric fluxes

- C twisted by trivector $R = \frac{1}{6} R^{ijk}(x) \partial_i \wedge \partial_j \wedge \partial_k$, with Roytenberg bracket:

$$[(Y_1, \alpha_1), (Y_2, \alpha_2)]_R := ([Y_1, Y_2]_{TM} + R(\alpha_1, \alpha_2, -), \\ \mathcal{L}_{Y_1} \alpha_2 - \mathcal{L}_{Y_2} \alpha_1 - \frac{1}{2} d(\alpha_2(Y_1) - \alpha_1(Y_2)))$$

- Brackets: $[dx^i, dx^j]_R = R^{ijk} \partial_k$, $[dx^i, dx^j, dx^k]_R = R^{ijk}$
- Integrate out ϕ_i :

$$S_R^{(2)} = \int_{\Sigma_2} \xi_i \wedge dX^i + \int_{\Sigma_3} \frac{1}{6} R^{ijk}(X) \xi_i \wedge \xi_j \wedge \xi_k \\ + \int_{\Sigma_2} \frac{1}{2} g^{ij}(X) \xi_i \wedge * \xi_j$$

- For constant R^{ijk} and g^{ij} , e.o.m. for $X^i \implies \xi_i = dP_i$

Generalized Poisson σ -model

- Pure boundary theory, linearize with auxiliary fields η_I :

$$S_R^{(2)} = \int_{\Sigma_2} \left(\eta_I \wedge dX^I + \frac{1}{2} \Theta^{IJ}(X) \eta_I \wedge \eta_J \right) + \int_{\Sigma_2} \frac{1}{2} G^{IJ} \eta_I \wedge * \eta_J$$

$$X = (X^1, \dots, X^d, P_1, \dots, P_d) \in T^*M, \quad \Theta = \begin{pmatrix} R^{ijk} p_k & \delta^i_j \\ -\delta_i^j & 0 \end{pmatrix},$$

$$G^{IJ} = \begin{pmatrix} g^{ij} & 0 \\ 0 & 0 \end{pmatrix} \quad \text{Effective target space} = \text{phase space}$$

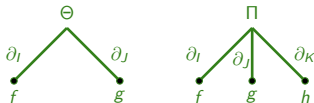
- H -twisted Poisson bivector: $\Pi := [\Theta, \Theta]_S = \wedge^3 \Theta^\sharp(H)$,
with $H = dB$, $B = \frac{1}{6} R^{ijk} p_k dp_i \wedge dp_j$
- Noncommutative/nonassociative phase space:

$$\{x^I, x^J\}_\Theta = \Theta^{IJ}(x), \quad \{x^I, x^J, x^K\}_\Theta := \Pi(x^I, x^J, x^K) = \begin{pmatrix} R^{ijk} & 0 \\ 0 & 0 \end{pmatrix}$$

Path integral quantization

- Suitable functional integrals reproduce Kontsevich's graphical expansion for global deformation quantization (Cattaneo & Felder '00).
Formality maps U_n : multivector fields $\longrightarrow$ differential operators:

$$U_n(\mathcal{X}_1, \dots, \mathcal{X}_n) = \sum_{\Gamma \in G_n} w_\Gamma D_\Gamma(\mathcal{X}_1, \dots, \mathcal{X}_n)$$



- Star-product and 3-bracket:

$$f \star g = \sum_{n=0}^{\infty} \frac{(\mathrm{i} \hbar)^n}{n!} U_n(\Theta, \dots, \Theta)(f, g) =: \Phi(\Theta)(f, g)$$

$$[f, g, h]_\star = \sum_{n=0}^{\infty} \frac{(\mathrm{i} \hbar)^n}{n!} U_{n+1}(\Pi, \Theta, \dots, \Theta)(f, g, h) =: \Phi(\Pi)(f, g, h)$$

Formality conditions

- ▶ U_n define L_∞ -morphisms relating Schouten brackets to Gerstenhaber brackets (Kontsevich '03)

- ▶ $[\Phi(\Theta), \star]_G = i\hbar \Phi([\Theta, \Theta]_S)$ quantifies nonassociativity:

$$(f \star g) \star h - f \star (g \star h) = \frac{\hbar}{2i} \Phi(\Pi)(f, g, h) = \frac{\hbar}{2i} [f, g, h]_\star$$

- ▶ $[\Phi(\Pi), \star]_G = i\hbar \Phi([\Pi, \Theta]_S)$ encodes quantum Leibnitz rule for Nambu–Poisson structure $\{f, g, h\}_\Pi := \Pi(df, dg, dh)$:

$$[f \star g, h, k]_\star - [f, g \star h, k]_\star + [f, g, h \star k]_\star = f \star [g, h, k]_\star + [f, g, h]_\star \star k$$

- ▶ Higher derivation properties encode quantum fundamental identity, giving quantized Nambu–Poisson structure

Dynamical nonassociativity

- Explicit dynamical **nonassociative** star-product:

$$f \star g = f \star_p g := \cdot \left(e^{\frac{i\hbar}{2} R^{ijk} p_k \partial_i \otimes \partial_j} e^{\frac{i\hbar}{2} (\partial_i \otimes \tilde{\partial}^i - \tilde{\partial}^i \otimes \partial_i)} (f \otimes g) \right)$$

- Replace dynamical variable p with constant $\tilde{p}$
 $\implies$ associative Moyal-type star-product $\tilde{\star} := \star_{\tilde{p}}$

- 3-bracket:

$$[f, g, h]_{\star} = \frac{4i}{\hbar} \left[\tilde{\star} \left(\sinh \left(\frac{\hbar^2}{4} R^{ijk} \partial_i \otimes \partial_j \otimes \partial_k \right) (f \otimes g \otimes h) \right) \right]_{\tilde{p} \rightarrow p}$$

- Trace property: $\int [f, g, h]_{\star} = 0$
- Agrees with 3-product from closed string vertex operators
 (Blumenhagen *et al.* '11)

Seiberg–Witten maps and noncommutative gerbes

- Poisson Θ , quantum Moser lemma $\equiv$ covariantizing map $\mathcal{D}$
(Jurčo, Schupp & Wess '00):

$$\begin{array}{ccc}
 B : & \Theta & \xrightarrow{\text{quant}} \star \\
 \rho \downarrow & \rho \downarrow & \downarrow \mathcal{D} \\
 B + F : & \Theta' & \xrightarrow{\text{quant}} \star'
 \end{array}$$

$$\Theta' = \Theta (1 + \hbar F \Theta)^{-1} , \quad \rho = \text{flow generated by } \Theta(A, -) , \\
 \mathcal{D}(f \star g) = \mathcal{D}f \star' \mathcal{D}g , \quad \mathcal{D}x =: x + \hat{A}$$

- H -twisted Θ : $H = dB_\alpha$, $B_\beta - B_\alpha = dA_{\alpha\beta}$

$$\text{Untwisted } \Theta_\alpha := \Theta (1 - \hbar B_\alpha \Theta)^{-1}$$

$$\Theta \xrightarrow{\text{quant}} \text{nonassociative } \star$$

$$\Theta_\alpha, \Theta_\beta \xrightarrow{\text{quant}} \text{associative } \star_\alpha, \star_\beta \text{ related by } \mathcal{D}_{\alpha\beta} \quad (\text{Aschieri et al. '10})$$

Generalised Seiberg–Witten maps for non-geometric fluxes

- ▶ Trivial gerbe on T^*M : $\alpha =$ constant momentum $\tilde{p}$:

$$\Theta_{\tilde{p}} = \begin{pmatrix} \hbar R^{ijk} \tilde{p}_k & \delta^i_j \\ -\delta^j_i & 0 \end{pmatrix}, \quad B_{\tilde{p}} = \begin{pmatrix} 0 & 0 \\ 0 & R^{ijk} (p_k - \tilde{p}_k) \end{pmatrix}$$

$\mathcal{D}_{\tilde{p}\tilde{p}'} : \tilde{\star} \longrightarrow \tilde{\star}'$ generated by $A_{\tilde{p}\tilde{p}'} = R^{ijk} p_i (\tilde{p}_k - \tilde{p}'_k) dp_j$;
for $\tilde{p} = 0$ canonical Moyal star-product $\star_0$

- ▶ $\mathcal{D}_{\tilde{p}0} : \text{associative} \longrightarrow \text{nonassociative}$ can be computed explicitly:

$$f \star g = [\mathcal{D}_{\tilde{p}} f \star_0 \mathcal{D}_{\tilde{p}} g]_{\tilde{p} \rightarrow p}$$

- ▶ Nonassociative generalization of Seiberg–Witten map:

$$A = \tilde{a}^i(x, p) dp_i:$$

- ▶ **Quantized diffeomorphisms:** $\Theta(A, -) = \tilde{a}^i(x, p) \partial_i$
"nonassociative gravity"
- ▶ **Quantized Nambu–Poisson maps:** $A = R(a_2, -)$, $a_2 \in \Omega^2(M)$
higher gauge theory

Categorified Weyl quantization

- ▶ Convolution quantization of Lie 2-algebras
- ▶ Reduction of standard R -space Courant algebroid:

$$[x^i, x^j]_Q = i R^{ijk} \tilde{p}_k, \quad [x^i, \tilde{p}_j]_Q = 0 = [\tilde{p}_i, \tilde{p}_j]_Q, \quad \langle x^i, p_j \rangle = \delta^i_j$$

- ▶ Induces skeletal 2-term L_∞ -algebra

$$V = (V_1 = \mathbb{R} \xrightarrow{d} V_0 = \mathfrak{g})$$

with classifying 3-cocycle $j : \mathfrak{g} \wedge \mathfrak{g} \wedge \mathfrak{g} \longrightarrow \mathbb{R}$,
 $j(x^i, x^j, x^k) = R^{ijk}$

- ▶ Integrates to Lie 2-group

$$\mathcal{G}_1 = G \times U(1) \xrightleftharpoons[t]{s} \mathcal{G}_0 = G$$

with associator integrating j

Quasi-Hopf cochain quantization

- ▶ Nonabelian Lie algebra $\mathfrak{g}$ of Bopp shifts generated by vector fields:

$$P_i = \partial_i, \quad \tilde{P}^i = \tilde{\partial}^i, \quad M_{ij} = p_i \partial_j - p_j \partial_i$$

- ▶ Quasi-Hopf deformation of $U(\mathfrak{g})$ by cochain twist:

$$\mathcal{F} = \exp \left[-\frac{i\hbar}{2} \left(\frac{1}{2} R^{ijk} M_{jk} \otimes P_i + P_i \otimes \tilde{P}^i - \tilde{P}^i \otimes P_i \right) \right]$$

- ▶ Quantization functor on category of quasi-Hopf module algebras generates nonassociative algebras through associator

$$\phi = \partial \mathcal{F} = \exp \left(\frac{\hbar^2}{2} R^{ijk} P_i \otimes P_j \otimes P_k \right)$$