Convergence of Star Products and the Nuclear Weyl Algebra

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- Beiser, S., Waldmann, S.: *Fréchet algebraic deformation quantization of the Poincaré disk.* Preprint (August 2011), 58 pages. To appear in Crelle's J. reine angew. Math.
- Waldmann, S.: *A Nuclear Weyl Algebra.* Preprint 1209.5551 (September 2012), 48 pages.

Plan of the talk

Introduction: Convergence of star products?

Main objective: Find examples and a framework where formal star products actually converge.
Needed for: reasonable theory of star exponentials, phase space reduction . . .

The set-up (most simplest example):

- A vector space V together with a bilinear form

$$\Lambda\colon V \times V \longrightarrow \mathbb{R}$$

  determines Poisson structure on the symmetric algebra $\mathrm{S}^\bullet(V)$.
- Formal Weyl star product gives formal deformation quantization of this.
- Alternatively: C^*-algebraic Weyl algebra.

Idea: Want something in between: $\hbar$ not formal anymore, but not yet C^*, since interesting topology on V should be remembered.

More precise set-up:

- V a locally convex topological vector space (Hausdorff), possibly continuously $\mathbb{Z}_2$-graded.
- Λ a continuous bilinear form (not separately continuous). No symmetry properties assumed, but only pairing between even–even and odd-odd parts of V. Continuity means there is a continuous seminorm p with

$$|\Lambda(v, w)| \leq \mathrm{p}(v)\mathrm{p}(w).$$

- Define $P_\Lambda \in \mathrm{End}(\mathrm{S}^\bullet(V) \otimes \mathrm{S}^\bullet(V))$ by

$$P_\Lambda(v_1 \cdots v_n \otimes w_1 \cdots w_m)$$
$$= \sum_{k,l} \Lambda(v_k, w_l) v_1 \cdots \overset{k}{\wedge} \cdots v_n \otimes w_1 \cdots \overset{l}{\wedge} \cdots w_m$$

plus some signs if in graded case.

- The Poisson bracket on $\mathrm{S}^\bullet(V)$

$$\{a, b\} = \mu \circ (P_\Lambda - \tau \circ P_\Lambda \circ \tau)(a \otimes b)$$

  with μ the symmetric tensor product.
- The (formal) Weyl product

$$a \star_{z\Lambda} b = \mu \circ \mathrm{e}^{zP_\Lambda}(a \otimes b)$$

  for z formal parameter or $z \in \mathbb{C}$. Here everything converges since we are on the symmetric algebra!

The topology on $\mathrm{S}^\bullet(V)$

Need a topology on $\mathrm{S}^\bullet(V)$ making the Poisson bracket and the star product continuous and allows for an interestingly large completion.

Work on the tensor algebra (easier) and induce the topology for $\mathrm{S}^\bullet(V) \subseteq T^\bullet(V)$ from there.
On each tensor power $V^{\otimes n}$ use the π-topology: for each continuous seminorm p on V define $\mathrm{p}^n = \mathrm{p} \otimes \cdots \otimes \mathrm{p}$ on $V^{\otimes n}$. Then all of these seminorms define the π-topology.
For $a = \sum_n a_n \in T^\bullet(V)$ define new seminorm

$$\mathrm{p}_R(a) = \sum_{n=0}^{\infty} n!^R \mathrm{p}^n(a_n)$$

and equip $T^\bullet(V)$ with all these seminorms, with parameter

$$R \geq 0.$$

Gives a locally convex space denoted by $T^\bullet_R(V)$, inducing a locally convex topology on the symmetric algebra, denoted by $\mathrm{S}^\bullet_R(V)$.

- Completion of $T^\bullet_R(V)$ and $\mathrm{S}^\bullet_R(V)$ can be described explicitly:

$$\widehat{T}^\bullet_R(V) = \left\{ a \in \prod_{n=0}^{\infty} V^{\hat{\otimes}_\pi n} \;\middle|\; \forall \mathrm{p}\colon \mathrm{p}_R(a) < \infty \right\}$$

- Elements in $\hat{\mathrm{S}}^\bullet_R(V)$ can be viewed as "analytic functions" on dual V' with certain growth conditions on Taylor coefficients, depending on the parameter R.
- Alternative versions of p_R using other ℓ^p-summations or even a sup-version does not yield anything new.

- For $R = 0$ the tensor algebra $\hat{T}^\bullet_{R=0}(V)$ is the free complete locally multiplicatively convex algebra generated by V.
- Unfortunately, there seems to be nothing like a free locally convex algebra generated by V: otherwise we could just take a quotient by the canonical commutation relations and we would be done.
  Note that we necessarily have to go beyond the case of sub-multiplicative seminorms!
- Simple examples show that we can not even expect an entire calculus for the canonical commutation relations: There are entire functions f such that $f(q)f(p)$ does not make any sense for q and p satisfying $[q, p] = \mathrm{i}\hbar$.

Continuity of $\star_{z\Lambda}$

Depending on the value of the parameter R, the product $\star_{z\Lambda}$ becomes continuous.

Lemma

Let $R \geq \frac{1}{2}$. Then for all p satisfying $|\Lambda(v, w)| \leq \mathrm{p}(v)\mathrm{p}(w)$ there exists a constant $c > 0$ with

$$\mathrm{p}_R(a \star_{z\Lambda} b) \leq c(c\mathrm{p})_R(a)(c\mathrm{p})_R(b)$$

Proof: this is a longer but straightforward estimate. The constant c depends on R and on z.

Definition

Let $R \geq \frac{1}{2}$. The symmetric algebra $\mathrm{S}^\bullet_R(V)$ equipped with the Weyl product $\star_{z\Lambda}$ is called the locally convex Weyl algebra $\mathcal{W}_R(V, \star_{z\Lambda})$.

Theorem

$\mathcal{W}_R(V, \star_{z\Lambda})$ is a locally convex algebra. It is first countable, if V is first countable.

Some first features of the construction:

- The star product $a \star_{z\Lambda} b$ for $a, b \in \mathcal{W}_R(V, \star_{z\Lambda})$ converges absolutely and gives a entire deformation.
- For a real vector space V and a complex-valued the complexified Weyl algebra $\mathcal{W}_R(V, \star_{\frac{\mathrm{i}}{\hbar}2\Lambda})$ with real $\hbar$ becomes a *-algebra with respect to complex conjugation if the symmetric part of Λ is imaginary and the antisymmetric part is real.
- For $R < 1$ the exponential series $\exp(v)$ for $v \in V$ belongs to the completion.

Further properties of the Weyl algebra

The construction of the locally convex Weyl algebra preserves many properties of the locally convex space V, quite like the C^*-algebraic construction.

Theorem

If V has an absolute Schauder basis, then $\mathcal{W}_R(V, \star_{z\Lambda})$ has an absolute Schauder basis, too.

Recall that $\{e_i\}_{i \in I} \subseteq V$ with $\{\varphi^i\}_{i \in I} \subseteq V'$ is called absolute Schauder basis if

$$v = \sum_{i \in I} \varphi^i(v) e_i$$

converges and for all continuous seminorms p

$$v \mapsto \sum_{i \in I} |\varphi^i(v)| \mathrm{p}(e_i)$$

is a continuous seminorm, too.

Theorem

The Weyl algebra $\mathcal{W}_R(V, \star_{z\Lambda})$ is nuclear iff V is nuclear.

Nuclearity has many equivalent definitions (technical) but the upshot is that nuclear spaces behave in many ways much nicer than general locally convex space, in particular concerning tensor products.

Corollary

If V is finite-dimensional, then the Weyl algebra $\mathcal{W}_R(V, \star_{z\Lambda})$ is nuclear and it has an absolute Schauder basis.

- This gives nice features of the Weyl algebra concerning deformation quantization in finite dimensions, i.e. for the usual star products.
- The absolute Schauder basis can e.g. be obtained from the symmetric tensor powers of a basis of V, i.e. the "monomials".
- In this case the construction reproduces results of Omori, Maeda, Miyasaki and Yoshioka. Moreover, a slight variant reproduces the results of Beiser, Römer, W.

Theorem

The construction of $\mathcal{W}_R(V, \star_{z\Lambda})$ *depends functorially on the pair* (V, Λ).

In particular, the continuous linear Poisson maps $(V, \Lambda) \longrightarrow (\tilde{V}, \tilde{\Lambda})$ lift to continuous algebra homomorphisms.

Let $\varphi \in V'$ be even. Then there is a unique algebra automorphism τ_φ of $\mathcal{W}_R(V, \star_{z\Lambda})$ with

$$\tau_\varphi(v) = v + \varphi(v)\mathbb{1}.$$

Geometrically, this is the pull-back of a polynomial by the translation by φ, if we interpret $\mathrm{S}^\bullet(V)$ as polynomials on V'.

Theorem

The automorphism τ_φ is continuous. If $R < 1$ and if φ is in the image of the musical homomorphism

$$\sharp\colon V \ni w \mapsto w^\sharp = \Lambda_-(w, \cdot\,) \in V'$$

then τ_φ is inner.

The inner element is a $\star_{z\Lambda}$-exponential series of a w with $w^\sharp = \varphi$.

Theorem

Suppose Λ, Λ' have the same antisymmetric part. Then the Weyl algebras $\mathcal{W}_R(V, \star_{z\Lambda})$ and $\mathcal{W}_R(V, \star_{z\Lambda})$ are isomorphic via a continuous isomorphism.

For formal star products this is nothing new, the emphasize lies on the word "continuous" here.

An example: the Peierls bracket and Free QFT

As a first application in infinite dimensions: (Q)FT on globally hyperbolic space-times.

The set-up:

- Take a globally hyperbolic space-times $M \cong \mathbb{R} \times \Sigma$ with a normally hyperbolic differential operator $D = \square^\nabla + B = D^*$ on a real vector bundle $E \longrightarrow M$ with fiber metric h.
- Let $F_M = F_M^+ - F_M^-$ be the propagator where

$$F_M^\pm \colon \Gamma_0^\infty(E^*) \longrightarrow \Gamma_{\mathrm{sc}}^\infty(E^*)$$

are the causal Green operators.

Canonical Poisson algebra: modelled on initial data on Σ

- Phase space will be $\Gamma_0^\infty(E_\Sigma) \oplus \Gamma_0^\infty(E_\Sigma)$.
- Take $V_\Sigma = \Gamma_0^\infty(E_\Sigma^*) \oplus \Gamma_0^\infty(E_\Sigma^*)$ as linear functions on the phase space (very few!)
- Take $\mathrm{S}^\bullet(V_\Sigma)$ as polynomial algebra on the phases space.
- Endow this with the constant (canonical) Poisson structure coming from

$$\Lambda_\Sigma\left((\varphi_0, \dot{\varphi}_0), (\psi_0, \dot{\psi}_0)\right) = \int_\Sigma \left(h_\Sigma^{-1}(\varphi_0, \dot{\psi}_0) - h_\Sigma^{-1}(\dot{\varphi}_0, \psi_0)\right) \mu_\Sigma$$

- This is a continuous bilinear form, so we can apply the Weyl algebra construction.

Now the covariant set-up:

- As covariant phase space we take $\Gamma^\infty_{\mathrm{sc}}(E)$.
- The observables will then be polynomials on this, modeled by the symmetric algebra over $\Gamma^\infty_0(E^*)$.
- On $\Gamma^\infty_0(E^*)$ we have the bilinear form

$$\Lambda(\varphi, \psi) = \int_M h^{-1}(F_M(\varphi), \psi)\mu.$$

- Again continuous.
- The corresponding Poisson bracket on $\mathrm{S}^\bullet(\Gamma^\infty_0(E^*))$ is the Peierls bracket.
- We can again apply the Weyl algebra construction for this.

The covariant Poisson bracket is now very much degenerate.

- The ideal generated by the kernel of F_M turns out to be a Poisson ideal.
- This ideal coincides with those polynomials which vanish when evaluated on solutions $u \in \Gamma^\infty_{\mathrm{sc}}(E)$ to the wave equation defined by D.

Theorem

The quotient of the covariant Poisson algebra modulo the ideal generated by $\ker F_M$ *is canonically isomorphic to the canonical Poisson algebra. The same holds true for the corresponding Weyl algebras.*

The Weyl algebra for the covariant Poisson algebra yields a local net of Weyl algebras satisfying time-slice and the usual locality axioms as required in AQFT.

Outlook

- Representation theory needs to be developed via usual techniques (GNS)
- Time evolution for quadratic elements more tricky: $\star$-exponential will not be in the completion (does not exists in a meaningful way).
- Still: it might give a one-parameter group of outer automorphisms.
- Go beyond the constant case: in finite dimensions possible for several examples via phase space reduction.
- Go beyond continuous case: separately continuous or bornological setting?

Thank you very much and have a good trip back!