BAYRISCHZELL WORKSHOP 2014

# CERTAIN ASPECTS OF NON-ASSOCIATIVE STRUCTURES IN PHYSICS

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Based on parts of arXiv:1309.3172 [hep-th], plus a bit more

## APPRECIATIONS AND OUTLINE

**I am honored and thankful to the organizers for inviting/admitting me to this nice annual series of workshops on non-commutative (and now also non-associative) geometry that was conceived/initiated by Julius Wess and his group of people and it keeps going strong ever since.**

**Thanks to Dieter, but also Ralph & Co, I began to appreciate the role of NC/NA in physics during my frequent visits to Munich.**

**Currently, most of the ongoing works focus on NC/NA of non-geometric closed string backgrounds and alike algebraic structures of DFT.**

**I will rather focus on certain analogues of NC/NA, letting the experts explain on their own the exciting new developments in string theory.**

**In particular, I will discuss the following topics:**

- **Magnetic field analogue of non-geometry in Maxwell-Dirac theory**
  - **NC/NA of momenta and the fate of angular symmetry.** **[classical and quantum]**
  - **NC/NA is distributed, not localized, and it also closely related to breakdown of integrability of charged point-particle motion.**

- **Deformation theory and cohomology**
  - **3-cocycles in Lie algebra and group cohomology.**
  - **NC/NA star-product as substitute for canonical quantization.**
  - **Universal characterization of obstructions.** **[new]**

- **NC/NA structures for integrability in $2+1$ dimensions.** **[in brief]**

# MAGNETIC FIELD ANALOGUE OF NC/NA

**A spinless point-particle (e, m) in magnetic field background $\vec{B}(\vec{x})$ has commutation relations among its coordinates and momenta:**

$$[x^i,\ p^j] = i\delta^{ij}, \qquad [x^i,\ x^j] = 0, \qquad [p^i,\ p^j] = ie\ \epsilon^{ijk} B_k(\vec{x})$$

**leading to non-commutativity of $p^i$ in Maxwell theory, $\vec{\nabla} \cdot \vec{B} = 0$.**

**In Dirac's generalization of Maxwell theory we have $\vec{\nabla} \cdot \vec{B} \neq 0$ and**

$$[[p^i,\ p^j],\ p^k]\ +\ \text{cyclic}\ \equiv\ [p^i,\ p^j,\ p^k]\ = -e\ \epsilon^{ijk} \vec{\nabla} \cdot \vec{B}\ \neq\ 0$$

**Associativity of momenta is lost in the presence of magnetic charges.**

**This provides a simple model for NC/NA of string theory with $x^i \leftrightarrow p^i$.**

**Consider a continuous spherically symmetric distribution of magnetic charge in space, $\rho(x)$, to study (some of) the implications of NC/NA in classical and quantum theory. Setting $x^2 = \vec{x} \cdot \vec{x}$, we have**

$$\vec{\nabla} \cdot \vec{B} = \rho(x), \qquad \vec{\nabla} \times \vec{B} = 0 \quad \text{(static)}.$$

**The particular solution of the inhomogeneous equation is expressed as**

$$\vec{B}(\vec{x}) = \frac{\vec{x}}{f(x)}, \qquad \rho(x) = \frac{3f(x) - xf'(x)}{f^2(x)}.$$

**Some notable example are:**

- $f(x) = x^3/g$ **so that** $\rho(x) = 4\pi g\ \delta(x)$ **[Dirac monopole with charge g]**
- $f(x) = 3/\rho$ **so that** $\rho(x) = \rho$ **is constant and** $\vec{B}(\vec{x}) = \rho\, \vec{x}/3$ **[cf $R$-flux]**

**Study the dynamics of point-particle for general profile function f(x).**

**Using the Hamiltonian $H = \vec{p}\cdot\vec{p}/2m$, the Lorentz force acting on the spinless particle $(e, m)$ in the magnetic field background is**

$$\frac{d\vec{p}}{dt} = i[H,\ \vec{p}\,] = \frac{e}{2m}(\vec{p}\times\vec{B} - \vec{B}\times\vec{p}\,)$$

**which for $\vec{B}(\vec{x}) = \vec{x}/f(x)$ takes the special form**

$$\frac{d^2\vec{x}}{dt^2} = -\frac{e}{mf(x)}\left(\vec{x}\times\frac{d\vec{x}}{dt}\right).$$

**Lorentz force is proportional to angular momentum and does no work.**

**Energy conservation provides one integral of motion, $E = A/2m$.**
**Simple manipulation shows that $x^2(t) = At^2 + D$, setting $x^2(0) = D$.**
**[$D$ provides the closest distance to the origin (perihelion of trajectory)]**
**Complete integrability requires three more integrals of motion.**

**For general choices of profile function $f(x)$, however, there are no additional integrals, since**

$$\frac{d}{dt}\left(\vec{x}\times\frac{d\vec{x}}{dt}\right) = -\frac{e}{mf(x)}\,\vec{x}\times\left(\vec{x}\times\frac{d\vec{x}}{dt}\right) = \frac{e\ x^3}{mf(x)}\,\frac{d\hat{x}}{dt}.$$

**Angular symmetry is broken in the presence of magnetic charges!**

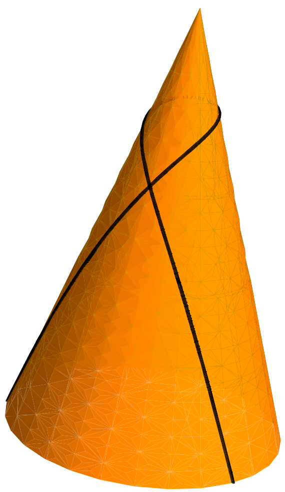
**The only exception is the Dirac monopole having $f(x) = x^3/g$. In this case, the improved angular momentum $\vec{J} = m\vec{K}$ is conserved, where**

$$\vec{K} \equiv \vec{x}\times\frac{d\vec{x}}{dt} - \frac{eg}{m}\,\hat{x}$$

**is the celebrated Poincaré vector.**

**In all other case, including constant $f(x)$, angular symmetry is broken and the classical motion of the particle appears to be non-integrable.**

**Trajectory of a spinless particle in the field of a magnetic monopole: the charged particle $(e, m)$ precesses with angular velocity $\vec{K}/(At^2 + D)$**



- **The magnetic monopole $g$ is located at the tip of the cone**
- **The Poincaré vector $\vec{K}$ provides the axis of the cone**
- $\vec{K} \cdot \hat{x} = -eg/m$ **determines the opening angle of the cone**

# ANOTHER LOOK AT ANGULAR SYMMETRY

**Try to follow as closely as possible the conventional definitions and algebraic structures of particle dynamics, without assuming particular representations nor Hilbert space [only that $\vec{p}$ acts as derivation].**

**Assume that $\vec{x}$ and $\vec{p}$ form a complete and irreducible set of observables for the point-particle in a static magnetic field $\vec{B}(\vec{x})$.**

**Angular momentum $\vec{J}$ ought to satisfy the algebraic relations**

$$[J^i,\ x^j\,] \;=\; i\epsilon^{ijk}\,x^k, \qquad [J^i,\ p^j\,] \;=\; i\epsilon^{ijk}\,p^k, \qquad [J^i,\ J^j\,] \;=\; i\epsilon^{ijk}\,J^k$$

**so that angular momentum is conserved, $[H,\ J^i] = 0$, in the background of any spherically symmetric magnetic field $\vec{B}(\vec{x})$.**

**Let $\vec{J}$ be the orbital angular momentum, plus an improvement term that accounts for the angular momentum of the electromagnetic field**

$$\vec{J} \;=\; \vec{x} \times \vec{p} - \vec{C}.$$

**Then, we obtain the following conditions for $\vec{C}$**

$$[x^i,\; C^j\;] \;=\; 0, \qquad [p^i,\; C^j\;] \;=\; ie\left(x^i B^j - \delta^{ij}(\vec{x}\cdot\vec{B})\right),$$

$$C^i \;=\; ex^i(\vec{x}\cdot\vec{B}) + \frac{i}{2}\,\epsilon^{ijk}\,[C^j,\; C^k\;].$$

**The only consistent solution corresponds to the magnetic field of a Dirac monopole, in which case $J = \vec{x} \times \vec{p} - eg\;\hat{x}$ [Poincaré vector].**

**Non-associativity is responsible for the violation of angular symmetry.**

**The apparent violation of non-associativity in a Dirac monopole field is eliminated by imposing the boundary conditions $\Psi(0) = 0$ on the wave-functions so that $\vec{p}$ (derivations) are represented by self-adjoint operators, even though they are defined in patches as $\vec{p} = -i\nabla - e\vec{A}$.**

**Rotations by an angle $\theta$ around an axis $\hat{n}$ (take $\hat{n} = \hat{x}$) are described by**

$$R(\hat{n} = \hat{x}, \theta) \; = \; e^{-i\theta \; \hat{x}\cdot\vec{J}} \; = \; e^{-ieg \; \theta}.$$

**Then, for a point-particle in a monopole field, single valuedness of R (up to a sign) yields Dirac's quantization condition $eg = n \in \mathbb{Z} \quad (\times\hbar/2)$. Finite translations in space also associate when $eg$ is quantized.**

**In all other cases, non-associativity is for real, obstructing canonical quantization. What can be used as substitute? $\longrightarrow$ star-product.**

# DEFORMATIONS AND COHOMOLOGY

**First, we focus on the case of constant magnetic charge density $\rho$ for which we have the basic commutation relations**

$$[p^i,\ p^j] = i\,\hbar e\,\frac{\rho}{3}\,\epsilon^{ijk}x_k, \qquad [x^i,\ p^j] \ = \ i\hbar\delta^{ij}, \qquad [x^i,\ x^j] \ = \ 0.$$

**They should be compared to the commutation relations of the so called parabolic $R$-flux model whose coordinates and momenta satisfy**

$$[x^i,\ x^j] = i\,\hbar\frac{1}{T^2}R\,\epsilon^{ijk}p_k, \qquad [x^i,\ p^j] \ = \ i\hbar\delta^{ij}, \qquad [p^i,\ p^j] \ = \ 0$$

**where $T$ is the string tension, $c = 1$, and $R$ is a 3-form constant flux. Associator/Jacobiator does not vanish when $\hbar$ and $\rho$ are not zero**

$$[p^1,\ p^2,\ p^3] \ = \ [[p^1,\ p^2],\ p^3] + \text{cycl. perm.} \ = \ -e\hbar^2\rho.$$

**Different contractions of commutation relations of constant $\rho$ model:**

$\hbar = 0$, **any** $\rho$: $[p^i,\ p^j] = 0$, $[x^i,\ p^j] = 0$ **(Algebra of translations $\mathbf{t}_6$)**

$\hbar \neq 0$, $\rho = 0$: $[p^i,\ p^j] = 0$, $[x^i,\ p^j] = i\hbar\delta^{ij}$ **(Heisenberg algebra g)**

**whereas in all cases the coordinates commute,** $[x^i,\ x^j] = 0$**.**

**The algebra of a charged point-particle moving in the magnetic field background** $\vec{B} \sim \vec{x}$ **is a deformation of the Lie algebras above.**

**Lie algebra cohomology characterizes the deformation which leads to non-associativity — Chevalley-Eilenberg cohomology**

- **cochains of Abelian algebra $\mathbf{t}_6$ with real values —** $H^*(\mathbf{t}_6,\ \mathbb{R})$
- **cochains of Heisenberg algebra g with values in g —** $H^*(\mathbf{g},\ \mathbf{g})$

$H^*(\mathbf{t}_6,\ \mathbb{R})$**: Let $T_I = x^i, p^i$ the generators of $\mathbf{t}_6$. Consider a 3-cochain with $c_3(p^1, p^2, p^3) = 1$, up to normalization, and $c_3(T_I, T_J, T_K) = 0$ for all other choices of generators (i.e., when at least one $T$ is $x$). We have**

$$[T_I,\ T_J,\ T_K]\ \sim\ c_3(T_I, T_J, T_K)$$

**and, thus, only the associator $[p^1, p^2, p^3]$ does not vanish.**

**The obstruction satisfies the 3-cocycle condition $dc_3(T_I, T_J, T_K, T_L) = 0$, since for any four elements of $\mathbf{t}_6$ we have**

$$c_3([T_I, T_J],\ T_K,\ T_L) - c_3([T_I, T_K],\ T_J,\ T_L) + c_3([T_I, T_L],\ T_J,\ T_K) +$$

$$c_3([T_J, T_K],\ T_I,\ T_L) - c_3([T_J, T_L],\ T_I,\ T_K) + c_3([T_K, T_L],\ T_I,\ T_J)\ =\ 0$$

- $c_3$ **is not a coboundary, i.e., $c_3 \neq df_2$ in $H^*(\mathbf{t}_6,\ \mathbb{R})$.**

$H^*(\mathbf{g},\ \mathbf{g})$**:** **Let** $c_2(p^i, p^j) = \epsilon^{ijk} x^k$**, up to a multiplicative constant, and** $c_2(x^i, p^j) = 0 = c_2(x^i, x^j)$**. Acting with the coboundary operator, we obtain**

$$dc_2(p^1, p^2, p^3) = -c_2([p^1,\ p^2],\ p^3) + c_2([p^1,\ p^3],\ p^2) - c_2([p^2,\ p^3],\ p^1)$$

$$+\pi(p^1)c_2(p^2, p^3) - \pi(p^2)c_2(p^1, p^3) + \pi(p^3)c_2(p^1, p^2)$$

**where** $\pi(\mathbf{g}) = \mathrm{Ad}_{\mathbf{g}} = [\mathbf{g},\ \cdot\ ]$**. Then, for the Heisenberg algebra, we have**

$$dc_2(p^1, p^2, p^3) = [p^1,\ c_2(p^2, p^3)] - [p^2,\ c_2(p^1, p^3)] + [p^3,\ c_2(p^1, p^2)]$$

**leading to alternative cohomological interpretation of non-associativity**

$$[p^1,\ p^2,\ p^3]\ \sim\ dc_2(p^1, p^2, p^3)$$

- **The cohomological interpretation depends on the module** ($\mathbb{R}$ **vs** $\mathbf{g}$)**.**

# LIE GROUP COHOMOLOGY

**Exponentiate the action of the position and momentum generators. The formal group elements**

$$U(\vec{a},\ \vec{b})\ =\ e^{i(\vec{a}\cdot\vec{x}+\vec{b}\cdot\vec{p})}$$

**satisfy the composition law, obtained by applying the BCH formula,**

$$U(\vec{a}_1,\vec{b}_1)U(\vec{a}_2,\vec{b}_2)\ =\ e^{-\frac{i}{2}(\vec{a}_1\cdot\vec{b}_2-\vec{a}_2\cdot\vec{b}_1)}\ e^{-i\frac{R}{2}(\vec{b}_1\times\vec{b}_2)\cdot\vec{x}}\ U(\vec{a}_1+\vec{a}_2,\ \vec{b}_1+\vec{b}_2).$$

**Successive composition of any three group elements** $U_i = U(\vec{a}_i,\vec{b}_i)$ **yields**

$$(U_1\ U_2)\ U_3\ =\ e^{-i\frac{R}{2}(\vec{b}_1\times\vec{b}_2)\cdot\vec{b}_3}\ U_1\ (U_2\ U_3)$$

**which do not associate when** $R \sim e\rho \neq 0$**, setting** $\hbar = 1$**.**

**If $R$ were zero, we would have a projective representation of the Abelian group of translations in phase space. The phase factor**

$$\varphi_2(\vec{a}_1,\vec{b}_1;\vec{a}_2,\vec{b}_2) = \vec{a}_1\cdot\vec{b}_2 - \vec{a}_2\cdot\vec{b}_1$$

**is a real-valued 2-cocycle in group cohomology, satisfying**

$$d\varphi_2(\vec{b}_1,\vec{b}_2,\vec{b}_3) \equiv \varphi_2(\vec{b}_2,\vec{b}_3)-\varphi_2(\vec{b}_1+\vec{b}_2,\vec{b}_3)+\varphi_2(\vec{b}_1,\vec{b}_2+\vec{b}_3)-\varphi_2(\vec{b}_1,\vec{b}_2) = 0$$

**and, thus, it does not show up in the associator, as in ordinary QM.**

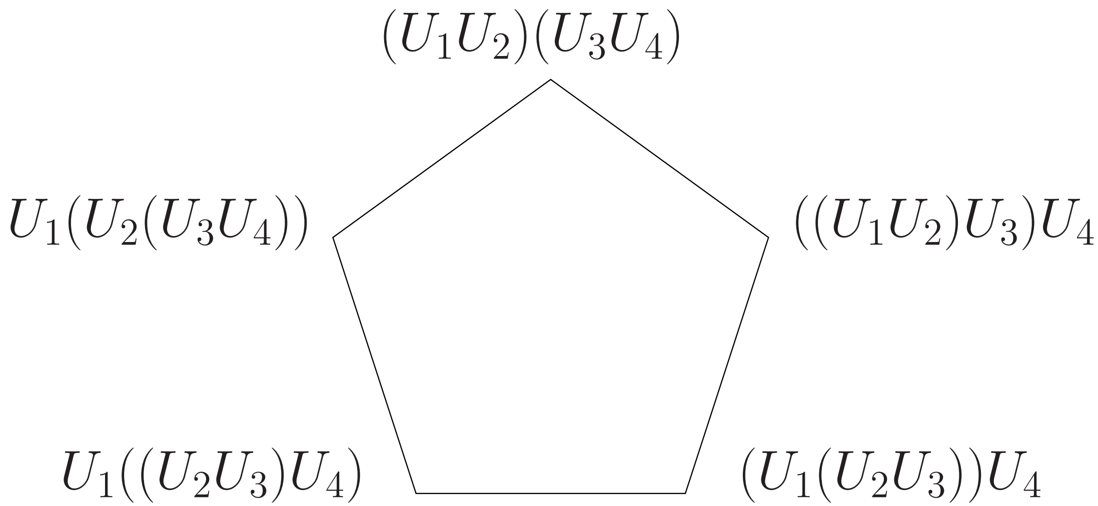
**If $R \neq 0$, there is an additional $x$-dependent factor in the composition law that gives rise to a phase in the associator of three group elements**

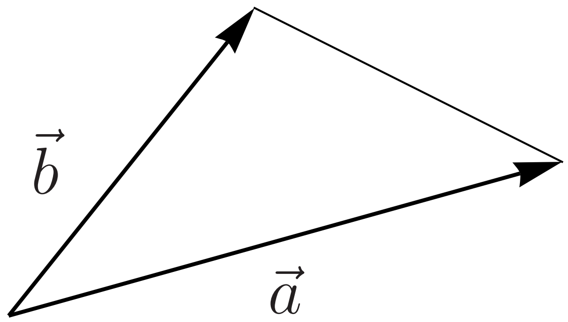
$$\varphi_3(\vec{b}_1,\vec{b}_2,\vec{b}_2) = (\vec{b}_1\times\vec{b}_2)\cdot\vec{b}_3$$

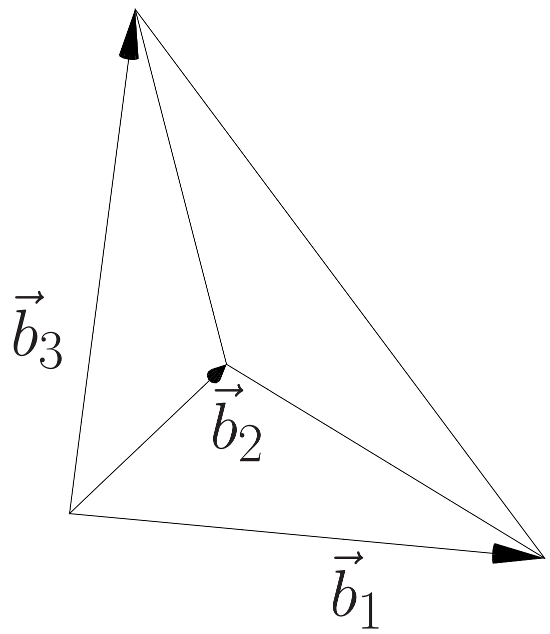
**The new phase is a real-valued 3-cocycle in the cohomology of the Abelian group of translations in phase space, satisfying**

$$d\varphi_3(\vec{b}_1,\vec{b}_2,\vec{b}_3,\vec{b}_4) \;\equiv\; \varphi_3(\vec{b}_2,\vec{b}_3,\vec{b}_4) - \varphi_3(\vec{b}_1+\vec{b}_2,\vec{b}_3,\vec{b}_4) + \varphi_3(\vec{b}_1,\vec{b}_2+\vec{b}_3,\vec{b}_4) - \varphi_3(\vec{b}_1,\vec{b}_2,\vec{b}_3+\vec{b}_4) + \varphi_3(\vec{b}_1,\vec{b}_2,\vec{b}_3) \;=\; 0 \;.$$

**A schematic representation is provided by Mac Lane's pentagon:**






**Geometric interpretation of the non-trivial cocycles $\varphi_2$ and $\varphi_3$:**

$$\text{Area}(\vec{a}, \vec{b}) = \frac{1}{2}\ |\vec{a} \times \vec{b}|$$

$$\text{Volume}(\vec{b}_1, \vec{b}_2, \vec{b}_3) = \frac{1}{6}\ |(\vec{b}_1 \times \vec{b}_2) \cdot \vec{b}_3|$$

**ALTERNATIVE INTERPRETATION of non-associativity is provided by the cohomology of Heisenberg-Weyl group with cochains taking values in the Heisenberg algebra. Introducing the elements**

$$U_W(g) \; = \; e^{i(\vec{a}\cdot\vec{x}+\vec{b}\cdot\vec{p}+c\mathbf{1})}$$

**the group composition law** $U(\vec{a}_1,\vec{b}_1)U(\vec{a}_1,\vec{b}_1)$ **takes the form**

$$U_W(g_1)\; U_W(g_2) \; = \; e^{-i\frac{R}{2}\varphi_2(g_1,g_2)}\; U_W(g_1 g_2)$$

**where** $\varphi_2(g_1,g_2) = (\vec{b}_1 \times \vec{b}_2)\cdot\vec{x}$ **takes values in the Heisenberg algebra. Then, the obstruction to associativity assumes the coboundary form**

$$(\vec{b}_1 \times \vec{b}_2)\cdot\vec{b}_3 = d\varphi_2(g_1,g_2,g_3).$$

**Group cohomology can be employed to define a NC/NA $\star$-product.**

# THE STAR PRODUCT

**When $R = 0$, all classical observables $f(x, p)$ are assigned to operators $\hat{F}(\hat{x}, \hat{p})$ acting on Hilbert space $\mathcal{H}$. Their product is non-commutative but associative.**

**An equivalent description is provided by Moyal star-product in phase space: Fourier analyse**

$$f(\vec{x}, \vec{p}) = \frac{1}{(2\pi)^3} \int d^3a d^3b \ \tilde{f}(\vec{a}, \vec{b}) e^{i(\vec{a} \cdot \vec{x} + \vec{b} \cdot \vec{p})}$$

**and apply Weyl's correspondence rule to assign self-adjoint operators**

$$\hat{F}(\hat{\vec{x}}, \hat{\vec{p}}) = \frac{1}{(2\pi)^3} \int d^3a d^3b \ \tilde{f}(\vec{a}, \vec{b}) \hat{U}(\vec{a}, \vec{b})$$

where

$$\hat{U}(\vec{a},\ \vec{b}) = e^{i(\vec{a}\cdot\hat{\vec{x}}+\vec{b}\cdot\hat{\vec{p}})}.$$

**The product of any two operators takes the form**

$$\hat{F}_1\cdot\hat{F}_2 = \frac{1}{(2\pi)^6}\int d^3a_1 d^3b_1 d^3a_2 d^3b_2\ \tilde{f}_1(\vec{a}_1,\vec{b}_1)\tilde{f}_2(\vec{a}_2,\vec{b}_2)\hat{U}(\vec{a}_1,\ \vec{b}_1)\hat{U}(\vec{a}_2,\ \vec{b}_2)$$

**and it can be worked out using the composition law**

$$\hat{U}(\vec{a}_1,\ \vec{b}_1)\hat{U}(\vec{a}_2,\ \vec{b}_2) = e^{-\frac{i}{2}(\vec{a}_1\cdot\vec{b}_2-\vec{a}_2\cdot\vec{b}_1)}\hat{U}(\vec{a}_1+\vec{a}_2,\ \vec{b}_1+\vec{b}_2).$$

**The 2-cocycle** $\varphi_2(\vec{a}_1,\vec{b}_1;\vec{a}_1,\vec{b}_1) = \vec{a}_1\cdot\vec{b}_2 - \vec{a}_2\cdot\vec{b}_1$ **makes the product of the corresponding phase space functions non-commutative but associative. The result turns out to be**

$$(f_1 \star f_2)(\vec{x},\vec{p}) = e^{\frac{i}{2}\left(\vec{\nabla}_{x_1}\cdot\vec{\nabla}_{p_2} - \vec{\nabla}_{x_2}\cdot\vec{\nabla}_{p_1}\right)} f_1(\vec{x}_1,\vec{p}_1) f_2(\vec{x}_2,\vec{p}_2)|_{\vec{x}_1=\vec{x}_2=\vec{x};\ \vec{p}_1=\vec{p}_2=\vec{p}}$$

**giving rise to the series expansion**

$$(f_1 \star f_2)(\vec{x},\vec{p}) = (f_1 \cdot f_2)(\vec{x},\vec{p}) + \frac{i}{2}\{f_1,\ f_2\} + \cdots \quad .$$

**Non-commutative geometry: the notion of point becomes fuzzy.**

**Quantum dynamics is equivalently described by the Moyal bracket**

$$\{\{f_1,\ f_2\}\} \equiv -i(f_1 \star f_2 - f_2 \star f_1) = \{f_1,\ f_2\} + \text{higher derivatives}$$

**acting as derivation**

$$\{\{f_1,\ f_2 \star f_3\}\} = f_2 \star \{\{f_1,\ f_3\}\} + \{\{f_1,\ f_2\}\} \star f_3.$$

**When $R \neq 0$, the rules of canonical quantization do not apply, but it is still possible to define a star-product non-commutative/non-associative.**

**We follow the same line of thought as before, assigning to $f(\vec{x},\vec{p})$**

$$F(\vec{x},\vec{p}) = \frac{1}{(2\pi)^3}\int d^3a d^3b\ \tilde{f}(\vec{a},\vec{b})U(\vec{a},\vec{b}),$$

**and using the generalized composition law,**

$$U(\vec{a}_1,\vec{b}_1)U(\vec{a}_2,\vec{b}_2)\ =\ e^{-\frac{i}{2}(\vec{a}_1\cdot\vec{b}_2-\vec{a}_2\cdot\vec{b}_1)}\,e^{-i\frac{R}{2}(\vec{b}_1\times\vec{b}_2)\cdot\vec{x}}\ U(\vec{a}_1+\vec{a}_2,\ \vec{b}_1+\vec{b}_2).$$

**The result is the NC/NA $x$-dependent star-product**

$$(f_1\star_x f_2)(\vec{x},\vec{p}) = e^{i\frac{R}{2}\ \vec{x}\cdot(\vec{\nabla}_{p_1}\times\vec{\nabla}_{p_2})}e^{\frac{i}{2}\left(\vec{\nabla}_{x_1}\cdot\vec{\nabla}_{p_2}-\vec{\nabla}_{x_2}\cdot\vec{\nabla}_{p_1}\right)}$$
$$f_1(\vec{x}_1,\vec{p}_1)f_2(\vec{x}_2,\vec{p}_2)|_{\vec{x}_1=\vec{x}_2=\vec{x};\ \vec{p}_1=\vec{p}_2=\vec{p}}\ .$$

**The substitute for quantum dynamics is provided by the bracket**

$$\{\{f_1,\ f_2\}\}_x \equiv -i(f_1 \star_x f_2 - f_2 \star_x f_1)$$

**which does not act as derivation, i.e.,**

$$\{\{f_1,\ f_2 \star_x f_3\}\}_x \ \neq \ f_2 \star_x \{\{f_1,\ f_3\}\}_x + \{\{f_1,\ f_2\}\}_x \star_x f_3.$$

**A related result is that the associator/Jacobiator does not vanish**

$$\{\{f_1(p),\ f_2(p),\ f_3(p)\}\}_x \ \neq \ 0.$$

• **Symmetries appear to be broken as consequence of non-associativity like the breakdown of angular symmetry discussed earlier.**

# FURTHER GENERALIZATIONS

**In the more general case, $[p^1, p^2, p^3] = -e\hbar^2\, \vec{\nabla} \cdot \vec{B}$ is not constant. Static spherically symmetric fields $\vec{B} = \vec{x}/f(x)$, with $x^2 = \vec{x} \cdot \vec{x}$, yield**

$$\vec{\nabla} \cdot \vec{B} \;=\; \rho(x) \;=\; \frac{3f(x) - xf'(x)}{f^2(x)}\,.$$

**Since $[p^i, p^j] = i\,\hbar e\, \epsilon^{ijk} B_k(\vec{x})$, one can still think of the basic commutation relations as deformation of the Heisenberg algebra g by a 2-cochain**

$$c_2(p^i, p^j) \;\sim\; \epsilon^{ijk}\, B_k(\vec{x})$$

**taking values in the space of (say) local smooth functions of $x$, which is a g-module, and let $c_2$ be zero otherwise. As before, we have**

$$dc_2(p^1,p^2,p^3) = -c_2([p^1,\ p^2],\ p^3) + c_2([p^1,\ p^3],\ p^2) - c_2([p^2,\ p^3],\ p^1)$$

$$+\pi(p^1)c_2(p^2,p^3) - \pi(p^2)c_2(p^1,p^3) + \pi(p^3)c_2(p^1,p^2)\,.$$

**For the Heisenberg algebra the momenta act as derivation and they commute, so the computation results to**

$$dc_2(p^1,p^2,p^3) = [p^1,\ c_2(p^2,p^3)] - [p^2,\ c_2(p^1,p^3)] + [p^3,\ c_2(p^1,p^2)] =$$

$$-i\frac{\partial}{\partial x^1}c_2(p^2,p^3) + i\frac{\partial}{\partial x^2}c_2(p^1,p^3) - i\frac{\partial}{\partial x^3}c_2(p^1,p^2) \ = \ -i\vec{\nabla}\cdot\vec{B}\,,$$

**while** $dc_2$ **vanishes identically for all other entries with one or more** $x$**.**

**This provides a universal cohomological interpretation of NC/NA, as**

$$[p^1,\ p^2,\ p^3]\ \sim\ dc_2(p^1,p^2,p^3)\,.$$

**The cohomological characterization of NC/NA can also be considered at group level and use it to define the star-product for more general distribution of magnetic charge,** $\rho(x)$**.**

**Question: Is there a one-to-one correspondence between magnetic field backgrounds and non-geometric closed string models?** [**under** $x \leftrightarrow p$]

- **A way to go about it is to find the closed string analogue of a Dirac monopole and use it to engineer more general string models for given distribution** $\rho$**.**

# NON-ASSOCIATIVITY FOR INTEGRABILITY

**Let me end with a brief discussion of yet another (less well known) aspect of NC/NA that works for rather than against integrability.**

**A number of two-dimensional models in statistical mechanics have been exactly solved using the so called triangle (or Yang-Baxter) relations. Schematically, they are of the following form,**

$$R_{ab}\, R_{ac}\, R_{bc} \;=\; R_{bc}\, R_{ac}\, R_{ab}$$

**and they can be regarded as the associativity condition for an $L$-operator algebra**

$$L_{1,a}\, L_{1,b}\, R_{ab} \;=\; R_{ab}\, L_{1,b}\, L_{1,a}\,.$$

**The triangle relations are the conditions for the row-to-row transfer matrices to commute. Alternatively, these models can be put into field theoretic form by considering the transfer matrix that adds a single face to the lattice and regarding this as an $S$-matrix. Then, the triangle relations become the condition for the $S$-matrix to factorize.**

**These structures can be generalized to three-dimensional models of statistical mechanics and field theory. Then, one has the so called tetrahedron (or Zamolodchikov) equations**

$$R_{abc}\,R_{ade}\,R_{bdf}\,R_{cef} \;=\; R_{cef}\,R_{bdf}\,R_{ade}\,R_{abc}$$

**regarded as compatibility condition of the $L$-operator algebra**

$$L_{12,a}\,L_{13,b}\,L_{23,c}R_{abc} \;=\; R_{abc}\,L_{23,c}\,L_{13,b}\,L_{12,a}\,.$$

The tetrahedron equations are the integrability conditions for the transfer matrices of statistical models to commute and the $S$-matrix of field theory models to be factorizable in three dimensions.
[there are also higher simplex equations in higher dimensions]

Theorem (Baez-Crans, 2003): Any semi-strict Lie 2-algebra provides a solution of the tetrahedron equations, just as any Lie algebra provides a solution of the triangle relations. [arXiv:0307263 [math.QA]]

Bottom line: These diverse results provide common ground for using homotopy associative algebras in string theory, Maxwell-Dirac theory, as well in higher dimensional integrable systems that is worth exploring further by putting them all together.

# THANK YOU!