

Non-associative Deformations of Geometry in DFT

Ralph Blumenhagen

Max-Planck-Institut für Physik, München



(Bhg, Fuchs, Hassler, Lüst, Sun , arXiv:1312.0719)



Introduction

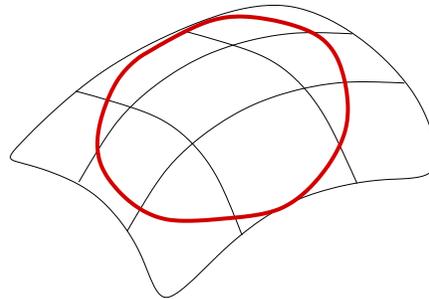
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Reconciling quantum theory with gravity \Rightarrow giving up the principle of **locality**

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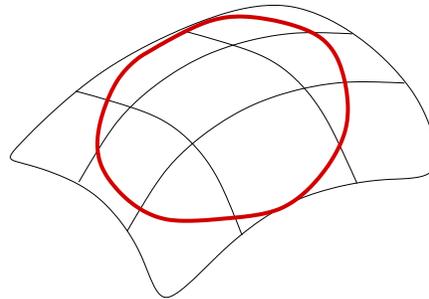
In string theory one smears out the **probe**



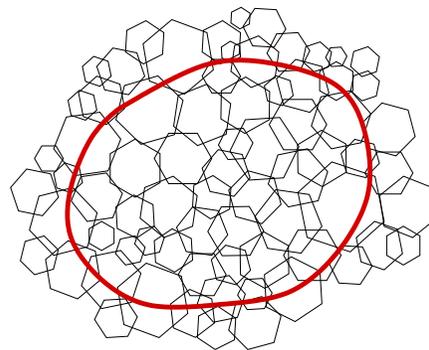
Introduction

Reconciling quantum theory with gravity \Rightarrow giving up the principle of **locality**

In string theory one smears out the **probe**



Could imagine smearing out underlying **space-time**, e.g. non-commutative geometry $\Delta x \Delta y > \ell^2$ (Wess et al.)



Introduction

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String theory is described by **2D non-linear sigma model**

$$\mathcal{S} = \frac{1}{2\pi\alpha'} \int_{\Sigma} d^2z (G_{ab} + B_{ab}) \partial X^a \bar{\partial} X^b + \dots ,$$

where **conformal invariance** provides the string equations of motion, which are captured by the effective action

$$S = \frac{1}{2\kappa^2} \int d^n x \sqrt{-|G|} e^{-2\phi} \left(R - \frac{1}{12} H_{abc} H^{abc} + 4\partial_a \phi \partial^a \phi \right) .$$

Leading order: Einstein gravity with a two-form B_{ab} and a scalar Φ .

There exist **conformal field theories** which **cannot** be identified with such simple large radius geometries.

Introduction

Introduction

These are left-right **asymmetric** like asymmetric orbifolds. Applying **T-duality** leads to the chain of fluxes (Shelton, Taylor, Wecht)

$$H_{abc} \leftrightarrow f_{ab}{}^c \leftrightarrow Q_a{}^{bc} \overset{?}{\leftrightarrow} R^{abc} ,$$

Q and R are non-geometric fluxes. What is the nature of R -flux?

There are claims that Type IIB orientifolds with **non-geometric** fluxes generically lead to **de-Sitter** vacua (Danielsson, Dibitetto), (Damian, Loaiza-Brito).

→ Need to better understand this regime of string theory.

Introduction

Introduction

String theory: D-brane with magnetic flux $F^{ij} \rightarrow$
non-commutative geometry with

$$[x^i, x^j] = x^i \star x^j - x^j \star x^i = i F^{ij}$$

with a non-commutative product on function space
(Seiberg/Witten 1999)

$$f \star g = f \cdot g + \frac{i}{2} F^{ij} \partial_i f \partial_j g + \dots$$

Does there exist a generalization to the **closed** string?

Introduction

Introduction

(Bouwknegt, Hannabuss, Mathai), (Bhg, Plauschinn), (Lüst), (Bhg, Deser, Lüst, Plauschinn, Rennecke), (Mylonas, Schupp, Szabo), (Chatzistavrakidis, Jonge)

Evidence has been collected for the existence of a **tri-bracket**

$$[x^i, x^j, x^k] = \begin{cases} 0 & H\text{-flux} \\ (\alpha')^2 R^{ijk} & R\text{-flux} \end{cases} .$$

The same result would arise from a (pre-) Lie-algebra

$$[x^i, x^j] = \frac{i}{\hbar} (\alpha')^2 R^{ijk} p_k, \quad [x^i, p_j] = i\hbar \delta^i_j .$$

so that the **Jacobiator** gives precisely the tri-bracket.

Whether tri-bracket is **fundamental** or a consequence of such a Jacobiator has **not** been agreed upon yet.

Introduction

Introduction

Computation of tachyon scattering amplitudes in a CFT with

$$G_{ij} = \delta_{ij} \text{ and } H_{ijk} = \text{const.}$$

(Bhg, Deser, Lüst, Plauschinn, Rennecke)

Introduction

Computation of tachyon scattering amplitudes in a CFT with $G_{ij} = \delta_{ij}$ and $H_{ijk} = \text{const.}$

(Bhg, Deser, Lüst, Plauschinn, Rennecke)

- Exact CFT up to linear order in H
- For both H -flux and R -flux the final scattering amplitude was associative
- Prior to invoking momentum conservation, for the R -flux there appeared world-sheet independent phase factors

Introduction

Computation of tachyon scattering amplitudes in a CFT with $G_{ij} = \delta_{ij}$ and $H_{ijk} = \text{const.}$

(Bhg, Deser, Lüst, Plauschinn, Rennecke)

Phases could be encoded in the tri-product

$$(f \Delta g \Delta h)(x) = \exp\left(\frac{(\alpha')^2}{6} R^{ijk} \partial_i^{x_1} \partial_j^{x_2} \partial_k^{x_3}\right) f(x_1) g(x_2) h(x_3) \Big|_x .$$

The tri-bracket then follows via

$$[x^i, x^j, x^k] = \sum_{\sigma \in P^3} \text{sign}(\sigma) x^{\sigma(i)} \Delta x^{\sigma(j)} \Delta x^{\sigma(k)} .$$

Non-constant fluxes on curved space

Non-constant fluxes on curved space

For the open string we have:

1. For on-shell string scattering amplitudes, the conformal $SL(2, \mathbb{R})$ symmetry group leaves the cyclic order of fields invariant.
2. In CFT on the disc we have crossing symmetry of N-point functions \rightarrow on-shell associativity.

(Herbst, Kling, Kreuzer),(Cornalba,Schiappa)

Non-constant fluxes on curved space

Non-constant fluxes on curved space

Therefore, we should get

$$\int d^n x \sqrt{G} f \star g = \int d^n x \sqrt{G} g \star f .$$

Indeed,

$$\begin{aligned} \int d^n x \sqrt{G} F^{ij} \partial_i f \partial_j g &= \int d^n x \partial_i \left(\sqrt{G} F^{ij} f \partial_j g \right) \\ &\quad - \int d^n x \partial_i \left(\sqrt{G} F^{ij} \right) f \partial_j g = 0 \end{aligned}$$

Equation of motion

$$\partial_i \left(\sqrt{G} F^{ij} \right) = \sqrt{G} \nabla_i F^{ij} = 0$$

Non-constant fluxes on curved space

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Equation of motion

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Interplay between equation of motion and cyclicity.

Questions

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Analogously, one can show associativity

$$\int d^n x \sqrt{G} (f \star g) \star h = \int d^n x \sqrt{G} f \star (g \star h).$$

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- How generic is this observation?
- Can it be generalized to non-abelian gauge fields or [gravity](#)?
- Does it allow to understand off-shell string theory?

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$$\int d^n x \sqrt{G} (f \star g) \star h = \int d^n x \sqrt{G} f \star (g \star h).$$

- How generic is this observation?
- Can it be generalized to non-abelian gauge fields or **gravity**?
- Does it allow to understand off-shell string theory?

For **gravity**, we found a way using the generalization to so-called **double field theory** (Siegel, Hull, Zwiebach, Hohm, etc).

Approach via DFT

Approach via DFT

- **Double Field Theory** provides a unified description of non-geometric fluxes.
- **No** notion of a **non-associative**, background dependent deformation of the geometry is visible.
- How can DFT be **reconciled** with a non-associative deformation of geometry?

Two important observations:

Approach via DFT

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Two important observations:

- Non-associativity is claimed to arise for an ***R*-flux background** contracted with ordinary partial derivatives $\partial/\partial x^i$.

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Two important observations:

- In quantum theories, where observables are operators acting on some Hilbert space, one can get non-commutativity, but the product of operators is **always associative**.

Approach via DFT

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- How can DFT be **reconciled** with a non-associative deformation of geometry?

Two important observations:

- Since CFTs are ordinary quantum theories, on-shell (in string theory) **no violation of associativity** → crossing symmetry of correlation function.

Flux formulation DFT

Flux formulation DFT

(Siegel, Aldazabal, Baron, Geissbühler, Grana, Marques, Nunez, Penas)

Doubled coordinates

$$X^M = (\tilde{x}_m, x^m)$$

Strong constraint

$$\partial_M \partial^M = 0, \quad \partial_M f \partial^M g = 0$$

Generalized bein E^A_M parametrized as

$$E^A_M = \begin{pmatrix} e_a^m & e_a^k B_{km} \\ e^a_k \beta^{km} & e^a_m + e^a_k \beta^{kl} B_{lm} \end{pmatrix},$$

with the ordinary bein $e_a^m s^{ab} e_b^n = G^{mn}$, two form B_{mn} and a two-vector β^{mn} . Flat derivative

$$\mathcal{D}^A = E^A_M \partial^M.$$

Flux formulation DFT

Flux formulation DFT

Generalized fluxes F_{ABC} defined as

$$\mathcal{F}_{ABC} = 3\Omega_{[ABC]}$$

in terms of the generalized Weitzenböck connection

$$\Omega_{ABC} = \mathcal{D}_A E_B^M E_{CM} .$$

Components of \mathcal{F}_{ABC} are geometric and non-geometric fluxes H, F, Q and R

$$\mathcal{F}_{abc} = H_{abc} , \quad \mathcal{F}^a{}_{bc} = F^a{}_{bc} , \quad \mathcal{F}_c{}^{ab} = Q_c{}^{ab} , \quad \mathcal{F}^{abc} = R^{abc} .$$

Flux formulation DFT

Flux formulation DFT

T-duality invariant **dilaton**

$$e^{-2d} = e^{-2\phi} \sqrt{g}$$

which is used to define the **flux**

$$\mathcal{F}_A = \Omega^B{}_{BA} + 2E_A{}^M \partial_M d.$$

The flat derivative satisfies the **commutation relations**

$$[\mathcal{D}_A, \mathcal{D}_B] = F^C{}_{AB} \mathcal{D}_C - \Omega^C{}_{AB} \mathcal{D}_C.$$

where the second term vanishes after invoking the strong constraint.

DFT: Symmetries

DFT: Symmetries

- Doubled diffeomorphisms, i.e.

$$X^M \rightarrow X^M + \xi^M$$

Acts via a generalized Lie-derivative on generalized tensors, e.g. on a doubled vector as

$$\mathcal{L}_\xi V^M = \xi^N \partial_N V^M + (\partial^M \xi_N - \partial_N \xi^M) V^N .$$

x^m dependence: standard diffeomorphisms and B -field gauge transformations

\tilde{x}_m dependence: β -field gauge transformations.

closure, i.e. $\Delta_{\xi_1} (\mathcal{L}_{\xi_2} V^M) = 0$, not automatic.

DFT: Symmetries

- local $O(D) \times O(D)$ symmetry acting on the bein as

$$\delta_{\Lambda} E_A^M = \Lambda_A^B E_B^M \quad \text{with} \quad \Lambda_A^C S_{CD} \Lambda_B^D = S_{AB}$$

- Global $O(D, D)$ symmetry:

$$\begin{aligned} (E_A)' &= (E_A)h, & d' &= d, \\ X' &= hX, & \partial' &= (h^t)^{-1} \partial, \end{aligned}$$

DFT: Action

DFT: Action

Invariant **action** of the flux formulation of DFT

$$S_{\text{DFT}} = \frac{1}{2} \int dX e^{-2d} \left[\mathcal{F}_{ABC} \mathcal{F}_{A'B'C'} \left(\frac{1}{4} S^{AA'} \eta^{BB'} \eta^{CC'} \right. \right. \\ \left. \left. - \frac{1}{12} S^{AA'} S^{BB'} S^{CC'} - \frac{1}{6} \eta^{AA'} \eta^{BB'} \eta^{CC'} \right) \right. \\ \left. + \mathcal{F}_A \mathcal{F}_{A'} \left(\eta^{AA'} - S^{AA'} \right) \right].$$

Form very similar to the scalar potential of **half maximal gauged supergravity** in 4D

Constraint in DFT

Constraint in DFT

- Strong constraint:
Weak and strong constraint

$$\partial_M \partial^M = 0, \quad \partial_M f \partial^M g = \mathcal{D}_A f \mathcal{D}^A g = 0.$$

- Locally, D dimensional **slice** in $2D$ doubled geometry
→ **SUGRA** frame
- Guarantee **closure** of the symmetry algebra
- Always implemented for the **uncompactified** directions. (Betz, Bhg, Lüst, Rennecke)

DFT: Bianchi identities

DFT: Bianchi identities

Fluxes satisfy the **generalized Bianchi identities**

$$\mathcal{D}_{[A} \mathcal{F}_{BCD]} - \frac{3}{4} \mathcal{F}_{[AB}{}^M \mathcal{F}_{CD]M} = \mathcal{Z}_{ABCD}$$

and

$$\mathcal{D}^M \mathcal{F}_{MAB} + 2\mathcal{D}_{[A} \mathcal{F}_{B]} - \mathcal{F}^M \mathcal{F}_{MAB} = \mathcal{Z}_{AB},$$

where the right hand sides are given by

$$\mathcal{Z}_{ABCD} = -\frac{3}{4} \Omega_{E[AB} \Omega^E{}_{CD]}$$

$$\mathcal{Z}_{AB} = \left(\partial^M \partial_M E_{[A}{}^N \right) E_{B]N} - 2 \Omega^C{}_{AB} \mathcal{D}_{CD}.$$

and vanish by the strong constraint.

DFT: Equations of motion

DFT: Equations of motion

Varying the action with respect to the **bein**, one obtains the equations of motion

$$\mathcal{G}^{[AB]} = 2S^{C[A} \mathcal{D}^{B]} \mathcal{F}_C + (\mathcal{F}_C - \mathcal{D}_C) \check{\mathcal{F}}^{C[AB]} + \check{\mathcal{F}}^{CD[A} \mathcal{F}_{CD}^{B]} = 0$$

with

$$\check{\mathcal{F}}^{ABC} = \check{S}^{ABCDEF} \mathcal{F}_{DEF}$$

and

$$\begin{aligned} \check{S}^{ABCDEF} = & \frac{1}{2} S^{AD} \eta^{BE} \eta^{CF} + \frac{1}{2} \eta^{AD} S^{BE} \eta^{CF} + \frac{1}{2} \eta^{AD} \eta^{BE} S^{CF} \\ & - \frac{1}{2} S^{AD} S^{BE} S^{CF} \end{aligned}$$

Dilaton eom: DFT Lagrangian vanishes.

Non-associative deformation

Non-associative deformation

In DFT, there exist two possible [tri-products](#):

Non-associative deformation

In DFT, there exist two possible **tri-products**:

$$f \Delta g \Delta h = f g h + \frac{(\lambda\alpha')^2}{6} \check{\mathcal{F}}^{ABC} \mathcal{D}_A f \mathcal{D}_B g \mathcal{D}_C h + O((\lambda\alpha')^4).$$

contains a term

$$H^{ijk} \partial_i f \partial_j g \partial_k h$$

with

$$H^{ijk} = g^{ii'} g^{jj'} g^{kk'} H_{i'j'k'}$$

Non-associative deformation

In DFT, there exist two possible **tri-products**:

$$f \Delta g \Delta h = f g h + \frac{(\lambda\alpha')^2}{6} \mathcal{F}_{ABC} \mathcal{D}^A f \mathcal{D}^B g \mathcal{D}^C h + O((\lambda\alpha')^4).$$

contains a term

$$R^{ijk} \partial_i f \partial_j g \partial_k h$$

Tri-product: \mathcal{F}^{ABC}

Tri-product: $\check{\mathcal{F}}^{ABC}$

Performing an integration by parts one gets

$$\int dX e^{-2d} \check{\mathcal{F}}^{ABC} \mathcal{D}_A f \mathcal{D}_B g \mathcal{D}_C h = \int dX \partial_M (\dots)^M + \int dX e^{-2d} \left[(\mathcal{F}_C - \mathcal{D}_C) \check{\mathcal{F}}^{C[AB]} + \check{\mathcal{F}}^{CD[A} \mathcal{F}_{CD}{}^{B]} \right] f \mathcal{D}_A g \mathcal{D}_B h.$$

The second term can be written as

$$\int dX e^{-2d} \left[\mathcal{G}^{[AB]} - 2S^{M[A} \mathcal{D}^{B]} \mathcal{F}_M \right] f \mathcal{D}_A g \mathcal{D}_B h$$

Vanishes by

- strong constraint
- EOM $\mathcal{G}^{[AB]} = 0 \Rightarrow$ **no** non-associativity **on-shell**
- needs to be adjusted for stringy α' corrections to EOM

Tri-product: \mathcal{F}^{ABC}

Tri-product: \check{F}^{ABC}

- Remarkable relation between principle of vanishing non-associativity and dynamics of the system.
- This is the closed string generalization of the open string story with F^{ij}
- This includes gravity!

Tri-product: \mathcal{F}_{ABC}

Tri-product: \mathcal{F}_{ABC}

Consider the second tri-product

$$f \Delta g \Delta h = f g h + \frac{(\lambda\alpha')^2}{6} \mathcal{F}_{ABC} \mathcal{D}^A f \mathcal{D}^B g \mathcal{D}^C h + O((\lambda\alpha')^4).$$

The order $(\lambda\alpha')^2$ term **vanishes** by the **strong** constraint.

Thus, in this case this tri-product is actually **trivial** and the coordinates are **associative**.

$$[x^i, x^j, x^k] = 0$$

To get a **non-trivial** tri-product the strong constraint (between background and fluctuations) needs to be **weakened**. E.g.

consider string states with $N - \overline{N} = p \cdot w \neq 0$.

Result

Result

In **CFT** it was found in various papers

$$[x^i, x^j, x^k] = \begin{cases} 0 & H\text{-flux due to L - R cancel.} \\ (\alpha')^2 \hat{R}^{ijk} & R\text{-flux} \end{cases} .$$

Evaluating the **DFT** results for a holonomic basis we get for the tri-bracket

$$[x^i, x^j, x^k] = \begin{cases} (\alpha')^2 H^{ijk} & H\text{-flux } (\check{\mathcal{F}}^{ABC}) \\ 0 & R\text{-flux due to strong constr.} \end{cases} .$$

Without imposing any constraint, $[x^i, x^j, x^k] = \hat{R}^{ijk}$.

Higher orders

Higher orders

At leading order in derivatives ($\mathcal{D}\check{\mathcal{F}}_{ABC}$), there is a natural candidate for the **all order** in $(\lambda\alpha')^2$ tri-product, namely

$$(f \Delta g \Delta h)(X) = \exp\left(\frac{\ell_s^4}{6} \check{\mathcal{F}}_{ABC} \mathcal{D}_{X_1}^A \mathcal{D}_{X_2}^B \mathcal{D}_{X_3}^C\right) f(X_1) g(X_2) h(X_3) \Big|_X .$$

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This product satisfies

$$\int dX e^{-2d} f \Delta g \Delta h = \int dX e^{-2d} f g h .$$

Higher orders

Higher orders

Analogously, at leading order in $(\mathcal{D}\check{\mathcal{F}}_{ABC})$ we define the K -fold tri-product as

$$(f_1 \Delta_K f_2 \Delta_K \dots \Delta_K f_K)(X) \stackrel{\text{def}}{=} \exp\left(\frac{\ell_s^4}{6} \check{\mathcal{F}}_{ABC} \sum_{1 \leq a < b < c \leq K} \mathcal{D}_{X_a}^A \mathcal{D}_{X_b}^B \mathcal{D}_{X_c}^C\right) f_1(X_1) f_2(X_2) \dots f_K(X_K) \Big|_X$$

One can prove that for each K

$$\int dX e^{-2d} f_1 \Delta_K f_2 \Delta_K \dots \Delta_K f_K = \int dX e^{-2d} f_1 f_2 \dots f_K .$$

Higher orders

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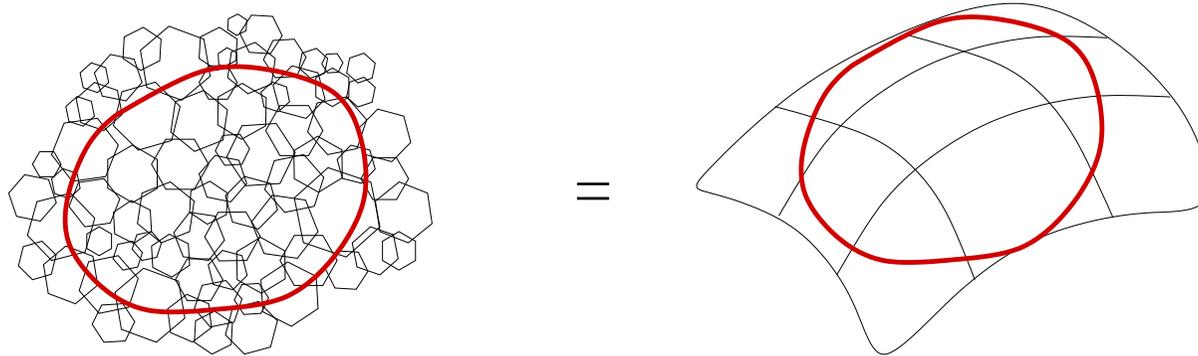
- provides the rule for deforming the product of K-terms in the action \Rightarrow no physical terms generated on-shell
- In contrast to the \star_K product in open string theory \Rightarrow non-commutative \star -product generates physical terms in the effective action (Seiberg-Witten map etc)

Outlook

Outlook

Showed that non-associative deformations of space-time can be **consistent** with DFT/string theory

In simple words: this deformation can be **under the radar** of the finite size strings



- How general is the observation that dynamics can be derived from the **principle** of vanishing non-associativity?
- Generalization to **M-theory**?
- Relation to string field theory resp. off-shell backgrounds?