Phase space quantization, non-commutativity and the gravitational field

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Main goal

Study the algebraic structure of phase space in non-commutative geometry, in the presence of a non-trivial frame (gravitational field). cf. Burić, Madore

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- → Parallelizability.
- → Symplectic structure.
- → Leibniz rule.
- → Jacobi identities.

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Main results

- ✓ Role of left and right acting operators, symplectic duality.
- Extended algebras, quadratic in momenta. Madore
- Class of non-trivial examples, symplectic nilmanifolds.



The plan

- Reminder on non-commutative plane and torus
- 2 Non-trivial frames and phase space
- 3 Application to symplectic nilmanifolds
- 4 Compact case
- 5 Main messages and future directions

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Classical vs. Quantum

In classical mechanics:

- Phase space of d-dimensional space \rightsquigarrow 2d-dimensional, (x^a, p_a) .
- Symplectic structure, $\omega = \delta_a^b \mathrm{d} x^a \wedge \mathrm{d} p_b$.
- Poisson bracket, $\{f,g\} = \delta_a^b(\partial_{x^b} f \partial_{p_a} g \partial_{p_a} f \partial_{x^b} g)$.

Players: $\mathrm{d} x^{a}, \mathrm{d} p_{a}, \partial_{x^{a}}, \partial_{p_{a}}$.

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In quantum mechanics:

- Hermitian operators \hat{x}^a and \hat{p}_a .
- "Complex structure": $\hat{x} \rightarrow \hat{p}, \ \hat{p} \rightarrow -\hat{x}$ (cf. role in matrix compactifications with flux) A.C., Jonke '12, '13
- Wave functions do not depend on both positions and momenta, but only on one set, depending on the chosen representation.

Simultaneous consideration of ∂_{x^a} , ∂_{p_a} is unnecessary and redundant.

Quantum vs. Non-commutative

Phase space of quantum mechanics (use flat indices i, j, \ldots ; curved indices are a, b, \ldots):

$$[\hat{x}^{i}, \hat{x}^{j}] = 0$$
, $[\hat{x}^{i}, \hat{p}_{j}] = i\hbar \delta^{i}_{j}$, $[\hat{p}_{i}, \hat{p}_{j}] = 0$.

- Position rep.: $x, -i\hbar\partial_x$.
- Momentum rep.: $p, i\hbar \partial_p$.
- Momenta are outer derivations of the position algebra (position rep.).

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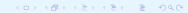
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Phase space of non-commutative quantum mechanics (in absence of magnetic sources):

$$[\hat{x}^i,\hat{x}^j]=i\theta^{ij}\ ,\quad [\hat{x}^i,\hat{p}_j]=i\hbar\delta^i_i\ ,\quad [\hat{p}_i,\hat{p}_j]=0\ .$$

Duval, Horvathy '00; Nair, Polychronakos '00; Morariu, Polychronakos '01; Horvathy '02

- θ^{ij} : constant parameters; components of symplectic 2-vector.
- $\hat{x}^i \in \mathcal{A}$, non-commutative, associative algebra (e.g. matrix algebra).
- \hat{p}_i : inner derivations. Adjoint action, $\hat{p}_i = \hbar \omega_{ij} [\hat{x}^j, \cdot]$. ω_{ij} : symplectic 2-form; $\theta^{ij} \omega_{jk} = -\delta^i_k$.
- Jacobi identities and Leibniz rule.



An alternative picture

Ansatz:
$$\hat{p}_i = \hbar \omega_{ij} (\hat{x}^j + \hat{y}^j)$$
 . Nair, Polychronakos '00

Assumption:
$$[\hat{x}^i, \hat{y}^j] = 0 \implies [\hat{y}^i, \hat{y}^j] = -i\theta^{ij}$$
.

 \rightarrow momenta need two mutually commuting copies of A.

Equivalence to previous picture:
$$\hat{y}^i = -\hat{x}_R^i \quad \Rightarrow \quad \hat{p}_i = \hbar \omega_{ij} (\hat{x}_L^j - \hat{x}_R^j)$$
.

Gross, Nekrasov '00

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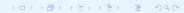
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Message

Actions via commutators always involve two copies of A, namely A_L and A_R .



Compact case - Torus

Periodicity condition: $\hat{x}^i \sim \hat{x}^i + 2\pi R^j \delta^i_j$.

 \rightsquigarrow \hat{x}^i are not the position operators anymore (not single-valued, not observables).

Physical operators of position: $X^i = e^{\frac{i\hat{X}^i}{R^i}}$.

Phase space of compact case: cf. Connes, Douglas, Schwarz '97

$$X^{i}X^{j} = e^{-\frac{i\theta^{ij}}{R^{i}R^{j}}}X^{j}X^{i},$$

$$\hat{p}_{i}X^{j} = X^{j}(\hat{p}_{i} + \frac{\hbar}{R^{j}}\delta^{j}_{i}),$$

$$[\hat{p}_{i}, \hat{p}_{i}] = 0.$$

Representations and quantum bundles...Brace, Morariu, Zumino '98; Morariu, Polychronakos '01

Q: Can we do more than planar and toroidal?

Interlude: Symplectic duality (or L vs. R)

Test R-operators on
$$f\in\mathcal{A}$$
 (note: curved indices): $[\hat{x}_R^a,\hat{x}_R^b]f=-i\theta^{ab}f+i[\theta^{ab},f]$.

Two assumptions:

- $f \in \mathcal{A}_L$. Justified.
- ullet $heta^{ab} \in \mathcal{A}_R$.

Full \hat{x} -relations:

$$[\hat{x}_L^a, \hat{x}_L^b] = i\theta^{ab} \;, \quad [\hat{x}_L^a, \hat{x}_R^b] = 0 \;, \quad [\hat{x}_R^a, \hat{x}_R^b] = -i\theta^{ab} \;.$$

Symplectic duality (if manifold with ω/θ , dual with $-\omega/-\theta$) . Bates, Weinstein

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Gravitational field

Assume a non-trivial frame e^{i}_{a} , as in the treatment of gravity as a gauge theory.

Extension of canonical commutation relations: Burić, Madore

$$[\hat{x}^a, \hat{p}_i] = i\hbar e^a_{\ i}(\hat{x}^b) \ ,$$

with non-commutative frame $e^{i}_{a}(\hat{x})$.

No assumption on L- or R-dependence of frame. Let consistency decide.

Assumptions

 $m extsf{ iny}$ Parallelizability of classical manifold. Global 1-forms $e^i=e^i_{\ a}(x){\rm d} x^a$,

$$\mathrm{d} e^i = - \tfrac{1}{2} f^i_{jk} e^j \wedge e^k \quad \Rightarrow \quad f^i_{jk} = 2 e^a_{\ [j} e^b_{\ k]} \partial_b e^i_{\ a} \; .$$

✓ Symplectic structure. Non-degenerate, closed 2-form $\omega = \frac{1}{2}\omega_{ij}e^i \wedge e^j$, constant in the global basis. 2-vector $\theta = \frac{1}{2}\theta^{ij}\theta_i \wedge \theta_j$ (θ_i : the dual vectors),

$$\theta^{ij} = -(\omega^{-1})^{ij} .$$

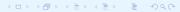
✓ Leibniz rule.

$$[f,gh]=g[f,h]+[f,g]h$$
, $f,g,h\in\mathcal{A}_L$.

Jacobi identities.

$$Jac(f,g,h) := [f,[g,h]] + [h,[f,g]] + [g,[h,f]] = 0$$
.

• Since A_R will play a role too, extend validity to full $A_L \times A_R$.



The operator algebra

• Position commutator \sim components of symplectic 2-vector.

$$[\hat{x}^a, \hat{x}^b] = i L^2 \theta^{ab} .$$

$$(L^2 \sim G_N \hbar \quad \text{or} \quad \alpha' \quad \text{or} \dots \text{ (here 1)}).$$

• Mixed commutator \sim extended canonical commutation relation.

$$[\hat{x}^a, \hat{p}_i] = i\hbar e^a_i$$
.

Momentum commutator ~ unspecified, to be determined.

$$[\hat{p}_i,\hat{p}_j]=i\frac{\hbar^2}{L^2}F_{ij} \ .$$

Determining the momenta

Consistency yields:

$$\hat{p}_i = \hbar e^a_{\ i} \omega_{ab} (\hat{x}^b_L - \hat{x}^b_R) \ ,$$

appropriately ordered when necessary. It requires $e^a_i(\hat{\mathbf{x}}^b_R)$.

The momentum commutator is determined and it is quadratic: cf. Madore

$$[\hat{p}_i,\hat{p}_j] = M_{ij} + N_{ij}^{k} \hat{p}_k + P_{ij}^{kl} \hat{p}_k \hat{p}_l.$$

Coefficients:

$$\begin{split} P^{kl}_{ij} &= e^k_{\ c} e^l_{\ d} L^{cd}_{[ij]} \;, \\ N_{ij}^{\ k} &= \hbar \omega_{bd} e^k_{\ c} \bigg(2 K^{cb}_{[i} e^d_{\ j]} + P^{ml}_{ij} \big(K^{cb}_{l} e^d_{\ m} + K^{db}_{(m} e^c_{\ l)} \big) \bigg) \;, \\ M_{ij} &= -\frac{\hbar^2}{4} \omega_{ac} \omega_{bd} P^{kl}_{ij} K^{ca}_{k} K^{db}_{l} \;, \end{split}$$

with the definitions: $[e^a_{\ i},\hat{x}^b_R]=K^{ab}_i$, $[e^a_{\ i},e^b_{\ j}]=L^{ab}_{ij}$.

Additionally: $[\hat{x}_R^a, \hat{p}_i] = i\hbar e^a_i - e^k_b K_i^{ba} \hat{p}_k \longrightarrow L$ and R asymmetry.



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Non-triviality?

Are the assumptions so strong that only the trivial (planar and toroidal) cases survive?

e.g. spheres are out. Quantized with different techniques Madore, Ramgoolam, ...

A large pool of candidates: nilmanifolds (parallelizable).

4D and 6D classification: 3+26 symplectic nilmanifolds (2+25 if tori are counted out).

Goze, Khakimdjanov '96

Odd-dimensional too, via symplectic leaves, with deformation quantization. e.g. Rieffel '89

Bird's eye view on nilmanifolds

As compact manifolds: Iterated toroidal fibrations.

Associated to nilpotent Lie algebras $(ab, cd, \ldots, yz) \rightarrow f^1_{ab}, f^2_{cd}, \ldots, f^d_{yz}$. Non-compact group manifolds.

Nipotency step \sim fiber iteration. Step 1 \rightarrow torus (Kähler nilmanifold).

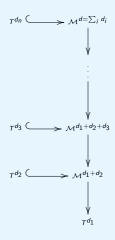
Benson, Gordon '88

Step 2 to d-1 \rightarrow non-Kähler, often symplectic.

Always exists a global basis e^i .

Frame components:

$$\begin{array}{lll} e^{i}_{a} & = & \delta^{i}_{a} + \#f^{i}_{ab}x^{b} + \#f^{i}_{bc}f^{b}_{ad}x^{c}x^{d} + \\ & + & \#f^{i}_{bc}f^{b}_{pd}f^{p}_{aq}x^{c}x^{d}x^{q} + \#f^{i}_{bc}f^{b}_{pd}f^{p}_{rq}f^{r}_{as}x^{c}x^{d}x^{q}x^{s} \ . \end{array}$$



Step classification of symplectic nilmanifolds

4D

<u>Class</u>	Step	Symplectic form
(0,0,0,12)	2	$e^{14} + e^{23}$
(0,0,42,12)	3	$e^{14} + e^{23}$

6D - Step 2

Class	Symplectic form
(0,0,0,0,0,12)	$e^{16} + e^{23} + e^{45}$
(0,0,0,0,13+42,14+23)	$e^{16} + e^{25} + e^{34}$
(0,0,0,0,12,13)	$e^{16} + e^{25} + e^{34}$
(0,0,0,0,12,34)	$e^{15} + e^{36} + e^{24}$
(0,0,0,0,12,14+23)	$e^{13} + e^{26} + e^{45}$
(0,0,0,12,13,23)	$e^{15} + e^{24} + e^{36}$

6D - Step 3

Class	Symplectic form
(0,0,0,0,12,14+25)	$e^{13} + e^{26} + e^{45}$
(0,0,0,0,12,15)	$e^{16} + e^{25} + e^{34}$
(0,0,0,12,14+23,13+42)	$e^{15} + 2e^{26} + e^{34}$
(0,0,0,12,14,13+42)	$e^{15} + e^{26} + e^{34}$
(0,0,0,12,14,23+24)	$e^{16} - e^{34} + e^{25}$
(0,0,0,12,13,14)	$e^{16} + e^{24} + e^{35}$
(0,0,0,12,13,24)	$e^{26} + e^{14} + e^{35}$
(0,0,0,12,13,14+23)	$e^{16} - 2e^{34} - e^{25}$

Step classification of symplectic nilmanifolds

6D - Step 4

Symplectic form
$e^{16} + e^{35} + e^{24}$
$e^{13} + e^{26} - e^{45}$
$e^{15} + e^{24} + e^{34} - e^{26}$
$e^{15} + e^{24} - e^{35} + e^{16}$
$e^{15} + e^{24} + e^{35} + e^{16}$

6D - Step 5

Class	Symplectic form
(0,0,12,13,14,15)	$e^{16} + e^{34} - e^{25}$
(0,0,12,13,14,15+23)	$e^{16} + e^{34} + e^{24} - e^{25}$
(0,0,12,13,14+23,15+24)	$e^{16} + 2e^{34} - e^{25}$

Particulars of step 2 cases

Frame is simple:
$$e^i_{\ a}=\delta^i_{\ a}+\frac{1}{2}\kappa_{(ab)}f^i_{\ ab}\hat{x}^b_R$$
 , $\kappa_{(ab)}+\kappa_{(ba)}=2$.

$$K$$
 and L parameters: $K_i^{ab} = -\frac{i}{2}\kappa_{(ic)}f^a_{\ ic}\theta^{bc}$ and $L^{ab}_{ij} = \frac{i}{4}\kappa_{(ic)}\kappa_{(jd)}f^a_{\ ic}f^b_{\ jd}\theta^{dc}$.

The coefficients of the momentum commutator:

$$M_{ij} = 0 \ , \quad N_{ij}^{\ k} = -i\hbar f^k_{\ ij} \ , \quad P^{kl}_{ij} = \tfrac{i}{4} \kappa_{(ic)} \kappa_{(jd)} f^k_{\ [i\underline{c}} f^l_{\ j]d} \theta^{dc} \ . \label{eq:mass_eq}$$

→ there are quadratic cases already at step 2.

Additionally:
$$[\hat{x}_R^a,\hat{p}_i]=i\hbar e^a_{\ i}+rac{i}{2}\kappa_{(ic)}f^k_{\ ic}\theta^{ac}\hat{p}_k$$
 .

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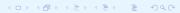
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 .

The Jacobi identities are always satisfied, but they are not always trivial. E.g., although $Jac(\hat{p}_i, \hat{p}_j, \hat{p}_k) = 0$ identically, on the other hand

$$\mathsf{Jac}(\hat{p}_{i},\hat{p}_{j},\hat{x}^{a}) = 0 \quad \Rightarrow \quad [e^{a}_{\ i},\hat{p}_{j}] - [e^{a}_{\ j},\hat{p}_{i}] = i\hbar f^{a}_{\ ij} - 2P^{kl}_{ij}e^{a}_{\ (k}\hat{p}_{l)} \ ,$$

which at this level is just a constraint.



A benchmark case

$$(0,0,0,0,13+42,14+23)$$
: 6D, step 2.

1-forms:

$$e^i = \mathrm{d} x^i \;,\; i = 1,...,4 \;, \quad e^5 = \mathrm{d} x^5 + x^3 \mathrm{d} x^1 - x^4 \mathrm{d} x^2 \;, \quad e^6 = \mathrm{d} x^6 + x^4 \mathrm{d} x^1 + x^3 \mathrm{d} x^2 \;.$$

Dual vectors:

$$\theta_1 = \partial_1 - x^3 \partial_5 - x^4 \partial_6 \ , \quad \theta_2 = \partial_2 + x^4 \partial_5 - x^3 \partial_6 \ , \quad \theta_i = \partial_i \ , \ i = 3, \ldots, 6 \ .$$

Symplectic 2-form: $e^{16} + e^{25} + e^{34}$.

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Inverse non-commutative frame components:

$$e_1^5 = -\hat{x}_R^3$$
, $e_2^5 = \hat{x}_R^4$, $e_1^6 = -\hat{x}_R^4$, $e_2^6 = -\hat{x}_R^3$.

The momenta:

$$\hat{\rho}_{1} = \hbar([\hat{x}^{6}, \cdot] + \hat{x}_{R}^{3}[\hat{x}^{2}, \cdot] + \hat{x}_{R}^{4}[\hat{x}^{1}, \cdot]), \quad \hat{\rho}_{2} = \hbar([\hat{x}^{5}, \cdot] - \hat{x}_{R}^{4}[\hat{x}^{2}, \cdot] + \hat{x}_{R}^{3}[\hat{x}^{1}, \cdot]),
\hat{\rho}_{3} = \hbar[\hat{x}^{4}, \cdot], \quad \hat{\rho}_{4} = -\hbar[\hat{x}^{3}, \cdot], \quad \hat{\rho}_{5} = -\hbar[\hat{x}^{2}, \cdot], \quad \hat{\rho}_{6} = -\hbar[\hat{x}^{1}, \cdot].$$

The commutation relations:

✓ Positions:

$$[\hat{x}^1, \hat{x}^6] = [\hat{x}^2, \hat{x}^5] = [\hat{x}^3, \hat{x}^4] = i$$
.

Mixed I:

$$[\hat{x}^5, \hat{p}_1] = [\hat{x}^6, \hat{p}_2] = -i\hbar \hat{x}_R^3 \ , \quad [\hat{x}^5, \hat{p}_2] = -[\hat{x}^6, \hat{p}_1] = i\hbar \hat{x}_R^4 \ .$$

Momenta:

$$\begin{split} \left[\hat{\rho}_{1}, \hat{\rho}_{3} \right] &= \left[\hat{\rho}_{4}, \hat{\rho}_{2} \right] = -i\hbar\hat{\rho}_{5} \;, \quad \left[\hat{\rho}_{1}, \hat{\rho}_{4} \right] = \left[\hat{\rho}_{2}, \hat{\rho}_{3} \right] = -i\hbar\hat{\rho}_{6} \;, \\ \left[\hat{\rho}_{1}, \hat{\rho}_{2} \right] &= i(\hat{\rho}_{5})^{2} + i(\hat{\rho}_{6})^{2} \qquad \rightsquigarrow \text{quadratic.} \end{split}$$

Mixed II:

$$[\hat{x}_R^3, \hat{p}_2] = [\hat{x}_R^4, \hat{p}_1] = i\hat{p}_5 , \qquad [\hat{x}_R^3, \hat{p}_1] = -[\hat{x}_R^4, \hat{p}_2] = i\hat{p}_6 , [\hat{x}_R^5, \hat{p}_1] = [\hat{x}_R^6, \hat{p}_2] = -i\hbar\hat{x}_R^3 , \qquad [\hat{x}_R^5, \hat{p}_2] = -[\hat{x}_R^6, \hat{p}_1] = i\hbar\hat{x}_R^4 .$$

✓ Jacobi identities: e.g.

$$\begin{aligned} [[\hat{\rho}_1, \hat{\rho}_2], \hat{x}^5] &= 2\hbar \hat{\rho}_5 , \\ [[\hat{x}^5, \hat{\rho}_1], \hat{\rho}_2] &= -\hbar \hat{\rho}_5 , \\ [[\hat{\rho}_2, \hat{x}^5], \hat{\rho}_1] &= -\hbar \hat{\rho}_5 . \end{aligned}$$

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Compactification of nilmanifolds

Classically

Tori: $\mathbb{R}^d/\mathbb{Z}^d$ \rightsquigarrow identifications: $x^i \sim x^i + 2\pi R^j \delta^i_i$.

Nilmanifolds: \mathcal{N}/Γ , \mathcal{N} nilpotent, Γ discrete co-compact.

Identifications? e.g. for step 2: $e^i = (\delta^i_a + \frac{1}{2}\kappa_{(ab)}f^i_{ab}x^b)\mathrm{d}x^a$.

$$\begin{array}{ll} \text{Invariance under shifts} & \Rightarrow & x^a \sim x^a + 2\pi R^i \tau^a_{\ i} \ , \\ \text{with} \ \tau^a_{\ i} = \delta^a_{\ i} + \frac{1}{2} \kappa_{(bi)} f^a_{\ ib} x^b & \text{(cf. } \mathrm{e}^a_{\ i} = \delta^a_{\ i} - \frac{1}{2} \kappa_{(bi)} f^a_{\ ib} x^b) \ . \end{array}$$

Periodicity: $\hat{x}_R^a \sim \hat{x}_R^a + 2\pi R^i \tau_i^a (\hat{x}_R^b)$.

Exponentiate position operators: $X^a = e^{\frac{i\hat{X}^a}{R^a}}$.

New algebra:

$$\begin{array}{rcl} X^a X^b & = & e^{-\frac{i\theta^{ab}}{R^aR^b}} X^b X^a \ , \\ \hat{p}_i X^a & = & X^a \big(\hat{p}_i + \frac{\hbar}{R^a} e^a_{\ i} \big) \ . \end{array}$$

"Non-commutatively"

Periodicity: $\hat{x}_{P}^{a} \sim \hat{x}_{P}^{a} + 2\pi R^{i} \tau^{a}_{i}(\hat{x}_{P}^{b})$.

Exponentiate position operators: $X^a = e^{\frac{i\hat{X}^a}{R^a}}$

New algebra:

$$\hat{p}_i X^a = X^a (\hat{p}_i + \frac{\hbar}{R^a} e^a_i)$$
 .

 $X^a X^b = e^{-\frac{i\theta^{ab}}{R^a R^b}} X^b X^a$.

Recall benchmark case: $e^5 = dx^5 + x^3 dx^1 - x^4 dx^2$, $e^6 = dx^6 + x^4 dx^1 + x^3 dx^2$, which means that classically:

$$x^{3} \to x^{3} + 2\pi R^{3} \Rightarrow x^{5} \to x^{5} - 2\pi R^{3} x^{1}, \quad x^{6} \to x^{6} - 2\pi R^{3} x^{2},$$

 $x^{4} \to x^{4} + 2\pi R^{4} \Rightarrow x^{5} \to x^{5} + 2\pi R^{4} x^{2}, \quad x^{6} \to x^{6} - 2\pi R^{4} x^{1}.$

Inverse frame (enters algebra)

$$e_1^5 = e_2^6 = -\hat{x}_R^3$$
,
 $e_2^5 = -e_1^6 = \hat{x}_R^4$.

The frame (enters algebra)
$$\tau$$
—coefficients (enter periodicity conditions) $\tau_1 = e^6_2 = -\hat{x}_R^3$, $\tau_2^5 = -e^6_1 = \hat{x}_R^4$. $\tau_3^5 = \tau_4^6 = -\hat{x}_R^1$, $\tau_4^5 = -\tau_3^6 = \hat{x}_R^2$.

Main messages

- ightharpoonup Phase space + non-trivial frame \rightarrow consistent algebraic structures.
- ✓ Many examples, at least the nilmanifold class.
- \checkmark Even though one might be looking at \mathcal{A}_L , all $\mathcal{A}_L \times \mathcal{A}_R$ is important.

Main messages

- ightharpoonup Phase space + non-trivial frame o consistent algebraic structures.
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Things to do

- Calculate curvature. E.g. for nilmanifolds, classically: $R_{ab} = \frac{1}{2} (\frac{1}{2} f_a^{\ cd} f_{bcd} f_{ac}^{\ d} f_{db}^{\ c} f_{ad}^{\ c} f_{bc}^{\ d})$. Corrections due to non-commutativity?
- Beyond symplectic.
 All 6D nilmanifolds admit generalized complex structures Cavalcanti, Gualtieri '04
 Dirac structures (classified), coexistence of all flux types A.C., Jonke, Lechtenfeld '13
- Include sources.
- More parallelizable examples; solvmanifold class. Beyond parallelizability.
- Dynamics of non-commutative phase space?