Phase space quantization, non-commutativity and the gravitational field

Athanasios Chatzistavrakidis

Institut für Theoretische Physik, Leibniz Universität Hannover

Based on: arXiv:1404.2812 [hep-th]

Bayrischzell Workshop 2014

Main goal

Study the algebraic structure of phase space in non-commutative geometry, in the presence of a non-trivial frame (gravitational field). cf. Burić, Madore

Main goal

Study the algebraic structure of phase space in non-commutative geometry, in the presence of a non-trivial frame (gravitational field). cf. Burić, Madore

Main assumptions

- $\rightsquigarrow$ Parallelizability.
- $\rightsquigarrow$ Symplectic structure.
- $\rightsquigarrow$ Leibniz rule.
- $\rightsquigarrow$ Jacobi identities.

Main goal

Study the algebraic structure of phase space in non-commutative geometry, in the presence of a non-trivial frame (gravitational field). cf. Burić, Madore

Main assumptions

- $\rightsquigarrow$ Parallelizability.
- $\rightsquigarrow$ Symplectic structure.
- $\rightsquigarrow$ Leibniz rule.
- $\rightsquigarrow$ Jacobi identities.

Main results

- ✔ Role of left and right acting operators, symplectic duality.
- ✔ Extended algebras, quadratic in momenta. Madore
- ✔ Class of non-trivial examples, symplectic nilmanifolds.

The plan

1. Reminder on non-commutative plane and torus
2. Non-trivial frames and phase space
3. Application to symplectic nilmanifolds
4. Compact case
5. Main messages and future directions

The plan

1. Reminder on non-commutative plane and torus
2. Non-trivial frames and phase space
3. Application to symplectic nilmanifolds
4. Compact case
5. Main messages and future directions

Classical vs. Quantum

In classical mechanics:

- Phase space of d-dimensional space $\rightsquigarrow$ 2d-dimensional, (x^a, p_a) .
- Symplectic structure, $\omega = \delta_a^b \mathrm{d}x^a \wedge \mathrm{d}p_b$.
- Poisson bracket, $\{f, g\} = \delta_a^b(\partial_{x^b} f \partial_{p_a} g - \partial_{p_a} f \partial_{x^b} g)$.

Players: $\mathrm{d}x^a, \mathrm{d}p_a, \partial_{x^a}, \partial_{p_a}$.

Classical vs. Quantum

In classical mechanics:

- Phase space of d-dimensional space $\rightsquigarrow$ 2d-dimensional, (x^a, p_a) .
- Symplectic structure, $\omega = \delta_a^b \mathrm{d}x^a \wedge \mathrm{d}p_b$.
- Poisson bracket, $\{f, g\} = \delta_a^b (\partial_{x^b} f \partial_{p_a} g - \partial_{p_a} f \partial_{x^b} g)$.

Players: $\mathrm{d}x^a, \mathrm{d}p_a, \partial_{x^a}, \partial_{p_a}$.

In quantum mechanics:

- Hermitian operators $\hat{x}^a$ and $\hat{p}_a$.
- "Complex structure": $\hat{x} \to \hat{p},\ \hat{p} \to -\hat{x}$

  (cf. role in matrix compactifications with flux) A.C., Jonke '12, '13
- Wave functions do not depend on both positions and momenta, but only on one set, depending on the chosen representation.

Simultaneous consideration of $\partial_{x^a}, \partial_{p_a}$ is unnecessary and redundant.

Quantum vs. Non-commutative

Phase space of quantum mechanics (use flat indices $i, j, \ldots$; curved indices are $a, b, \ldots$):

$$[\hat{x}^i, \hat{x}^j] = 0 \ , \quad [\hat{x}^i, \hat{p}_j] = i\hbar\delta^i_j \ , \quad [\hat{p}_i, \hat{p}_j] = 0 \ .$$

- Position rep.: $x, -i\hbar\partial_x$.
- Momentum rep.: $p, i\hbar\partial_p$.
- Momenta are outer derivations of the position algebra (position rep.).

Quantum vs. Non-commutative

Phase space of quantum mechanics (use flat indices $i, j, \ldots$; curved indices are $a, b, \ldots$):

$$[\hat{x}^i, \hat{x}^j] = 0 \ , \quad [\hat{x}^i, \hat{p}_j] = i\hbar\delta^i_j \ , \quad [\hat{p}_i, \hat{p}_j] = 0 \ .$$

- Position rep.: $x, -i\hbar\partial_x$.
- Momentum rep.: $p, i\hbar\partial_p$.
- Momenta are outer derivations of the position algebra (position rep.).

Phase space of non-commutative quantum mechanics (in absence of magnetic sources):

$$[\hat{x}^i, \hat{x}^j] = i\theta^{ij} \ , \quad [\hat{x}^i, \hat{p}_j] = i\hbar\delta^i_j \ , \quad [\hat{p}_i, \hat{p}_j] = 0 \ .$$

Duval, Horvathy '00; Nair, Polychronakos '00; Morariu, Polychronakos '01; Horvathy '02

- θ^{ij}: constant parameters; components of symplectic 2-vector.
- $\hat{x}^i \in \mathcal{A}$, non-commutative, associative algebra (e.g. matrix algebra).
- $\hat{p}_i$: inner derivations. Adjoint action, $\hat{p}_i = \hbar\omega_{ij}[\hat{x}^j, \cdot]$.
  ω_{ij}: symplectic 2-form; $\theta^{ij}\omega_{jk} = -\delta^i_k$.
- Jacobi identities and Leibniz rule.

An alternative picture

Ansatz: $\hat{p}_i = \hbar\omega_{ij}(\hat{x}^j + \hat{y}^j)$. Nair, Polychronakos '00

Assumption: $[\hat{x}^i, \hat{y}^j] = 0 \quad \Rightarrow \quad [\hat{y}^i, \hat{y}^j] = -i\theta^{ij}$.

$\rightsquigarrow$ momenta need two mutually commuting copies of $\mathcal{A}$.

Equivalence to previous picture: $\hat{y}^i = -\hat{x}^i_R \quad \Rightarrow \quad \hat{p}_i = \hbar\omega_{ij}(\hat{x}^j_L - \hat{x}^j_R)$.

Gross, Nekrasov '00

An alternative picture

Ansatz: $\hat{p}_i = \hbar\omega_{ij}(\hat{x}^j + \hat{y}^j)$. Nair, Polychronakos '00

Assumption: $[\hat{x}^i, \hat{y}^j] = 0 \quad \Rightarrow \quad [\hat{y}^i, \hat{y}^j] = -i\theta^{ij}$.

$\rightsquigarrow$ momenta need two mutually commuting copies of $\mathcal{A}$.

Equivalence to previous picture: $\hat{y}^i = -\hat{x}^i_R \quad \Rightarrow \quad \hat{p}_i = \hbar\omega_{ij}(\hat{x}^j_L - \hat{x}^j_R)$.

Gross, Nekrasov '00

Message

Actions via commutators always involve two copies of $\mathcal{A}$, namely $\mathcal{A}_L$ and $\mathcal{A}_R$.

Compact case - Torus

Periodicity condition: $\hat{x}^i \sim \hat{x}^i + 2\pi R^j \delta^i_j$.

$\rightsquigarrow$ $\hat{x}^i$ are not the position operators anymore (not single-valued, not observables).

Physical operators of position: $X^i = e^{\frac{i\hat{x}^i}{R^i}}$.

Phase space of compact case: cf. Connes, Douglas, Schwarz '97

$$
\begin{aligned}
X^i X^j &= e^{-\frac{i\theta^{ij}}{R^i R^j}} X^j X^i \ , \\
\hat{p}_i X^j &= X^j (\hat{p}_i + \tfrac{\hbar}{R^j} \delta^j_i) \ , \\
[\hat{p}_i, \hat{p}_j] &= 0 \ .
\end{aligned}
$$

Representations and quantum bundles...Brace, Morariu, Zumino '98; Morariu, Polychronakos '01

Q: Can we do more than planar and toroidal?

Interlude: Symplectic duality (or L vs. R)

Test R-operators on $f \in \mathcal{A}$ (note: curved indices): $[\hat{x}_R^a, \hat{x}_R^b] f = -i\theta^{ab} f + i[\theta^{ab}, f]$.

Two assumptions:

- $f \in \mathcal{A}_L$. Justified.
- $\theta^{ab} \in \mathcal{A}_R$.

Full $\hat{x}$-relations:

$$[\hat{x}_L^a, \hat{x}_L^b] = i\theta^{ab} \ , \quad [\hat{x}_L^a, \hat{x}_R^b] = 0 \ , \quad [\hat{x}_R^a, \hat{x}_R^b] = -i\theta^{ab} \ .$$

Symplectic duality (if manifold with ω/θ, dual with $-\omega/-\theta$) . Bates, Weinstein

The plan

Gravitational field

Assume a non-trivial frame ${e^i}_a$, as in the treatment of gravity as a gauge theory.

Extension of canonical commutation relations: Burić, Madore

$$[\hat{x}^a, \hat{p}_i] = i\hbar {e^a}_i(\hat{x}^b) \ ,$$

with non-commutative frame ${e^i}_a(\hat{x})$.

No assumption on L- or R-dependence of frame. Let consistency decide.

Assumptions

✔ Parallelizability of classical manifold. Global 1-forms $e^i = e^i{}_a(x)\mathrm{d}x^a$,

$$\mathrm{d}e^i = -\tfrac{1}{2} f^i{}_{jk} e^j \wedge e^k \quad \Rightarrow \quad f^i{}_{jk} = 2 e^a{}_{[j} e^b{}_{k]} \partial_b e^i{}_a \ .$$

✔ Symplectic structure. Non-degenerate, closed 2-form $\omega = \frac{1}{2}\omega_{ij} e^i \wedge e^j$, constant in the global basis. 2-vector $\theta = \frac{1}{2}\theta^{ij}\theta_i \wedge \theta_j$ (θ_i: the dual vectors),

$$\theta^{ij} = -(\omega^{-1})^{ij} \ .$$

✔ Leibniz rule.

$$[f, gh] = g[f, h] + [f, g]h \ , \quad f, g, h \in \mathcal{A}_L \ .$$

✔ Jacobi identities.

$$\mathrm{Jac}(f, g, h) := [f, [g, h]] + [h, [f, g]] + [g, [h, f]] = 0 \ .$$

- Since $\mathcal{A}_R$ will play a role too, extend validity to full $\mathcal{A}_L \times \mathcal{A}_R$.

The operator algebra

- Position commutator $\sim$ components of symplectic 2-vector.

$$[\hat{x}^a, \hat{x}^b] = iL^2\theta^{ab} \ .$$

$(L^2 \sim G_N\hbar \quad \text{or} \quad \alpha' \quad \text{or} \ \dots \ \text{(here 1)})$.

- Mixed commutator $\sim$ extended canonical commutation relation.

$$[\hat{x}^a, \hat{p}_i] = i\hbar e^a{}_i \ .$$

- Momentum commutator $\sim$ unspecified, to be determined.

$$[\hat{p}_i, \hat{p}_j] = i\frac{\hbar^2}{L^2}F_{ij} \ .$$

Determining the momenta

Consistency yields:

$$\hat{p}_i = \hbar e^a{}_i \omega_{ab} (\hat{x}^b_L - \hat{x}^b_R) \ ,$$

appropriately ordered when necessary. It requires $e^a{}_i(\hat{x}^b_R)$.

The momentum commutator is determined and it is quadratic: cf. Madore

$$[\hat{p}_i, \hat{p}_j] = M_{ij} + N_{ij}{}^k \hat{p}_k + P^{kl}_{ij} \hat{p}_k \hat{p}_l \ .$$

Coefficients:

$$\begin{aligned} P^{kl}_{ij} &= e^k{}_c e^l{}_d L^{cd}_{[ij]} \ , \\ N_{ij}{}^k &= \hbar \omega_{bd} e^k{}_c \left(2 K^{cb}_{[i} e^d{}_{j]} + P^{ml}_{ij} (K^{cb}_l e^d{}_m + K^{db}_{(m} e^c{}_{l)}) \right) \ , \\ M_{ij} &= -\frac{\hbar^2}{4} \omega_{ac} \omega_{bd} P^{kl}_{ij} K^{ca}_k K^{db}_l \ , \end{aligned}$$

with the definitions: $[e^a{}_i, \hat{x}^b_R] = K^{ab}_i \ , \quad [e^a{}_i, e^b{}_j] = L^{ab}_{ij}$.

Additionally: $[\hat{x}^a_R, \hat{p}_i] = i\hbar e^a{}_i - e^k{}_b K^{ba}_i \hat{p}_k \quad \rightsquigarrow$ L and R asymmetry.

The plan

Non-triviality?

Are the assumptions so strong that only the trivial (planar and toroidal) cases survive?

e.g. spheres are out. Quantized with different techniques Madore, Ramgoolam, ...

A large pool of candidates: nilmanifolds (parallelizable).

4D and 6D classification: 3+26 symplectic nilmanifolds (2+25 if tori are counted out).

Goze, Khakimdjanov '96

Odd-dimensional too, via symplectic leaves, with deformation quantization. e.g. Rieffel '89

Bird's eye view on nilmanifolds

As compact manifolds:
Iterated toroidal fibrations.

$$\begin{tikzcd}
T^{d_n} \arrow[r, hook] & \mathcal{M}^{d=\sum_i d_i} \arrow[d] \\
& \vdots \arrow[d] \\
T^{d_3} \arrow[r, hook] & \mathcal{M}^{d_1+d_2+d_3} \arrow[d] \\
T^{d_2} \arrow[r, hook] & \mathcal{M}^{d_1+d_2} \arrow[d] \\
& T^{d_1}
\end{tikzcd}$$

Associated to nilpotent Lie algebras
$(ab, cd, \dots, yz) \to f^1{}_{ab}, f^2{}_{cd}, \dots, f^d{}_{yz}$.
Non-compact group manifolds.

Nipotency step $\sim$ fiber iteration.
Step 1 $\to$ torus (Kähler nilmanifold).
Benson, Gordon '88

Step 2 to d-1$\to$ non-Kähler, often symplectic.

Always exists a global basis e^i.

Frame components:

$$\begin{aligned} e^i{}_a &= \delta^i{}_a + \# f^i{}_{ab} x^b + \# f^i{}_{bc} f^b{}_{ad} x^c x^d + \\ &+ \# f^i{}_{bc} f^b{}_{pd} f^p{}_{aq} x^c x^d x^q + \# f^i{}_{bc} f^b{}_{pd} f^p{}_{rq} f^r{}_{as} x^c x^d x^q x^s \ . \end{aligned}$$

Step classification of symplectic nilmanifolds

4D

Class	Step	Symplectic form
(0,0,0,12)	2	$e^{14} + e^{23}$
(0,0,42,12)	3	$e^{14} + e^{23}$

6D - Step 2

Class	Symplectic form
(0,0,0,0,0,12)	$e^{16} + e^{23} + e^{45}$
(0,0,0,0,13+42,14+23)	$e^{16} + e^{25} + e^{34}$
(0,0,0,0,12,13)	$e^{16} + e^{25} + e^{34}$
(0,0,0,0,12,34)	$e^{15} + e^{36} + e^{24}$
(0,0,0,0,12,14+23)	$e^{13} + e^{26} + e^{45}$
(0,0,0,12,13,23)	$e^{15} + e^{24} + e^{36}$

6D - Step 3

Class	Symplectic form
(0,0,0,0,12,14+25)	$e^{13} + e^{26} + e^{45}$
(0,0,0,0,12,15)	$e^{16} + e^{25} + e^{34}$
(0,0,0,12,14+23,13+42)	$e^{15} + 2e^{26} + e^{34}$
(0,0,0,12,14,13+42)	$e^{15} + e^{26} + e^{34}$
(0,0,0,12,14,23+24)	$e^{16} - e^{34} + e^{25}$
(0,0,0,12,13,14)	$e^{16} + e^{24} + e^{35}$
(0,0,0,12,13,24)	$e^{26} + e^{14} + e^{35}$
(0,0,0,12,13,14+23)	$e^{16} - 2e^{34} - e^{25}$

Step classification of symplectic nilmanifolds

6D - Step 4

Class	Symplectic form
(0,0,0,12,14-23,15+34)	$e^{16}+e^{35}+e^{24}$
(0,0,0,12,14,15)	$e^{13}+e^{26}-e^{45}$
(0,0,0,12,14,15+24)	$e^{13}+e^{26}-e^{45}$
(0,0,0,12,14,15+23+24)	$e^{13}+e^{26}-e^{45}$
(0,0,0,12,14,23+15)	$e^{13}+e^{26}-e^{45}$
(0,0,12,13,23,14)	$e^{15}+e^{24}+e^{34}-e^{26}$
(0,0,12,13,23,14-25)	$e^{15}+e^{24}-e^{35}+e^{16}$
(0,0,12,13,23,14+25)	$e^{15}+e^{24}+e^{35}+e^{16}$

6D - Step 5

Class	Symplectic form
(0,0,12,13,14,15)	$e^{16}+e^{34}-e^{25}$
(0,0,12,13,14,15+23)	$e^{16}+e^{34}+e^{24}-e^{25}$
(0,0,12,13,14+23,15+24)	$e^{16}+2e^{34}-e^{25}$

Particulars of step 2 cases

Frame is simple: $e^i{}_a = \delta^i{}_a + \frac{1}{2}\kappa_{(ab)} f^i{}_{ab} \hat{x}^b_R$, $\qquad \kappa_{(ab)} + \kappa_{(ba)} = 2$.

K and L parameters: $K^{ab}_i = -\frac{i}{2}\kappa_{(ic)} f^a{}_{ic} \theta^{bc}$ and $L^{ab}_{ij} = \frac{i}{4}\kappa_{(ic)}\kappa_{(jd)} f^a{}_{ic} f^b{}_{jd} \theta^{dc}$.

The coefficients of the momentum commutator:

$$M_{ij} = 0 \ , \quad N_{ij}{}^k = -i\hbar f^k_{ij} \ , \quad P^{kl}_{ij} = \frac{i}{4}\kappa_{(ic)}\kappa_{(jd)} f^k{}_{[i\underline{c}} f^l{}_{j]d} \theta^{dc} \ .$$

$\rightsquigarrow$ there are quadratic cases already at step 2.

Additionally: $[\hat{x}^a_R, \hat{p}_i] = i\hbar e^a{}_i + \frac{i}{2}\kappa_{(ic)} f^k{}_{ic} \theta^{ac} \hat{p}_k$.

Particulars of step 2 cases

Frame is simple: $e^i{}_a = \delta^i{}_a + \frac{1}{2}\kappa_{(ab)} f^i{}_{ab} \hat{x}^b_R$, $\qquad \kappa_{(ab)} + \kappa_{(ba)} = 2$.

K and L parameters: $K^{ab}_i = -\frac{i}{2}\kappa_{(ic)} f^a{}_{ic}\theta^{bc}$ $\quad$ and $\quad L^{ab}_{ij} = \frac{i}{4}\kappa_{(ic)}\kappa_{(jd)} f^a{}_{ic} f^b{}_{jd}\theta^{dc}$.

The coefficients of the momentum commutator:

$$M_{ij} = 0 \ , \quad N_{ij}{}^k = -i\hbar f^k_{ij} \ , \quad P^{kl}_{ij} = \frac{i}{4}\kappa_{(ic)}\kappa_{(jd)} f^k{}_{[i\underline{c}} f^l{}_{j]d}\theta^{dc} \ .$$

$\rightsquigarrow$ there are quadratic cases already at step 2.

Additionally: $[\hat{x}^a_R, \hat{p}_i] = i\hbar e^a{}_i + \frac{i}{2}\kappa_{(ic)} f^k{}_{ic}\theta^{ac}\hat{p}_k$.

The Jacobi identities are always satisfied, but they are not always trivial.
E.g., although $\mathrm{Jac}(\hat{p}_i, \hat{p}_j, \hat{p}_k) = 0$ identically, on the other hand

$$\mathrm{Jac}(\hat{p}_i, \hat{p}_j, \hat{x}^a) = 0 \quad \Rightarrow \quad [e^a{}_i, \hat{p}_j] - [e^a{}_j, \hat{p}_i] = i\hbar f^a_{ij} - 2P^{kl}_{ij} e^a{}_{(k}\hat{p}_{l)} \ ,$$

which at this level is just a constraint.

A benchmark case

$(0,0,0,0,13+42,14+23)$: 6D, step 2.

1-forms:

$$e^i = \mathrm{d}x^i \ , \ i = 1, \ldots, 4 \ , \quad e^5 = \mathrm{d}x^5 + x^3\mathrm{d}x^1 - x^4\mathrm{d}x^2 \ , \quad e^6 = \mathrm{d}x^6 + x^4\mathrm{d}x^1 + x^3\mathrm{d}x^2 \ .$$

Dual vectors:

$$\theta_1 = \partial_1 - x^3\partial_5 - x^4\partial_6 \ , \quad \theta_2 = \partial_2 + x^4\partial_5 - x^3\partial_6 \ , \quad \theta_i = \partial_i \ , \ i = 3, \ldots, 6 \ .$$

Symplectic 2-form: $e^{16} + e^{25} + e^{34}$.

A benchmark case

$(0,0,0,0,13+42,14+23)$: 6D, step 2.

1-forms:

$$e^i = \mathrm{d}x^i \ , \ i = 1, ..., 4 \ , \quad e^5 = \mathrm{d}x^5 + x^3 \mathrm{d}x^1 - x^4 \mathrm{d}x^2 \ , \quad e^6 = \mathrm{d}x^6 + x^4 \mathrm{d}x^1 + x^3 \mathrm{d}x^2 \ .$$

Dual vectors:

$$\theta_1 = \partial_1 - x^3 \partial_5 - x^4 \partial_6 \ , \quad \theta_2 = \partial_2 + x^4 \partial_5 - x^3 \partial_6 \ , \quad \theta_i = \partial_i \ , \ i = 3, \ldots, 6 \ .$$

Symplectic 2-form: $e^{16} + e^{25} + e^{34}$.

Inverse non-commutative frame components:

$$e^5{}_1 = -\hat{x}^3_R \ , \quad e^5{}_2 = \hat{x}^4_R \ , \quad e^6{}_1 = -\hat{x}^4_R \ , \quad e^6{}_2 = -\hat{x}^3_R \ .$$

The momenta:

$$\begin{aligned} \hat{p}_1 &= \hbar([\hat{x}^6, \cdot] + \hat{x}^3_R[\hat{x}^2, \cdot] + \hat{x}^4_R[\hat{x}^1, \cdot]) \ , \quad \hat{p}_2 = \hbar([\hat{x}^5, \cdot] - \hat{x}^4_R[\hat{x}^2, \cdot] + \hat{x}^3_R[\hat{x}^1, \cdot]) \ , \\ \hat{p}_3 &= \hbar[\hat{x}^4, \cdot] \ , \quad \hat{p}_4 = -\hbar[\hat{x}^3, \cdot] \ , \quad \hat{p}_5 = -\hbar[\hat{x}^2, \cdot] \ , \quad \hat{p}_6 = -\hbar[\hat{x}^1, \cdot] \ . \end{aligned}$$

The commutation relations:

- Positions:

$$[\hat{x}^1, \hat{x}^6] = [\hat{x}^2, \hat{x}^5] = [\hat{x}^3, \hat{x}^4] = i \ .$$

- Mixed I:

$$[\hat{x}^5, \hat{p}_1] = [\hat{x}^6, \hat{p}_2] = -i\hbar \hat{x}^3_R \ , \quad [\hat{x}^5, \hat{p}_2] = -[\hat{x}^6, \hat{p}_1] = i\hbar \hat{x}^4_R \ .$$

- Momenta:

$$\begin{aligned} [\hat{p}_1, \hat{p}_3] &= [\hat{p}_4, \hat{p}_2] = -i\hbar \hat{p}_5 \ , \quad [\hat{p}_1, \hat{p}_4] = [\hat{p}_2, \hat{p}_3] = -i\hbar \hat{p}_6 \ , \\ [\hat{p}_1, \hat{p}_2] &= i(\hat{p}_5)^2 + i(\hat{p}_6)^2 \qquad \rightsquigarrow \text{quadratic.} \end{aligned}$$

- Mixed II:

$$\begin{aligned} &[\hat{x}^3_R, \hat{p}_2] = [\hat{x}^4_R, \hat{p}_1] = i\hat{p}_5 \ , \qquad [\hat{x}^3_R, \hat{p}_1] = -[\hat{x}^4_R, \hat{p}_2] = i\hat{p}_6 \ , \\ &[\hat{x}^5_R, \hat{p}_1] = [\hat{x}^6_R, \hat{p}_2] = -i\hbar \hat{x}^3_R \ , \quad [\hat{x}^5_R, \hat{p}_2] = -[\hat{x}^6_R, \hat{p}_1] = i\hbar \hat{x}^4_R \ . \end{aligned}$$

- Jacobi identities: e.g.

$$\begin{aligned} [[\hat{p}_1, \hat{p}_2], \hat{x}^5] &= 2\hbar \hat{p}_5 \ , \\ [[\hat{x}^5, \hat{p}_1], \hat{p}_2] &= -\hbar \hat{p}_5 \ , \\ [[\hat{p}_2, \hat{x}^5], \hat{p}_1] &= -\hbar \hat{p}_5 \ . \end{aligned}$$

The plan

Compactification of nilmanifolds

Classically

Tori: $\mathbb{R}^d/\mathbb{Z}^d \quad \rightsquigarrow \quad$ identifications: $x^i \sim x^i + 2\pi R^j \delta^i_j$.

Nilmanifolds: $\mathcal{N}/\Gamma$, $\mathcal{N}$ nilpotent, Γ discrete co-compact.

Identifications? e.g. for step 2: $e^i = (\delta^i_a + \frac{1}{2}\kappa_{(ab)} f^i{}_{ab} x^b)\mathrm{d}x^a$.

Invariance under shifts $\quad \Rightarrow \quad x^a \sim x^a + 2\pi R^i \tau^a{}_i$,
with $\tau^a{}_i = \delta^a{}_i + \frac{1}{2}\kappa_{(bi)} f^a{}_{ib} x^b$ (cf. $e^a{}_i = \delta^a{}_i - \frac{1}{2}\kappa_{(ib)} f^a{}_{ib} x^b$) .

“Non-commutatively”

Periodicity: $\hat{x}^a_R \sim \hat{x}^a_R + 2\pi R^i \tau^a{}_i(\hat{x}^b_R)$.

Exponentiate position operators: $X^a = e^{\frac{i\hat{x}^a}{R^a}}$.

New algebra:

$$
\begin{aligned}
X^a X^b &= e^{-\frac{i\theta^{ab}}{R^a R^b}} X^b X^a \, , \\
\hat{p}_i X^a &= X^a (\hat{p}_i + \tfrac{\hbar}{R^a} e^a{}_i) \, .
\end{aligned}
$$

"Non-commutatively"

Periodicity: $\hat{x}^a_R \sim \hat{x}^a_R + 2\pi R^i \tau^a{}_i(\hat{x}^b_R)$.

Exponentiate position operators: $X^a = e^{\frac{i\hat{x}^a}{R^a}}$.

New algebra:

$$\begin{aligned} X^a X^b &= e^{-\frac{i\theta^{ab}}{R^a R^b}} X^b X^a , \\ \hat{p}_i X^a &= X^a(\hat{p}_i + \tfrac{\hbar}{R^a} e^a{}_i) . \end{aligned}$$

Recall benchmark case: $e^5 = \mathrm{d}x^5 + x^3\mathrm{d}x^1 - x^4\mathrm{d}x^2$, $e^6 = \mathrm{d}x^6 + x^4\mathrm{d}x^1 + x^3\mathrm{d}x^2$, which means that classically:

$$\begin{aligned} x^3 \to x^3 + 2\pi R^3 \quad &\Rightarrow \quad x^5 \to x^5 - 2\pi R^3 x^1 , \quad x^6 \to x^6 - 2\pi R^3 x^2 , \\ x^4 \to x^4 + 2\pi R^4 \quad &\Rightarrow \quad x^5 \to x^5 + 2\pi R^4 x^2 , \quad x^6 \to x^6 - 2\pi R^4 x^1 . \end{aligned}$$

Inverse frame (enters algebra)

$e^5{}_1 = e^6{}_2 = -\hat{x}^3_R$,
$e^5{}_2 = -e^6{}_1 = \hat{x}^4_R$.

τ-coefficients (enter periodicity conditions)

$\tau^5{}_3 = \tau^6{}_4 = -\hat{x}^1_R$,
$\tau^5{}_4 = -\tau^6{}_3 = \hat{x}^2_R$.

Main messages

- ✔ Phase space + non-trivial frame $\rightarrow$ consistent algebraic structures.
- ✔ Many examples, at least the nilmanifold class.
- ✔ Even though one might be looking at $\mathcal{A}_L$, all $\mathcal{A}_L \times \mathcal{A}_R$ is important.

Main messages

- ✔ Phase space + non-trivial frame → consistent algebraic structures.
- ✔ Many examples, at least the nilmanifold class.
- ✔ Even though one might be looking at $\mathcal{A}_L$, all $\mathcal{A}_L \times \mathcal{A}_R$ is important.

Things to do

- Calculate curvature.
  E.g. for nilmanifolds, classically: $R_{ab} = \frac{1}{2}\left(\frac{1}{2} f_a{}^{cd} f_{bcd} - f^d{}_{ac} f_{db}{}^c - f^c{}_{ad} f^d{}_{bc}\right)$.
  Corrections due to non-commutativity?
- Beyond symplectic.
  All 6D nilmanifolds admit generalized complex structures Cavalcanti, Gualtieri '04
  Dirac structures (classified), coexistence of all flux types A.C., Jonke, Lechtenfeld '13
- Include sources.
- More parallelizable examples; solvmanifold class. Beyond parallelizability.
- Dynamics of non-commutative phase space?