Abelian e Non-abelian Twisted Multiparticle Systems

Zhanna Kuznetsova

CMCC - Federal University of ABC
Santo Andre - Sao Paulo, Brazil

Noncommutative Workshop
Bayrischzell, May, 2014

Outline

Outline

Outline

Based on

- Z. K., M. Rojas and F. Toppan, **Int. J. Mod. Phys. A**23 (2008) 309.
- B. Chakraborty, Z. K. and F. Toppan, **JMP 51** (2010) 112102.
- P.G. Castro, B. Chakraborty, Z. K. and F. Toppan, **Central Eur. J. Phys. 9** (2011) 841.
- P.G. Castro, B. Chakraborty, R. Kullock and F. Toppan, **JMP 52** (2011) 032102.
- Z. K. and F. Toppan, **Eur. Phys. J. C 73** (2013) 2483.
- Z. K. and F. Toppan, *Nonabelian twist and Calogero conformal systems*, paper in preparation.

Introduction

Spacetime geometry is expected to exhibit discrete features at the Plank scale. A schematic approach to these phenomena is provided by noncommutative models in which spacetime coordinates are replaced by noncommuting operators.

In this talk we will consider space noncommutativity in non-relativistic systems

$$[x_i, x_j] = (\mathit{something})$$

This "something" is obtained as a result of a Drinfel'd twist deformation of a universal enveloping algebra $\mathcal{U}(\mathcal{G})$ with a Hopf algebra structure, where $\mathcal{G}$ is a Lie algebra containing the Heisenberg algebra $\mathcal{H} = \{\hbar, x_i, p_j\}$ as a subalgebra.

Notice that here the Planck constant $\hbar$ should be considered as an element of the algebra and not as a multiple of identity.

Hopf algebras

Let A be an associative algebra over a field $\mathcal{F}(= \mathbf{C})$ with an identity map $i : \mathcal{F} \to A$ and a product $\mu : A \otimes A \to A$.

A is called a **Hopf algebra** if

a) there exists a homomorphism $\Delta : A \to A \otimes A$, called coproduct, satisfying the coassociativity condition $(id \otimes \Delta)(A) = (\Delta \otimes id)(A)$, and

b) there exists a homomorphism $\varepsilon : A \to \mathcal{F}$ called counit and an antihomorphism $S : A \to A$ called antipode.

Diagrams relating coproduct, counit and antipode can be found, for example, in E. Abe, Hopf algebras, Cambridge tracts in mathematics 74, Cambridge Univ. Press, 1980.

The noncommutativity is obtained by deformation of the **universal enveloping algebra** $\mathcal{U}(\mathcal{G})$ with its Hopf algebraic structure.
The generators g_i of $\mathcal{G}$ are called the **primitive elements**.
For the primitive elements $g_i \in \mathcal{G}$ the undeformed costructures and the antipode are

$$\begin{aligned} \Delta(g_i) &= g_i \otimes \mathbf{1} + \mathbf{1} \otimes g_i \\ \varepsilon(g_i) &= 0 \\ S(g_i) &= -g_i \end{aligned}$$

For the identity $\mathbf{1} \in \mathcal{U}(\mathcal{G})$ the costructures and the antipode are

$$\begin{aligned} \Delta(\mathbf{1}) &= \mathbf{1} \otimes \mathbf{1} \\ \varepsilon(\mathbf{1}) &= 1 \\ S(\mathbf{1}) &= \mathbf{1} \end{aligned}$$

The other elements of $\mathcal{U}(\mathcal{G})$ are called composite. The coproduct for, for example, $g_1 g_2$ ($g_1, g_2 \in \mathcal{G}$) is computed as

$$\Delta(g_1 g_2) = \Delta(g_1)\Delta(g_2) = (\mathbf{1} \otimes g_1 + g_1 \otimes \mathbf{1})(\mathbf{1} \otimes g_2 + g_2 \otimes \mathbf{1})$$
$$= \mathbf{1} \otimes g_1 g_2 + g_1 g_2 \otimes \mathbf{1} + g_1 \otimes g_2 + g_2 \otimes g_1$$

From the formula above it is clear that in physical models, when one applies the Hopf algebraic structure, all additive observable operators should be primitive elements of $\mathcal{G}$ and not composite.

Example: the energy of a 2-particle state $E^{(2)} = E_1 + E_2$.
If the Hamiltonian is a primitive element, this can be expressed as $\Delta(H) = \mathbf{1} \otimes H + H \otimes \mathbf{1}$.

Then, using the Drinfel'd twist, one is forced to twist not only $\mathcal{U}(\mathcal{H})$ (the enveloping Heisenberg algebra), but $\mathcal{U}(\mathcal{G})$ with $\mathcal{G}$ containing all elements which correspond to additive operators in commutative (non-deformed) case. We call this $\mathcal{G}$ the **unfolded Lie algebra**.

EXAMPLES:

- For the harmonic oscillator Hamiltonian we have

$$\mathcal{G} = \{x_i, p_i, \hbar, P_{ii}, X_{ii}, M_{ii}\},$$

  with $X_{ii} = \frac{1}{\hbar} x_i^2$, $P_{ii} = \frac{1}{\hbar} p_i^2$ and $M_{ii} = \frac{1}{\hbar} x_i p_i$.
- For a particle moving in constant electric $\vec{E}$ and magnetic B fields one should add $M_{ij} = \frac{1}{\hbar} x_i p_j + p_j x_i$ to the set of primitive elements:

$$\mathcal{G} = \{x_i, p_i, \hbar, P_{ii}, X_{ii}, M_{ij}\}.$$

- If at least one of the potential terms in the Hamiltonian contains a k-linear term (for $k \geq 3$), the enlarged algebra is necessarily **infinite-dimensional**. Examples: the anharmonic oscillator (x_i^4) or the Coulomb $\frac{1}{r}$ potential.

Hopf algebra structure $\mathcal{U}(\mathcal{G})$.

Let V be a Hilbert space.

Mapping

$$\rho \quad : \quad \mathcal{U}(\mathcal{G}) \to End(V).$$

Then

$$\Omega \in \mathcal{U}(\mathcal{G}) \quad \mapsto \quad \widehat{\Omega} = \rho(\Omega) \in End(V).$$

We have:

$\widehat{\Omega^{(1)}} = \widehat{\Omega}$, 1-particle operator,
$\widehat{\Omega^{(2)}} = \widehat{\Delta(\Omega)}$, 2-particle operator,
$\widehat{\Omega^{(3)}} = \widehat{(\Delta \oplus \mathbf{1})}(\Omega) = \widehat{(\mathbf{1} \oplus \Delta)}(\Omega)$, 3-particle operator,
$\ldots$.

Drinfel'd twist

Just to remember:

The **Drinfel'd twist** $\mathcal{F} \in \mathcal{U}(\mathcal{G}) \otimes \mathcal{U}(\mathcal{G})$ should satisfy cocycle and counitarity conditions
1) $(\mathbf{1} \otimes \mathcal{F})(id \otimes \Delta)\mathcal{F} = (\mathcal{F} \otimes \mathbf{1})(\Delta \otimes id)\mathcal{F}$
2) $(\varepsilon \otimes id)\mathcal{F} = \mathbf{1} = (id \otimes \varepsilon)\mathcal{F}$

The **deformed co-structures**, for $a \in \mathcal{U}(\mathcal{G})$, are:
$\Delta^{\mathcal{F}}(a) = \mathcal{F}\Delta(a)\mathcal{F}^{-1}$
$S^{\mathcal{F}}(a) = \chi S(a)\chi^{-1}$
with $\chi = \mu(id \otimes S)\mathcal{F}$.

In order for find linear subspace of $\mathcal{U}(\mathcal{G})$ one should calculate the **deformed generators** $g_i \mapsto g_i^{\mathcal{F}} = \bar{f}^{\alpha}(g_i)\bar{f}_{\alpha}$.
Here the Sweedler notation $\mathcal{F}^{-1} \equiv \bar{f}^{\alpha} \otimes \bar{f}_{\alpha}$ has been used.
Multiplication law for functions on NC space:
$m(f \otimes g) = fg \mapsto m^{\mathcal{F}}(f \otimes g) = m\left(\mathcal{F}^{-1} \triangleright (f \otimes g)\right) = f \star g$

Abelian Drinfel'd twist

$$\mathcal{F} = e^{i\alpha\epsilon_{ij}p_i \otimes p_j} \equiv f^{\beta} \otimes f_{\beta}$$

In $d = 2$ the twisted generators of $\mathcal{H}^{\mathcal{F}}$ have the form

$$\begin{aligned} x_1^{\mathcal{F}} &= x_1 - \alpha\hbar p_2 \\ x_2^{\mathcal{F}} &= x_2 + \alpha\hbar p_1 \end{aligned}$$

the other twisted generators are unchanged: $p_i^{\mathcal{F}} = p_i$ and $\hbar^{\mathcal{F}} = \hbar$.
The coordinate commutators are

$$[x_1^{\mathcal{F}}, x_2^{\mathcal{F}}] = 2i\hbar^2\alpha.$$

The **second order in** $\hbar$ in the r.h.s. shows that, in this model, the **NC appears in higher order in** $\hbar$ **(**$\mathcal{O}(\hbar^2)$**)**, while the ordinary quantum effects are of first order in $\hbar$ ($\mathcal{O}(\hbar)$).

Unfolded Lie algebra

$$\hat{\mathcal{G}} = \{x_i, p_j, \hbar, X_{ii}, X_S, P_{jj}, P_S, M_{ij}\}$$

where

$$\begin{aligned}
X_{ii} &= \frac{1}{\hbar} x_i^2 \\
X_S &= \frac{1}{\hbar}(x_1 x_2 + x_2 x_1) \\
P_{jj} &= \frac{1}{\hbar} p_j^2 \\
P_S &= \frac{1}{\hbar}(p_1 p_2 + p_2 p_1) \\
M_{ij} &= \frac{1}{\hbar}(x_i p_j + p_j x_i)
\end{aligned}$$

$$
\begin{aligned}
[x_i, p_j] &= i\hbar\delta_{ij} \\
[x_i, P_{ii}] &= 2ip_i \\
\left[x_i, M_{ji}\right] &= 2ix_j \quad (i \neq j) \\
[p_i, X_{ii}] &= -2ix_i \\
\left[p_i, M_{ij}\right] &= -2ip_j \quad (i \neq j) \\
[X_{ii}, P_{ii}] &= 2iM_{ii} \\
\left[X_{ii}, M_{ji}\right] &= 2iX_S \\
[X_S, P_{11}] &= 2iM_{21} \\
[X_S, P_{22}] &= 2iM_{12} \\
\left[X_S, M_{ij}\right] &= 2iM_{ii} \\
\left[P_{ii}, M_{ij}\right] &= -2iP_S \quad (i \neq j) \\
[P_S, M_{ii}] &= -2iP_S \\
\left[P_S, M_{ij}\right] &= -4iP_{jj} \\
\left[M_{ii}, M_{ij}\right] &= 2i\epsilon_{ij}M_{ij}
\end{aligned}
$$

A class of primitive elements $\Omega \in \mathcal{G}$:

$$\Omega \quad = \quad a(P_{11} + P_{22}) + b(X_{11} + X_{22}) + c(M_{12} - M_{21}) + dx_1 + fp_2,$$

for a, b, c, d, f arbitrary real parameters.
The $\widehat{\Omega}$ operators are Hermitian.

Three special cases:

i) for $b = 1$ and $a = c = d = f = 0$, Ω coincides with the **"squared radius"** $R^2 = X_{11} + X_{22}$;

ii) for $a = \frac{1}{2}$ and $b = \frac{1}{2}\omega^2$, Ω is associated to the Hamiltonian of the **harmonic oscillator** $H = \frac{1}{2}(P_{11} + P_{22}) + \frac{1}{2}\omega^2(X_{11} + X_{22})$;

iii) for $a = \frac{1}{2m}$, $b = \frac{m{\omega_c}^2}{8}$, $c = \frac{\omega_c}{4}$, $d = eE$, $f = 0$, we obtain the **Quantum Hall Effect Hamiltonian** in the presence of constant electric (E) and magnetic (B) fields (e is the electron's charge, $\omega_c = \frac{eB}{mc}$ is the cyclotron frequence).

The Untwisted General Operator

$$\Omega = a\,(P_{11} + P_{22}) + b(X_{11} + X_{22}) + c(M_{12} - M_{21}) + dx_1 + fp_2$$

For different choices of the constants a, b, c, d, f corresponds to **(I), (II), (III)**.

Twisted General Operator

$$\begin{aligned}\Omega^{\mathcal{F}} &= a(P^{11} + P^{22}) + b(X^{11} + X^{22}) + c(M^{12} - M^{21}) + dx_1 \\ &+ \alpha\left[2b(x_2p_1 - x_1p_2) - 2c(p_1^2 + p_2^2) - d\hbar p_2\right] \\ &+ \alpha^2 b\hbar(p_1^2 + p_2^2)\end{aligned}$$

Single-particle Quantization

$$\Omega^{\mathcal{F}} \in \mathcal{U}(\mathcal{G}) \to \widehat{\Omega^{\mathcal{F}}} \in End(V):$$

.

$$\widehat{\Omega}^{\mathcal{F}} = s(N+1) + tZ \qquad (s \geq |t|),$$

in terms of the commuting operators N, Z ($[N, Z] = 0$)

$$N = \hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2, \quad Z = i(\hat{a}_2 \hat{a}_1^\dagger - \hat{a}_1 \hat{a}_2^\dagger).$$

Eigenvalues: $n = 0, 1, 2, \ldots,$
$z = -n + 2j$ ($j = 0, 1, \ldots, n$).

For $s = |t|$, the vacuum is infinitely degenerate.
A unique vacuum solution exists for $s > |t|$.

Creation and annihilation operators

$$a_i^{(\lambda)} := \frac{1}{\sqrt{2}}(\lambda x_i + i\frac{p_i}{\lambda}) \quad , \quad a_i^{(\lambda)\dagger} := \frac{1}{\sqrt{2}}(\lambda x_i - i\frac{p_i}{\lambda}),$$

with λ suitably chosen.

The three cases:

i) the deformed squared radius operator $\widehat{R^2}^{\mathcal{F}}$

$$\lambda = \frac{1}{\sqrt{\alpha}} \quad , \quad s = t = 2\alpha$$

(one should note the singular limit for $\alpha \to 0$);

ii) the deformed hamiltonian $\widehat{H}^{\mathcal{F}}$ of the harmonic oscillator

$$\lambda = \sqrt[4]{\frac{\omega^2}{1+\alpha^2\omega^2}} \quad , \quad s = \omega\sqrt{1+\alpha^2\omega^2}, \qquad t = \alpha\omega^2;$$

iii) the deformed hamiltonian $\widehat{H_{QHE}}^{\mathcal{F}}$, in the presence of a constant magnetic field B

$$\lambda = \sqrt[4]{\frac{m\omega_c}{2 - m\alpha\omega_c}} \quad , \quad s = -t = \frac{1}{2}\omega_c(1 - \frac{\alpha\omega_c}{4}).$$

Different NC-quantizations for single-particle case in the literature:

i) the deformed squared radius operator
Scholtz, Gouba, Hafver, Rohwer, J. Phys. A (2009)
(quantization of the configuration space).

ii) the deformed hamiltonian of the harmonic oscillator
Kijanka, Kosinski, Phys. Rev. D (2004)

iii) the NC-quantum Hall Effect Hamiltonian
Dayi, Jellal, J. Math. Phys. (2002).

For the single-particle spectrum the “Unfolded Quantization” recovers the results in the literature.

Eigenvalues

The eigenvalues of N are $\{0, 1, 2, 3, \ldots, n, \ldots\}$.
For each engenvalue n of N there are $(n+1)$ eigenvalues of Z:
$\{-n, -n+2, \ldots, n-4, n-2, n\}$.

$$
\begin{aligned}
|v\rangle \quad &\longleftrightarrow \quad N|v\rangle = n|v\rangle \\
&\qquad\qquad\quad Z|v\rangle = j|v\rangle \\
|w\rangle = (\alpha a_1 + \beta a_2^\dagger)|v\rangle \quad &\longleftrightarrow \quad N|w\rangle = (n+1)|w\rangle \\
&\qquad\qquad\quad Z|w\rangle = (j-1)|w\rangle, \quad \textit{for} \quad \beta = i\alpha \\
&\qquad\qquad\quad Z|w\rangle = (j+1)|w\rangle, \quad \textit{for} \quad \beta = -i\alpha
\end{aligned}
$$

At n fixed the minimal eigenvalue of the operator Ω is for $j = -n$.
$\Omega = s(N+1) + tZ \Leftrightarrow$ min eigenvalue: $(s-t)n + s$.

- For $s > t$ (H_{osc} and H_{em}) there is a unique minimal eigenvalue for $n = 0$.
- For $s = t$ (R^2 operator) there is ∞ number of eigenstates which correspond to minimal eigenvalue (infinitely degenerate).

Square distance operator

Twisted square distance operator

$$(R^2)^{\mathcal{F}} = X_{11} + X_{22} + 2\alpha(x_2 p_1 - x_1 p_2) + \alpha^2 \hbar (p_1^2 + p_2^2)$$

Other set of oscillators ($[b, b^\dagger] = \hbar^2$)

$$b = \frac{1}{2\sqrt{\alpha}}(x_1^{\mathcal{F}} + i x_2^{\mathcal{F}})$$
$$b^\dagger = \frac{1}{2\sqrt{\alpha}}(x_1^{\mathcal{F}} - i x_2^{\mathcal{F}}),$$

and

$$(R^2)^{\mathcal{F}} = 2\alpha(bb^\dagger + b^\dagger b)$$

with eigenvalues $4\alpha n_b + 2\alpha$ for $N_b = b^\dagger b$.
This results are in accordance with another description of noncommutativity given by F. G. Scholtz, L. Gouba, A. Hafver and C. M. Rohwer, J. Phys. A 42, 175303 (2009).

Twisted Hamiltonians. One-particle state

Let us define $\hat{\Omega} \equiv \Omega^{\mathcal{F}}$ acting on the Hilbert space of one-particle states.
Here $\hbar = 1$.

$$\begin{aligned} \hat{\Omega} &= (a - 2\alpha c + \alpha^2 b)(p_1^2 + p_2^2) + b(x_1^2 + x_2^2) \\ &+ (2c - 2\alpha b)(x_1 p_2 - x_2 p_1) + d x_1 + (f - \alpha d) p_2 \end{aligned}$$

For the case (III), for example, $a = 1/2m$, $b = M^2/2m$, $c = M/2m$, $d = eE$, $f = 0$. ($M = \frac{eB}{2c}$)
The set of eigenvalues coincides with the one obtained in "fundamental" approach (many papers), see for example O. F. Dayi and A. Jellal, Hall Effect in Noncommutative Coordinates (2001).

2-particle state

The 2-particle state operator $((\Omega)^{\mathcal{F}})^{(2)}$ is obtained from $\Delta(\Omega^{\mathcal{F}})$.

$$\begin{aligned}\Omega^{\mathcal{F}} &= a(P^{11}+P^{22})+b(X^{11}+X^{22})+c(M^{12}-M^{21})+dx_1 \\ &+ \alpha\left[2b(x_2p_1-x_1p_2)-2c(p_1^2+p_2^2)-d\hbar p_2\right] \\ &+ \alpha^2 b\hbar(p_1^2+p_2^2)\end{aligned}$$

Defining Ω_0, Ω_1 and Ω_2 as

$$\Omega^{\mathcal{F}} \equiv \Omega_0+\alpha\Omega_1+\alpha^2\Omega_2$$

we have
$\Delta(\Omega^{\mathcal{F}}) = \Delta(\Omega_0)+2\alpha b\Delta(x_2p_1-x_1p_2)-2\alpha c\Delta(p_1^2+p_2^2)$
$-\alpha d\Delta(\hbar p_2)+\alpha^2 b\Delta[\hbar(p_2^2+p_2^2)]$

The resulting 2-particle operator splits into:

$$\begin{aligned}\left(\Omega^{\mathcal{F}}\right)^{(2)} &\equiv \Delta(\Omega^{\mathcal{F}}) \\ &= \Omega^{\mathcal{F}} \otimes 1 + 1 \otimes \Omega^{\mathcal{F}} + \Omega_r \otimes 1 + 1 \otimes \Omega_r + \hat{\Omega}_{mixed}\end{aligned}$$

where
$\Omega_r = -\alpha d p_2 + \alpha^2 b(p_1^2 + p_2^2)$ and
$\hat{\Omega}_{mixed} = -2\alpha b \epsilon_{ij}(x_i \otimes p_j + p_j \otimes x_i) + 4(-\alpha c + \alpha^2 b)(p_1 \otimes p_1 + p_2 \otimes p_2)$
The last term includes contributions from both particles.

Denoting $\Omega^{(2)} \equiv \hat{\Omega}' + \hat{\Omega}_{mixed}$, one could check whether $\hat{\Omega}'$ and $\hat{\Omega}_{mixed}$ commute.

$$
\begin{aligned}
& [\hat{\Omega}_{mixed}, \hat{\Omega}'] = -8i\alpha b(\alpha b - c)(x_i \otimes p_i + p_i \otimes x_i) \\
+ \quad & 4i\alpha d(-2\alpha b + c)(p_1 \otimes 1 + 1 \otimes p_1) - 2i\alpha bd(x_1 \otimes 1 + 1 \otimes x_2)
\end{aligned}
$$

The second line is equal to zero in the absence of electric field ($d = eE = 0$).

The first term is equal to zero if the noncommutative parameter is related to the magnetic field B as

$$\alpha = \frac{c}{b} = \frac{1}{|B|}\left(\frac{2c}{e}\right)^2.$$

”Fundamental” and twisted NC, 2-particle state

Let us consider the case (II) (Harmonic oscillator) and let $m = 1$.

Defining center of mass coordinates: $x_i^{c.m.} = 1/2(x_i^{(1)} + x_i^{(2)})$,

momenta $p_i^{c.m.} = p_i^{(1)} + p_i^{(2)}$ $(i = 1, 2)$,

relative coordinates: $x_i^{rel} = 1/2(x_i^{(2)} - x_i^{(1)})$,

and relative momenta $p_i^{rel} = p_i^{(2)} - p_i^{(1)}$.

- **(A)** Undeformed Hamiltonian (harmonic oscillator):
  $H = \frac{1}{2}(p_i^{c.m.})^2 + 2\omega^2(x_i^{c.m.})^2 + \frac{1}{2}(p_i^{rel})^2 + 2\omega^2(x_i^{rel})^2$
- **(B)** ”Fundamental” NC Hamiltonian:
  $H = \frac{1}{2}(1 + \alpha^2\omega^2)(p_i^{c.m.})^2 + 2\omega^2(x_i^{c.m.})^2 - 2\alpha^2\omega^2\epsilon_{ij}(x_i^{c.m.}p_j^{c.m})$
  $+\frac{1}{2}(1 + \alpha^2\omega^2)(p_i^{rel})^2 + 2\omega^2(x_i^{rel})^2 - 2\alpha\omega^2\epsilon_{ij}x_i^{rel}p_j^{rel}$
- **(C)** Twist deformed NC Hamiltonian:
  $H = (\frac{1}{2} + 2\alpha^2\omega^2)(p_i^{c.m.})^2 + 2\omega^2(x_i^{c.m.})^2 - 4\alpha\omega^2\epsilon_{ij}(x_i^{c.m.}p_j^{c.m.})$
  $+\frac{1}{2}(p_i^{rel})^2 + 2\omega^2(x_i^{rel})^2$
  In the 2-particle Hamiltonian obtained via twist, contrary to the ”fundamental” NC, the deformation appears only in the center of mass dynamics.

In one-particle state sector the ”fundamental” and ”twist-induced” noncommutative models coincide.

The difference appears in the multi-particle sector.

This is because the energy of multiparticle state is no longer additive but satisfies an associativity condition, induced by the coassociativity of the coproduct

$$\Delta(H^{\mathcal{F}}) = Id \otimes H^{\mathcal{F}} + H^{\mathcal{F}} \otimes Id + (\ldots),$$

where $(\ldots)$ **denotes the nonadditive extra terms.**

Energy eigenvalues in 2-particle case

- **(A)** Undeformed Hamiltonian:

$$E_{12} = 2\omega(n_1 + n_2) = 4\omega$$

- **(B)** ”Fundamental” NC Hamiltonian:

$$E_{12} = 2\omega\sqrt{1+\alpha^2\omega^2}(n_1 + n_2 + 2) + 2\alpha\omega^2(j_1 + j_2)$$

- **(C)** Twist deformed NC Hamiltonian:

$$E_{12} = 2\omega\sqrt{1+4\alpha^2\omega^2}(n_1 + 1) + 2\omega(n_2 + 1) + 4\alpha\omega^2 j_1$$

Here n_1 is associated with the center of mass coordinates while n_2 is asoociated with the relative coordinates
and $j = -n, -n+2, \ldots, n-2, n$.

Energy levels

”fundamental” NC	$4\omega\sqrt{1+\alpha^2\omega^2}$ $6\omega\sqrt{1+\alpha^2\omega^2}-2\alpha\omega^2$ $6\omega\sqrt{1+\alpha^2\omega^2}+2\alpha\omega^2$ $8\omega\sqrt{1+\alpha^2\omega^2}-4\alpha\omega^2$ $8\omega\sqrt{1+\alpha^2\omega^2}$ $8\omega\sqrt{1+\alpha^2\omega^2}+4\alpha\omega^2$ $\ldots$
”twist” NC	$4\omega+2\omega\sqrt{1+4\alpha^2\omega^2}$ $4\omega\sqrt{1+4\alpha^2\omega^2}\mp 4\alpha\omega^2$ $6\omega+2\omega\sqrt{1+4\alpha^2\omega^2}$ $4\omega+4\omega\sqrt{1+4\alpha^2\omega^2}\mp 4\alpha\omega^2$ $2\omega+6\omega\sqrt{1+4\alpha^2\omega^2}-8\alpha\omega^2$ $2\omega+6\omega\sqrt{1+4\alpha^2\omega^2}$ $2\omega+6\omega\sqrt{1+4\alpha^2\omega^2}+8\alpha\omega^2$ $\ldots$

Jordanian Twist

There are only two inequivalent deformations of $sl(2)$.
The first one (P. P. Kulish and N. Yu. Reshetikhin, J. Sov. Math. **23** (1983) 2435; M. Jimbo, Lett. Math. Phys. **10** (1985) 63), depends on a non-dimensional parameter q; it leads to the quantum group $\mathcal{U}_q(sl(2))$ and cannot be obtained from Drinfel'd twist technique.

The second one is called the Jordanian deformation of $sl(2)$. It can be obtained from the twist

$$\mathcal{F} = \exp\left(-iD \times \sigma\right)$$

where $\sigma = \ln(\mathbf{1} + \xi H)$, and H, D, K are $sl(2)$ generators.

Dubois-Violette and Launer (1990), Ohn (1992), Ogievetsky (1993), Kulish and Celeghini (1998), Boroviec, Lukierski and Tolstoy (2003).

In $sl(2) = \{D, H, K\}$ algebra

D is a dilatation operator, H is a positive and K is a negative root

$$\begin{aligned} [D, H] &= iH, \\ [D, K] &= -iK, \\ [H, K] &= -2iD. \end{aligned}$$

Differential realizations of $sl(2)$

.

- 1st order differential realization:

$$\begin{aligned} H &= i\partial_t, \\ D &= -it\partial_t + \beta, \\ K &= it^2\partial_t - 2\beta t; \end{aligned}$$

  from the hermiticity condition we have $\beta = -i/2 + \lambda$, $\lambda \in \mathbf{R}$.

- 2nd order differential realization:

$$\begin{aligned} H &= -\partial_x^2 + \frac{\rho}{x^2}, \\ D &= -\frac{i}{2}x\partial_x + c, \\ K &= \frac{1}{4}x^2, \end{aligned}$$

  for arbitrary ρ and $c = i/4$.
  The second one can act on d-dimensional space $(x_1, \ldots, x_d)$.

Two different cases for 2nd order differential realizations of $sl(2)$:

1. $\bar{\rho} = 0$ - free particle or harmonic oscillator Hamiltonian in non-deformed case.
   Unfolded algebra $\mathcal{G} = \{\hbar, x_i, p_j, H, D, K\}$.
2. $\bar{\rho} \neq 0$ - "Calogero type" Hamiltonian.
   Unfolded algebra $\mathcal{G}$ is infinite-dimensional
   $$\mathcal{G} = \{\hbar, x_i, p_j, H, D, K, \frac{1}{r^4}, \frac{x_i}{r^4}, \frac{1}{r^4}p_i + p_i\frac{1}{r^4}, \frac{x_i x_j + x_j x_i}{r^6}, \frac{x_i x_j + x_j x_i}{r^6}p_j + p_j\frac{x_i x_j + x_j x_i}{r^6}, \ldots\}$$

   All combinations written above should be multiplied by certain powers of $\hbar$ and considered as primitive elements of the algebra.

Connection with conformal mechanics

Different systems can be considered depending on the choice of Hamiltonian $\mathbf{H}$ (ρ is inside the twist)

1. free particle
   $\mathbf{H} \equiv H_0$, $\rho = 0$,
   $\mathbf{H} \equiv H_0$, $\rho \neq 0$.
2. harmonic oscillator
   $\mathbf{H} \equiv H_0 + K$, $\rho = 0$,
   $\mathbf{H} \equiv H_0 + K$, $\rho \neq 0$.
3. Calogero type of potential
   $\bar{\rho}/x^2 \to$ *Calogero* (Bellucci, Galajinsky and Krivonos, 2003)
   $\mathbf{H} \equiv H_{\bar{\rho}}$, $\rho = \bar{\rho}$,
   $\mathbf{H} \equiv H_{\bar{\rho}}$, $\rho \neq \bar{\rho}$.
4. Calogero potential in harmonic external force
   connection with free particle system: Brzezinski, C. Gonera, Maslanka, 1998.
   $\mathbf{H} \equiv H_{\bar{\rho}} + K$, $\rho = \bar{\rho}$,
   $\mathbf{H} \equiv H_{\bar{\rho}} + K$, $\rho \neq \bar{\rho}$.

$d = 1, 2, 3$ realizations.

d = 1. There is no noncommutativity in nonrelativistic models. But Jordanian transformation can be defined and investigated for better understanding of unfolded algebras.
The unfolded algebra is a subalgebra of the algebra of integer potentials of two types of generators (plus central element) which are either $[x, p] = i\hbar$ or $[a, a^\dagger] = \hbar$.

Denoting as a, b the operators and c the central element we have $\{a, b\} = c$ for Poisson brackets and $[a, b] = c$ for commutators.

d = 2,3,... we have Snyder noncommutative space.

- **For Poisson brackets** we have in $d = 1$
  $\{b^n a^p, b^m a^q\} = (mp - nq) b^{n+m-1} a^{p+q-1} c$
  or with $W_{n,p} \equiv \dfrac{b^n a^p}{c^{n+p-1}}$
  it can be written as
  $\{W_{n,p}, W_{m,q}\} = (mp - nq) W_{n+m-1,p+q-1}$.

  If the second index is $= 1$, defining $U_n \equiv W_{n+1,1}$ we see that it reproduces the centerless Virasoro algebra
  $\{U_n, U_m\} = (m - n) U_{m+n}$.
- **For commutators**

$$
\begin{aligned}
[b^n a^p, b^m a^q] &= b^n [a^p, b^m] a^q + b^m [b^n, a^q] a^p \\
[a^p, b^n] &= \sum_{j=1}^{n} \frac{p!}{(p-j)!} \begin{pmatrix} n \\ j \end{pmatrix} c^j b^{n-j} a^{p-j} \equiv c_{p,n}
\end{aligned}
$$

If $c = \hbar$ we have other potentials of $\hbar$ as it should be expected in quantization.

The Jordanian twist of $\mathcal{U}(\mathcal{G})$

The twist induces a deformation $g \mapsto g^{\mathcal{F}}$.
For any ρ the deformed generators are given by

$$\begin{aligned} x_i^{\mathcal{F}} &= x_i e^{\frac{\sigma}{2}}, \quad p_i^{\mathcal{F}} = p_i e^{-\frac{\sigma}{2}}, \quad \hbar^{\mathcal{F}} = \hbar \\ H^{\mathcal{F}} &= H e^{-\sigma}, \quad K^{\mathcal{F}} = K e^{\sigma}, \quad D^{\mathcal{F}} = D. \end{aligned}$$

The commutator of the deformed position variables has the form:

$$[x_i^{\mathcal{F}}, x_j^{\mathcal{F}}] = -\frac{i\xi}{2}\left(x_i^{\mathcal{F}} p_j^{\mathcal{F}} - x_j^{\mathcal{F}} p_i^{\mathcal{F}}\right) + \rho\mathcal{O}(\xi^3).$$

For $\rho = 0$ (or up to third order in ξ) we have the noncommutativity introduced by Snyder in H. S. Snyder, Phys. Rev. **71** (1947) 38.

Twisted commutators

The nonvanishing commutators of the deformed generators at $\rho = 0$:

$$[x_i^{\mathcal{F}}, p_j^{\mathcal{F}}] = i\hbar\delta_{ij} - (i\xi/2)p_i^{\mathcal{F}} p_j^{\mathcal{F}}$$
$$[x_i^{\mathcal{F}}, x_j^{\mathcal{F}}] = (i\xi/2)\left(x_j^{\mathcal{F}} p_i^{\mathcal{F}} - x_i^{\mathcal{F}} p_j^{\mathcal{F}}\right)$$

$$[x_i^{\mathcal{F}}, H^{\mathcal{F}}] = ip_i^{\mathcal{F}}\left(1 - \xi H^{\mathcal{F}}\right)$$
$$[x_i^{\mathcal{F}}, K^{\mathcal{F}}] = i\xi\left(K^{\mathcal{F}} p_i^{\mathcal{F}} - x_i^{\mathcal{F}} D^{\mathcal{F}}\right) + (3/4)\xi^2 x_i^{\mathcal{F}} H^{\mathcal{F}}$$
$$[x_i^{\mathcal{F}}, D^{\mathcal{F}}] = (i/2)x_i^{\mathcal{F}}\left(1 - \xi H^{\mathcal{F}}\right)$$
$$[p_i^{\mathcal{F}}, K^{\mathcal{F}}] = -i\left(x_i^{\mathcal{F}} + \xi p_i^{\mathcal{F}} D^{\mathcal{F}}\right) + (\xi^2/4)p_i^{\mathcal{F}} H^{\mathcal{F}}$$
$$[p_i^{\mathcal{F}}, D^{\mathcal{F}}] = -ip_i^{\mathcal{F}}\left(1 - (\xi/2)H^{\mathcal{F}}\right)$$

$$[D^{\mathcal{F}}, H^{\mathcal{F}}] = iH^{\mathcal{F}}\left(1 - \xi H^{\mathcal{F}}\right)$$
$$[D^{\mathcal{F}}, K^{\mathcal{F}}] = -iK^{\mathcal{F}}\left(1 - \xi H^{\mathcal{F}}\right)$$
$$[K^{\mathcal{F}}, H^{\mathcal{F}}] = 2iD^{\mathcal{F}}\left(1 + \xi H^{\mathcal{F}}\right) + 2\xi H^{\mathcal{F}} - 2\xi^2\left(H^{\mathcal{F}}\right)^2$$

Pseudo-Hermiticity of the Hamiltonian

$$\begin{aligned}
\mathbf{H} &= H_\rho + (\bar{\rho} - \rho)K^{-1} + \lambda K, \\
\mathbf{H}^{\mathcal{F}} &= H_\rho^{\mathcal{F}} + (\bar{\rho} - \rho)(K^{-1})^{\mathcal{F}} + \lambda K^{\mathcal{F}} \\
&= H_\rho T^{-2} + (\bar{g} - g)K^{-1}T^{-2} + \lambda K T^2
\end{aligned}$$

where $T = e^{\frac{\sigma}{2}}$.

Two types of η-hermiticity:

1. $\bar{\rho} - \rho \neq 0$, $\lambda = 0$
   $\left[H_\rho^{\mathcal{F}} + (\bar{\rho} - \rho)(K^{-1})^{\mathcal{F}}\right]^\dagger = \eta\left(H_\rho^{\mathcal{F}} + (\bar{\rho} - \rho)(K^{-1})^{\mathcal{F}}\right)\eta^{-1}$
   with $\eta = T^{-2}$,
2. $\bar{\rho} - \rho = 0$, $\lambda \neq 0$
   $\left[H_\rho^{\mathcal{F}} + \lambda K^{\mathcal{F}}\right]^\dagger = \eta\left(H_\rho^{\mathcal{F}} + \lambda K^{\mathcal{F}}\right)\eta^{-1}$ with $\eta = T^2$

Multi-particle operators

here $\lambda = 1$, $\bar{\rho} = 0$

2-particle state:

$$\begin{aligned}\Delta\left(\mathbf{H}^{\mathcal{F}}\right) &= Ke^{\sigma} \otimes e^{\sigma} + e^{\sigma} \otimes Ke^{\sigma} - \xi^2 \left(KH \otimes H + H \otimes KH\right) \\ &+ \sum_{n=1}^{\infty}(-\xi^{n-1}) \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) H^k \otimes H^{n-k}\end{aligned}$$

3-particle state:

$$\begin{aligned}& (id \otimes \Delta)\, \Delta\left(\mathbf{H}^{\mathcal{F}}\right) = (\Delta \otimes id)\, \Delta\left(\mathbf{H}^{\mathcal{F}}\right) = \\ = \;& (K \otimes \mathbf{1} \otimes \mathbf{1} + perm) \left[e^{\sigma} \otimes e^{\sigma} \otimes e^{\sigma} - \xi^2 \left(H \otimes H \otimes \mathbf{1} + perm\right)\right. \\ - \;& \left.\xi^4 \left(H \otimes H \otimes H\right)\right] \\ + \;& \sum_{n=1}^{\infty}(-\xi)^{n-1} \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) \sum_{l=0}^{k} \left(\begin{array}{c} k \\ l \end{array} \right) H^l \otimes H^{k-l} \otimes H^{n-k}\end{aligned}$$

Conclusions and open questions

- Drinfel'd twist prescribes a consistent way to obtain noncommutative deformations on space(time)
- for to make deformation physical one should include all additive operators as primitive elements into unfolded Lie algebra
- nontrivial consequences from coproduct on deformed multiparticle states

- Jordanian twist and its multiplication law for functions
- Snyder:"... while expressions for self-energy, polarization of the vacuum, and possibly nuclear forces will be considerably altered..."
- twist deformations on curved spaces (Ballesteros, Herranz, Meusburger, Naranjo, arXiv:1403.4773 generalized κ-Poincare)
- relastivistic case, different ways of investigations

Thank you for the attention