Inner perturbations in noncommutative geometry

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Overview

- Spectral (noncommutative) geometry
- Gauge theory from spectral triples
- Gauge group, semi-group of inner perturbations
- Examples: Yang–Mills, SM, Beyond SM

References

A. Chamseddine, Alain Connes, WvS. Inner Fluctuations in Noncommutative Geometry without the first order condition. *J. Geom. Phys.* 73 (2013) 222-234 [arXiv:1304.7583]

A. Chamseddine, Alain Connes, WvS. Beyond the Spectral Standard Model: Emergence of Pati-Salam Unification. *JHEP* 11 (2013) 132. [arXiv:1304.8050]

WvS. The spectral model of particle physics, *Nieuw Archief voor Wiskunde*, June 2014.

WvS. *Noncommutative Geometry and Particle Physics*. Mathematical Physics Studies, Springer, July 2014.

and also: `http://www.noncommutativegeometry.nl`

Spectral geometry

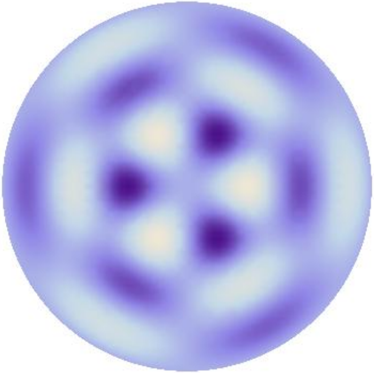
"Can one hear the shape of a drum?" (Kac, 1966)

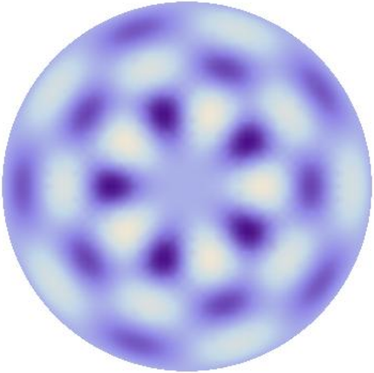
Or, more precisely, given a Riemannian manifold M, does the spectrum of wave numbers k in the Helmholtz equation

$$\boxed{\Delta_M u = k^2 u}$$

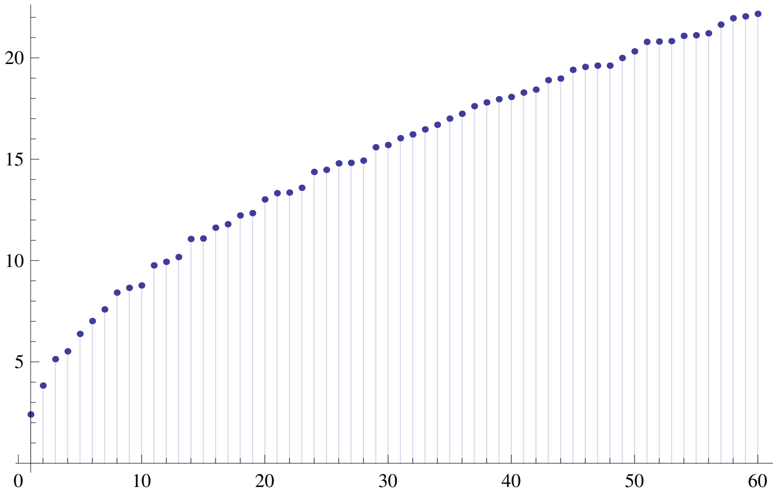
determine the geometry of M?

The disc

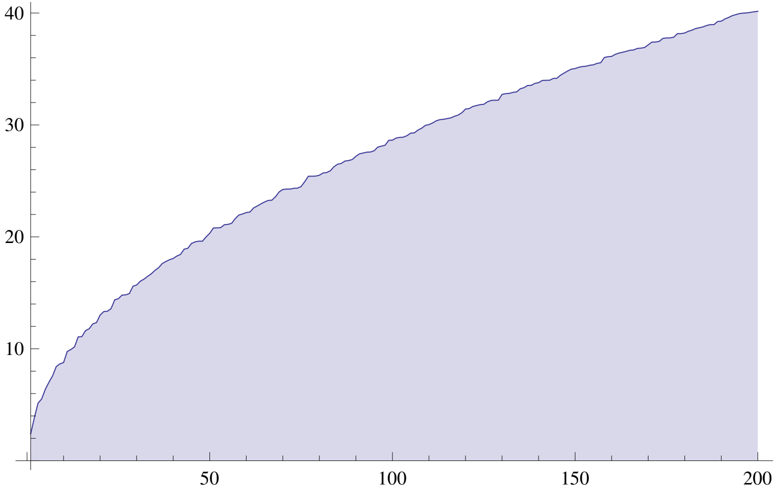




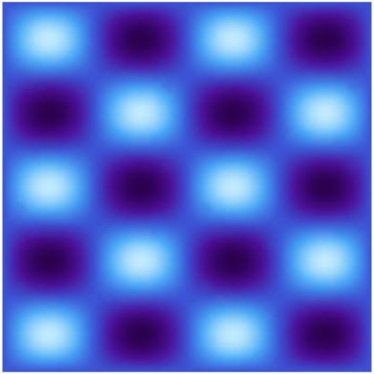
Wave numbers on the disc

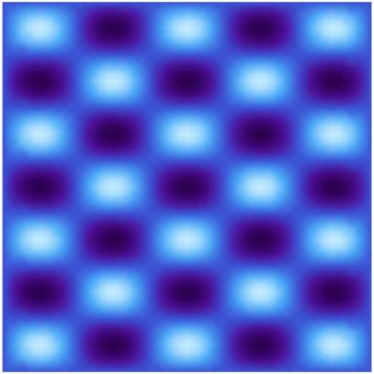


Wave numbers on the disc: high frequencies

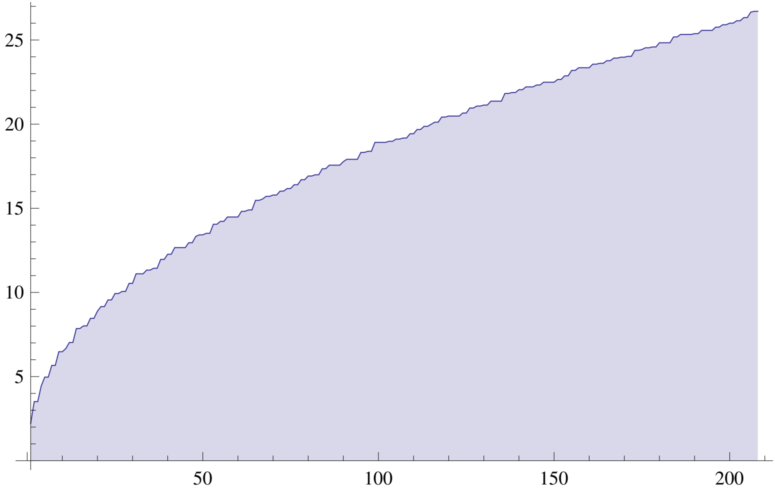


The square



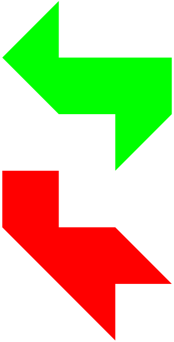


Wave numbers on the square



Isospectral domains

But, there are isospectral domains in $\mathbb{R}^2$:



(Gordon, Webb, Wolpert, 1992)

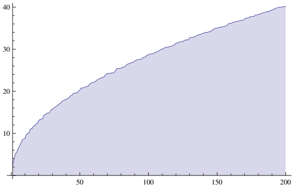
so the answer to Kac's question is no.

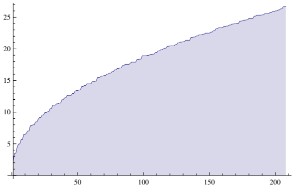
Weyl's estimate

Nevertheless, certain information can be extracted from spectrum, such as dimension n of M:

$$N(\Lambda) = \#\text{wave numbers } \leq \Lambda$$
$$\sim \frac{\Omega_n \mathrm{Vol}(M)}{n(2\pi)^n} \Lambda^n$$

For the disc and square this is confirmed by the parabolic shapes ($\sqrt{\Lambda}$):





Dirac operator

Recall that k^2 is an eigenvalue of the Laplacian in the Helmholtz equation.

- The Dirac operator D_M is a 'square-root' of the Laplacian, so that its spectrum consists of the wave numbers k.
- Exists on any Riemannian spin manifold M.

Spectral action functional

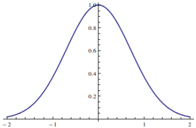
- Reconsider Weyl's estimate, in a smooth version:

$$\mathrm{Tr}\ f\left(\frac{D_M}{\Lambda}\right) = \sum_\lambda f\left(\frac{\lambda}{\Lambda}\right)$$

for a smooth cutoff function $f : \mathbb{R} \to \mathbb{R}$.

- For example, with a Gaussian cutoff function

$$f(x) = e^{-x^2}$$

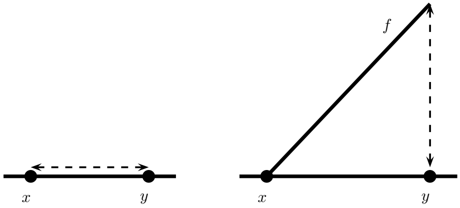


we can use heat asymptotics: $\mathrm{Tr}\ e^{-D_M^2/\Lambda^2} \sim \frac{\mathrm{Vol}(M)\Lambda^n}{(4\pi)^{n/2}}$

Hearing the shape of a drum

- As said, the geometry of M is not fully determined by spectrum of D_M.
- This can be improved by considering besides D_M also the algebra $C^\infty(M)$ of smooth functions on M, with pointwise product and addition
- In fact, the distance function on M is equal to

$$d(x,y) = \sup_{f \in C^\infty(M)} \{|f(x) - f(y)| : \text{ gradient } f \leq 1\}$$



- The gradient of f is given by the commutator $[D_M, f] = D_M f - f D_M$.

Finite spaces

- Finite space F, discrete topology

$$F = \quad {}_1\bullet \qquad {}_2\bullet \qquad \cdots\cdots \qquad {}_N\bullet$$

- Smooth functions on F are given by N-tuples in $\mathbb{C}^N$, and the corresponding algebra $C^\infty(F)$ corresponds to diagonal matrices

$$\begin{pmatrix} f(1) & 0 & \cdots & 0 \\ 0 & f(2) & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & f(N) \end{pmatrix}$$

- The finite Dirac operator is an arbitrary hermitian matrix D_F, giving rise to a distance function on F as

$$d(p,q) = \sup_{f \in C^\infty(F)} \{ |f(p) - f(q)| : \| [D_F, f] \| \leq 1 \}$$

Example: two-point space

$$F = \ {}_1\bullet \qquad {}_2\bullet$$

- Then the algebra of smooth functions

$$C^\infty(F) := \left\{ \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \middle| \lambda_1, \lambda_2 \in \mathbb{C} \right\}$$

- A finite Dirac operator is given by

$$D_F = \begin{pmatrix} 0 & \overline{c} \\ c & 0 \end{pmatrix}; \qquad (c \in \mathbb{C})$$

- The distance formula then becomes

$$d(p,q) = \begin{cases} |c|^{-1} & p \neq q \\ 0 & p = q \end{cases}$$

Finite **noncommutative** spaces

The geometry of F gets much more interesting if we allow for a *noncommutative* structure at each point of F.

- Instead of diagonal matrices, we consider block diagonal matrices

$$A = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & a_N \end{pmatrix},$$

  where the $a_1, a_2, \dots a_N$ are square matrices of size $n_1, n_2, \dots, n_N$.
- Hence we will consider the matrix algebra

$$\mathcal{A}_F := M_{n_1}(\mathbb{C}) \oplus M_{n_2}(\mathbb{C}) \oplus \cdots \oplus M_{n_N}(\mathbb{C})$$

- A finite Dirac operator is still given by a hermitian matrix.

Example: **noncommutative** two-point space

The two-point space can be given a noncommutative structure by considering the algebra $\mathcal{A}_F$ of 3×3 block diagonal matrices of the following form

$$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & a_{11} & a_{12} \\ 0 & a_{21} & a_{22} \end{pmatrix}$$

A finite Dirac operator for this example is given by a hermitian 3×3 matrix, for example

$$D_F = \begin{pmatrix} 0 & \overline{c} & 0 \\ c & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Spectral triples

Noncommutative Riemannian spin manifolds

$$(\mathcal{A}, \mathcal{H}, D)$$

- Extended to real spectral triple:
  - $J : \mathcal{H} \to \mathcal{H}$ real structure (anti-unitary)

  such that

$$J^2 = \pm 1; \qquad JD = \pm DJ$$

- Action of $\mathcal{A}^{\text{op}}$ on $\mathcal{H}$: $a^{\text{op}} = Ja^* J^{-1}$ and

$$[a^{\text{op}}, b] = 0; \qquad a, b \in \mathcal{A}$$

- D is said to satisfy first-order condition if

$$[[D, a], b^{\text{op}}] = 0$$

Spectral invariants

$$\mathrm{Tr}\ f(D/\Lambda) + \frac{1}{2}\langle J\widetilde{\psi}, D\widetilde{\psi}\rangle$$

- Invariant under unitaries $u \in \mathcal{U}(\mathcal{A})$ acting as

$$D \mapsto UDU^*; \qquad U = uJuJ^{-1}$$

- Gauge group: $\mathcal{G}(\mathcal{A}) := \{uJuJ^{-1} : u \in \mathcal{U}(\mathcal{A})\}$.
- Compute *rhs*:

$$D \mapsto D + u[D, u^*] + \hat{u}[D, \hat{u}^*] + \hat{u}[u[D, u^*], \hat{u}^*]$$

with $\hat{u} = JuJ^{-1}$ and blue term vanishes if D satisfies first-order condition

Semi-group of inner perturbations

$$\mathrm{Pert}(\mathcal{A}) := \left\{ \sum_j a_j \otimes b_j^{\mathrm{op}} \in \mathcal{A} \otimes \mathcal{A}^{\mathrm{op}} \;\middle|\; \begin{array}{l} \sum_j a_j b_j = 1 \\ \sum_j a_j \otimes b_j^{\mathrm{op}} = \sum_j b_j^* \otimes a_j^{*\mathrm{op}} \end{array} \right\}$$

with semi-group law inherited from product in $\mathcal{A} \otimes \mathcal{A}^{\mathrm{op}}$.

- $\mathcal{U}(\mathcal{A})$ maps to $\mathrm{Pert}(\mathcal{A})$ by sending $u \mapsto u \otimes u^{*\mathrm{op}}$.
- $\mathrm{Pert}(\mathcal{A})$ acts on D:

$$D \mapsto \sum_j a_j D b_j = D + \sum_j a_j [D, b_j]$$

- For real spectral triples we use the map $\mathrm{Pert}(\mathcal{A}) \to \mathrm{Pert}(\mathcal{A} \otimes \hat{\mathcal{A}})$ sending $A \mapsto A \otimes \hat{A}$ so that

$$D \mapsto \sum_{i,j} a_i \hat{a}_j D b_i \hat{b}_j$$

Perturbation semigroup for matrix algebras

Proposition

Let $\mathcal{A}_F$ be the algebra of block diagonal matrices (fixed size). Then the perturbation semigroup of $\mathcal{A}_F$ is

$$\mathrm{Pert}(\mathcal{A}_F) \simeq \left\{ \sum_j A_j \otimes B_j \in \mathcal{A}_F \otimes \mathcal{A}_F \;\middle|\; \begin{array}{l} \sum_j A_j (B_j)^t = \mathbb{I} \\ \sum_j A_j \otimes B_j = \sum_j \overline{B_j} \otimes \overline{A_j} \end{array} \right\}$$

The semigroup law in $\mathrm{Pert}(\mathcal{A}_F)$ is given by the matrix product in $\mathcal{A}_F \otimes \mathcal{A}_F$:

$$(A \otimes B)(A' \otimes B') = (AA') \otimes (BB').$$

- The two conditions in the above definition,

$$\sum_j A_j (B_j)^t = \mathbb{I} \qquad \sum_j A_j \otimes B_j = \sum_j \overline{B_j} \otimes \overline{A_j}$$

  are called normalization and self-adjointness condition, respectively.

- Let us check that the normalization condition carries over to products,

$$\left(\sum_j A_j \otimes B_j\right)\left(\sum_k A'_k \otimes B'_k\right) = \sum_{j,k} (A_j A'_k) \otimes (B_j B'_k)$$

  for which indeed

$$\sum_{j,k} A_j A'_k (B_j B'_k)^t = \sum_{j,k} A_j A'_k (B'_k)^t (B_j)^t = \mathbb{I}$$

Example: perturbation semigroup of two-point space

- Now $\mathcal{A}_F = \mathbb{C}^2$, the algebra of diagonal 2×2 matrices.
- In terms of the standard basis of such matrices

$$e_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad e_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

  we can write an arbitrary element of $\mathrm{Pert}(\mathbb{C}^2)$ as

$$z_1 e_{11} \otimes e_{11} + z_2 e_{11} \otimes e_{22} + z_3 e_{22} \otimes e_{11} + z_4 e_{22} \otimes e_{22}$$

- Matrix multiplying e_{11} and e_{22} yields for the normalization condition:

$$z_1 = 1 = z_4.$$

- The self-adjointness condition reads

$$z_2 = \overline{z_3}$$

  leaving only one free complex parameter so that $\mathrm{Pert}(\mathbb{C}^2) \simeq \mathbb{C}$.
- More generally, $\mathrm{Pert}(\mathbb{C}^N) \simeq \mathbb{C}^{N(N-1)/2}$ with componentwise product.

Example: perturbation semigroup of $M_2(\mathbb{C})$

- Let us consider a noncommutative example, $\mathcal{A}_F = M_2(\mathbb{C})$.
- We can identify $M_2(\mathbb{C}) \otimes M_2(\mathbb{C})$ with $M_4(\mathbb{C})$ so that elements in $\mathrm{Pert}(M_2(\mathbb{C}))$ are 4×4-matrices satisfying the normalization and self-adjointness condition. In a suitable basis:

$$\mathrm{Pert}(M_2(\mathbb{C})) = \left\{ \begin{pmatrix} 1 & v_1 & v_2 & iv_3 \\ 0 & x_1 & x_2 & ix_3 \\ 0 & x_4 & x_5 & ix_6 \\ 0 & ix_7 & ix_8 & x_9 \end{pmatrix} \middle| \begin{array}{l} v_1, v_2, v_3 \in \mathbb{R} \\ x_1, \ldots x_9 \in \mathbb{R} \end{array} \right\}$$

and one can show that

$$\mathrm{Pert}(M_2(\mathbb{C})) \simeq \mathbb{R}^3 \rtimes S.$$

- More generally (B.Sc. thesis Niels Neumann),

$$\mathrm{Pert}(M_N(\mathbb{C})) \simeq W \rtimes S'.$$

Example: noncommutative two-point space

- Consider noncommutative two-point space described by $\mathbb{C} \oplus M_2(\mathbb{C})$
- It turns out that

$$\mathrm{Pert}(\mathbb{C} \oplus M_2(\mathbb{C})) \simeq M_2(\mathbb{C}) \times \mathrm{Pert}(M_2(\mathbb{C}))$$

- Only $M_2(\mathbb{C}) \subset \mathrm{Pert}(\mathbb{C} \oplus M_2(\mathbb{C}))$ acts non-trivially on D_F:

$$D_F = \begin{pmatrix} 0 & \overline{c} & 0 \\ c & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 & \overline{c}\overline{\phi_1} & \overline{c}\overline{\phi_2} \\ c\phi_1 & 0 & 0 \\ c\phi_2 & 0 & 0 \end{pmatrix}$$

- The group of unitary block diagonal matrices is now $U(1) \times U(2)$ and an element (λ, u) therein acts as

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \mapsto \overline{\lambda} u \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}.$$

Example: perturbation semigroup of a manifold

Recall, for any involutive algebra $\mathcal{A}$

$$\mathrm{Pert}(\mathcal{A}) := \left\{ \sum_j a_j \otimes b_j^{\mathrm{op}} \in \mathcal{A} \otimes \mathcal{A}^{\mathrm{op}} \;\middle|\; \begin{array}{l} \sum_j a_j b_j = 1 \\ \sum_j a_j \otimes b_j^{\mathrm{op}} = \sum_j b_j^* \otimes a_j^{*\mathrm{op}} \end{array} \right\}$$

- We can consider functions in the tensor product $C^\infty(M) \otimes C^\infty(M)$ as functions of two variables, *i.e.* elements in $C^\infty(M \times M)$.
- The normalization and self-adjointness condition in $\mathrm{Pert}(C^\infty(M))$ translate accordingly and yield

$$\mathrm{Pert}(C^\infty(M)) = \left\{ f \in C^\infty(M \times M) \;\middle|\; \begin{array}{l} f(x,x) = 1 \\ f(x,y) = \overline{f(y,x)} \end{array} \right\}$$

- The action of $\mathrm{Pert}(C^\infty(M))$ on the partial derivatives appearing in a Dirac operator D_M is given by

$$\frac{\partial}{\partial x_\mu} \mapsto \frac{\partial}{\partial x_\mu} + \frac{\partial}{\partial y_\mu} f(x,y)\Big|_{y=x} =: \partial_\mu + A_\mu$$

Physical applications: Yang–Mills theory

On a 4-dimensional background:

- $\mathcal{A} = C^\infty(M) \otimes M_n(\mathbb{C})$
- $\mathcal{H} = L^2(S) \otimes M_n(\mathbb{C})$
- $D = D_M \otimes 1$
- $J = C \otimes (.)^*$

Proposition (Chamseddine-Connes, 1996)

- $\mathrm{Tr}\ f(D)$: *pure gravity (including higher-derivatives)*
- *The perturbations of D are given by hermitian $\gamma^\mu A_\mu$, describing an $\mathfrak{su}(n)$-gauge field on M.*
- *Gauge group $\mathcal{G}(\mathcal{A}) \simeq C^\infty(M, SU(n))$*
- *The spectral action of perturbed Dirac operator is given by*

$$\mathrm{Tr}\ f(D') \sim (\cdots) + \frac{f(0)}{24\pi^2} \int_M \mathrm{Tr}\ F_{\mu\nu} F^{\mu\nu}$$

Example beyond first-order

$$
\begin{aligned}
\mathcal{A}'_F &= \mathbb{C}_R \oplus \mathbb{C}_L \oplus M_2(\mathbb{C}) \\
\mathcal{H}_F &= (\mathbb{C}_R \oplus \mathbb{C}_L) \otimes (\mathbb{C}^2)^\circ \oplus \mathbb{C}^2 \otimes (\mathbb{C}_R^\circ \oplus \mathbb{C}_L^\circ) \\
J_F &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \circ C \qquad (C : \text{complex conjugation}), \\
D_F &= \begin{pmatrix} 0 & \overline{c} \otimes 1_2 & \begin{smallmatrix} \overline{d} & 0 \\ 0 & 0 \end{smallmatrix} & 0 \\ c \otimes 1_2 & 0 & 0 & \\ \begin{smallmatrix} d & 0 \\ 0 & 0 \end{smallmatrix} & 0 & 0 & 1_2 \otimes c \\ 0 & 0 & 1_2 \otimes \overline{c} & 0 \end{pmatrix}
\end{aligned}
$$

The algebra action of $(\lambda_R, \lambda_L, m) \in \mathcal{A}'_F$ on $\mathcal{H}_F$ is given explicitly by

$$
\pi(\lambda_R, \lambda_L, m) = \begin{pmatrix} \lambda_R 1_2 & & & \\ & \lambda_L 1_2 & & \\ & & m & \\ & & & m \end{pmatrix}, \pi^\circ(\lambda_R, \lambda_L, m) = \begin{pmatrix} m^t & & & \\ & m^t & & \\ & & \lambda_R 1_2 & \\ & & & \lambda_L 1_2 \end{pmatrix}.
$$

Proposition

The largest subalgebra $\mathcal{A}_F \subset \mathcal{A}'_F \equiv \mathbb{C}_R \oplus \mathbb{C}_L \oplus M_2(\mathbb{C})$ *for which the first-order condition holds (for the above* $\mathcal{H}_F, D_F$ *and* J_F*) is given by*

$$\mathcal{A}_F = \left\{ \left(\lambda_R, \lambda_L, \begin{pmatrix} \lambda_R & 0 \\ 0 & \mu \end{pmatrix} \right) : (\lambda_R, \lambda_L, \mu) \in \mathbb{C}_R \oplus \mathbb{C}_L \oplus \mathbb{C} \right\}$$

Proposition

The perturbed Dirac operator D'_F *is parametrized by three complex scalar fields* ϕ, σ_1, σ_2*:*

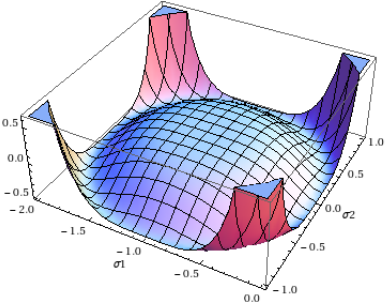
$$D'_F = \begin{pmatrix} 0 & \bar{c}\bar{\phi}\otimes 1_2 & \bar{d}\bar{v}\cdot\bar{v}^t & 0 \\ c\phi\otimes 1_2 & 0 & 0 & \\ dv\cdot v^t & 0 & 0 & 1_2\otimes c\phi \\ 0 & 0 & 1_2\otimes\bar{c}\bar{\phi} & 0 \end{pmatrix}$$

with $v = \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix}$.

Spectral action functional

Spectral action functional gives rise to a scalar potential

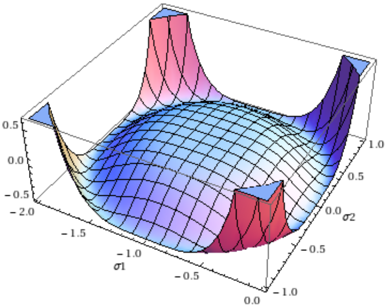
$$V(\phi, \sigma_1, \sigma_2) = -\frac{f_2}{\pi^2}\Lambda^2\left(4|c|^2|\phi|^2 + |d|^2(|\sigma_1|^2 + |\sigma_2|^2)^2\right) + \frac{f_0}{4\pi^2}\Big(4|c|^4|\phi|^4 + 4|c|^2|d|^2|\phi|^2(|\sigma_1|^2 + |\sigma_2|^2)^2 + |d|^4(|\sigma_1|^2 + |\sigma_2|^2)^4\Big)$$



Spontaneous symmetry breaking to first-order

Proposition

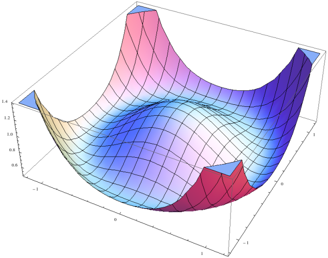
The potential $V(\phi = 0, \sigma_1, \sigma_2)$ has a local minimum at $(\sigma_1, \sigma_2) = (\sqrt{w}, 0)$ with $w = \sqrt{2f_2\Lambda^2/(f_0|d|^2)}$ and this point spontaneously breaks the symmetry group $\mathcal{U}(\mathcal{A}'_F)$ to $\mathcal{U}(\mathcal{A}_F)$.



“Usual” SSB

After the fields (σ_1, σ_2) have reached their vevs $(\sqrt{w}, 0)$, there is a remaining potential for the ϕ-field:

$$V(\phi) = -\frac{2f_2}{\pi^2}\Lambda^2|c|^2|\phi|^2 + \frac{f_0}{\pi^2}|c|^4|\phi|^4.$$



Selecting one of the minima of $V(\phi)$ spontaneously breaks the symmetry further from $\mathcal{U}(\mathcal{A}_F) = U(1)_R \times U(1)_L \times U(1)$ to $U(1)_L \times U(1)$, and generates mass terms for the $L - R$ abelian gauge field.

Beyond the Standard Model

One starts with the algebra

$$\mathcal{A}_{\mathrm{PS}} := \mathbb{H}_R \oplus \mathbb{H}_L \oplus M_4(\mathbb{C})$$

and an off-diagonal Dirac operator

$$D_F := \begin{pmatrix} S & T^* \\ T & \overline{S} \end{pmatrix}$$

- The largest 'first-order' subalgebra of $\mathcal{A}_{\mathrm{PS}}$ is $\mathbb{C} \oplus \mathbb{H}_L \oplus M_3(\mathbb{C})$.
- Symmetry breaking from Pati–Salam $SU(2)_R \times SU(2)_L \times SU(4)$ to Standard Model $U(1) \times SU(2)_L \times SU(3)$.
- Perturbation semigroup of $\mathcal{A}_{\mathrm{PS}}$ gives rise to many new scalar fields, including a real scalar singlet σ which is coupled to the Higgs sector:

$$V(\sigma, h) = -\frac{4g_2^2}{\pi^2} f_2 \Lambda^2 (h^2 + \sigma^2) + \frac{1}{24}\lambda_h h^4 + \frac{1}{2} h^2 \sigma^2 + \frac{1}{4}\lambda_\sigma \sigma^4$$

which allows for $m_h = 125.5\mathrm{GeV}$ and $m_\sigma \sim 10^{12}\mathrm{GeV}$.