Riemannian curvature of the NC 3-sphere

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Bayrischzell Workshop
May, 2015

Reference

Riemannian curvature of the noncommutative 3-sphere via pseudo-Riemannian calculi

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`arXiv:1505.07330`

Introduction

- We are interested in various aspects of (analogues of) the Riemannian curvature of noncommutative manifolds.
- How far one can get with a naive approach to curvature? Connection, curvature tensor, Ricci and scalar curvature?
- To find out what one can (and can *not* do), let us assume everything we need to be able to setup a Riemannian calculus.
- Although restrictive, the theory is not void, as a few noncommutative manifolds fit into the framework.
- Furthermore, we believe that these type of investigations shed light on the role of Riemannian geometry in the noncommutative setting. What one should, and should not expect.

Scalar curvature in noncommutative geometry

In recent years, there has been progress in understanding the (Riemannian) scalar curvature of noncommutative manifolds.

For a Riemannian manifold, the scalar curvature can be found as a coefficient in the asymptotic expansion of the heat kernel $e^{-t\Delta}$, which have a direct noncommutative analogue.

With the help of pseudo-differential calculus, developed for noncommutative tori, the scalar curvature for these noncommutative manifolds could be computed and, furthermore, a Gauss-Bonnet theorem was proven.

Scalar curvature for the noncommutative two torus (M. Khalkhali, F. Fathizadeh)

The Gauss-Bonnet theorem for the noncommutative two torus (A. Connes, P. Tretkoff)

Scalar curvature for noncommutative four-tori (M. Khalkhali, F. Fathizadeh)

Modular curvature and Morita equivalence (M. Lesch, H. Moscovici)

Curvature tensor?

Having a scalar curvature raises the question if there is a curvature tensor whose scalar curvature coincides with the ones found through the heat kernel expansion?

The role of the Riemannian curvature tensor in noncommutative geometry is not clear. Our work is an attempt to shed more light on this question.

In a recent interesting paper by J. Rosenberg, an algebraic approach to curvature is taken, which produces similar results as the (much more involved) analytical computations for the heat kernel.

Derivation based differential calculus

In the following, we assume that $\mathcal{A}$ is a unital $*$-algebra.

Let M be a right $\mathcal{A}$-module. A connection on M is a map $\nabla_d : M \to M$ for d in (possibly a Lie subalgebra of) $\mathrm{Der}(\mathcal{A})$, such that

$$\begin{aligned}
\nabla_d(\lambda U + V) &= \lambda \nabla_d U + \nabla_d V \\
\nabla_d(Ua) &= \nabla_d(U)a + Ud(a) \\
\nabla_{\lambda d + d'} U &= \lambda \nabla_d U + \nabla_{d'} U
\end{aligned}$$

for $a \in \mathcal{A}$, $U, V \in M$, $d, d' \in \mathrm{Der}(\mathcal{A})$ and $\lambda \in \mathbb{C}$.

Metric modules

A hermitian form on the right $\mathcal{A}$-module M is a map $h : M \times M \to \mathcal{A}$ such that for $a \in \mathcal{A}$ and $U, V, W \in M$

$$\begin{aligned} h(U, V + W) &= h(U, V) + h(U, W) \\ h(U, Va) &= h(U, V)a \\ h(U, V)^* &= h(V, U). \end{aligned}$$

We say that h is nondegenerate if $h(U, V) = 0$ for all $V \in M$ implies that $U = 0$.

If h is a nondegenerate hermitian form on M, we say that the pair (M, h) is a (right) metric $\mathcal{A}$-module.

Real metric calculus

With the notion of a *real metric calculus* we formalize the idea that a set of generators of M can act as derivations.

Definition

Let (M, h) be a (right) metric $\mathcal{A}$-module, let $\mathfrak{g} \subseteq \mathrm{Der}(\mathcal{A})$ be a (real) Lie algebra of hermitian derivations and let $\varphi : \mathfrak{g} \to M$ be a $\mathbb{R}$-linear map. Denoting the pair $(\mathfrak{g}, \varphi)$ by $\mathfrak{g}_\varphi$, the triple $(M, h, \mathfrak{g}_\varphi)$ is called a *real metric calculus* if it holds that

1. the image $M_\varphi = \varphi(\mathfrak{g})$ generates M as a (right) $\mathcal{A}$-module,
2. $h(E_1, E_2)^* = h(E_1, E_2)$ for all $E_1, E_2 \in M_\varphi$.

Data: Module M, metric h, derivations $\mathfrak{g}$, and $\varphi : \mathfrak{g} \to M$.

Real connection calculus

Definition

Let $(M, h, \mathfrak{g}_\varphi)$ be a real metric calculus and let $\nabla : \mathfrak{g} \times M \to M$ denote an affine connection on M. If it holds that

$$h(\nabla_d E_1, E_2) = h(\nabla_d E_1, E_2)^*$$

for all $E_1, E_2 \in M_\varphi$ and $d \in \mathfrak{g}$ then $(M, h, \mathfrak{g}_\varphi, \nabla)$ is called a *real connection calculus*.

Pseudo-Riemannian calculus

Definition

Let $(M, h, \mathfrak{g}_\varphi, \nabla)$ be a real connection calculus over M. The calculus is *metric* if

$$d\big(h(U, V)\big) = h\big(\nabla_d U, V\big) + h\big(U, \nabla_d V\big)$$

for all $d \in \mathfrak{g}$, $U, V \in M$, and *torsionfree* if

$$\nabla_{d_1}\varphi(d_2) - \nabla_{d_2}\varphi(d_1) - \varphi\big([d_1, d_2]\big) = 0$$

for all $d_1, d_2 \in \mathfrak{g}$. A metric and torsionfree real connection calculus over M is called a *pseudo-Riemannian calculus over M*.

Uniqueness of the pseudo-Riemannian calculus

Given a real metric calculus, there is no guarantee that one may find a torsionfree and metric connection, but if one exists, it is unique in the following sense.

Proposition

Let $(M, h, \mathfrak{g}_\varphi)$ be a real metric calculus over M. Then there exists at most one connection ∇ on M, such that $(M, h, \mathfrak{g}_\varphi, \nabla)$ is a pseudo-Riemannian calculus (i.e., such that ∇ is a real, torsionfree and metric connection).

Curvature

Given a pseudo-Riemannian calculus, one defines the curvature operator:

$$R(d_1, d_2)U = \nabla_{d_1}\nabla_{d_2}U - \nabla_{d_2}\nabla_{d_1}U - \nabla_{[d_1,d_2]}U$$

for $d_1, d_2 \in \mathfrak{g}$ and $U \in M$.

Proposition

Let $(M, h, \mathfrak{g}_\varphi, \nabla)$ be a pseudo-Riemannian calculus with curvature operator R. Then it holds that

1. $h(E_1, R(d_1, d_2)E_2) = -h(E_1, R(d_2, d_1)E_2)$
2. $R(d_1, d_2)\varphi(d_3) + R(d_2, d_3)\varphi(d_1) + R(d_3, d_1)\varphi(d_2) = 0,$

for $E_1, E_2 \in M_\varphi$ and $d_1, d_2, d_3 \in \mathfrak{g}$.

More symmetries of the curvature?

In general, it is not true that the curvature operator enjoys all the symmetries as in classical differential geometry. The reason is that even though we assume that $h(\nabla_d E_1, E_2)$ is hermitian, there is no guarantee that

$$h(\nabla_{d_1} \nabla_{d_2} E_1, E_2)$$

is hermitian for $E_1, E_2 \in M_\varphi$ and $d_1, d_2 \in \mathfrak{g}$. However, assuming this to be true one may prove the following:

Extended symmetries of the curvature

Proposition

Let $(M, h, \mathfrak{g}_\varphi, \nabla)$ *be a pseudo-Riemannian calculus, with curvature operator* R*, such that* $h(\nabla_{d_1}\nabla_{d_2}E_1, E_2)$ *is hermitian for all* $d_1, d_2 \in \mathfrak{g}$ *and* $E_1, E_2 \in M_\varphi$*. Then it holds that*

1. $h(E_1, R(d_1, d_2)E_2) = -h(E_1, R(d_2, d_1)E_2)$,
2. $h(E_1, R(d_1, d_2)E_2) = -h(E_2, R(d_1, d_2)E_1)$,
3. $R(d_1, d_2)\varphi(d_3) + R(d_2, d_3)\varphi(d_1) + R(d_3, d_1)\varphi(d_2) = 0$,
4. $h\big(\varphi(d_1), R(d_3, d_4)\varphi(d_2)\big) = h\big(\varphi(d_3), R(d_1, d_2)\varphi(d_4)\big)$,

for all $E_1, E_2 \in M_\varphi$ *and* $d_1, d_2, d_3, d_4 \in \mathfrak{g}$.

Examples

Are there any examples of noncommutative manifolds, for which one may construct a pseudo-Riemannian calculus? In this talk, I will discuss the noncommutative 3-sphere but there are other manifolds which fit into the framework (noncommutative torus, noncommutative 4-sphere).

Given the algebra of the noncommutative 3-sphere, there are possibly many ways of choosing $M, h, \mathfrak{g}, \varphi$ to construct a real metric calculus. The choices we make is in close analogy with the vector fields on the classical 3-sphere.

Hopf coordinates for the 3-sphere

The 3-sphere can be described as embedded in $\mathbb{C}^2$ by two complex coordinates $z = x^1 + ix^2$ and $w = x^3 + ix^4$, fulfilling $|z|^2 + |w|^2 = 1$, which can be realized by

$$z = e^{i\xi_1} \sin\eta \qquad w = e^{i\xi_2} \cos\eta,$$

giving

$$\begin{aligned} x^1 &= \cos\xi_1 \sin\eta & x^2 &= \sin\xi_1 \sin\eta \\ x^3 &= \cos\xi_2 \cos\eta & x^4 &= \sin\xi_2 \cos\eta. \end{aligned}$$

At a point, the tangent space is spanned by the three vectors

$$\begin{aligned} E_1 &= \partial_{\xi_1}\vec{x} = (-x^2, x^1, 0, 0) \\ E_2 &= \partial_{\xi_2}\vec{x} = (0, 0, -x^4, x^3) \\ E_\eta &= \partial_\eta\vec{x} = (\cos\xi_1 \cos\eta, \sin\xi_1 \cos\eta, -\cos\xi_2 \sin\eta, -\sin\xi_2 \sin\eta). \end{aligned}$$

The noncommutative 3-sphere

We consider the 3-sphere as defined by K. Matsumoto: Let S^3_θ be the $*$-algebra generated by two **normal** elements Z, W satisfying

$$WZ = qZW \qquad W^*Z = \bar{q}ZW^* \qquad WW^* + ZZ^* = \mathbb{1},$$

and introduce

$$X^1 = \frac{1}{2}(Z + Z^*) \qquad X^2 = \frac{1}{2i}(Z - Z^*)$$
$$X^3 = \frac{1}{2}(W + W^*) \qquad X^4 = \frac{1}{2i}(W - W^*),$$

implying $(X^1)^2 + (X^2)^2 + (X^3)^2 + (X^4)^2 = \mathbb{1}$. Normality of Z, W is equivalent to $[X^1, X^2] = [X^3, X^4] = 0$.

The noncommutative 3-sphere

Let us collect a few simple properties of S^3_θ in the following:

Proposition

If $A \in S^3_\theta$ then it holds that

1. $ZZ^*A = 0$ *implies that* $A = 0$,
2. $WW^*A = 0$ *implies that* $A = 0$,

and, moreover, ZZ^* and WW^* lie in the center of S^3_θ.

For notational convenience, we introduce

$$|Z|^2 = ZZ^* \qquad |W|^2 = WW^*$$

The tangent module

Can we find analogues of the tangent vectors

$$E_1 = \partial_{\xi_1}\vec{x} = (-x^2, x^1, 0, 0)$$
$$E_2 = \partial_{\xi_2}\vec{x} = (0, 0, -x^4, x^3)$$
$$E_\eta = \partial_\eta\vec{x} = (\cos\xi_1\cos\eta, \sin\xi_1\cos\eta, -\cos\xi_2\sin\eta, -\sin\xi_2\sin\eta)?$$

Clearly, we could take $E_1 = (-X^2, X^1, 0, 0)$, $E_2 = (0, 0, -X^4, X^3)$ in $(S^3_\theta)^4$, but what about E_η? Let us introduce

$$E_3 = |z||w|\partial_\eta\vec{x} = (x^1|w|^2, x^2|w^2|, -x^3|z|^2, -x^4|z|^2),$$

which has a natural analogue

$$E_3 = (X^1|W|^2, X^2|W|^2, -X^3|Z|^2, -X^4|Z|^2).$$

Now, we let M be the submodule of $(S^3_\theta)^4$ generated by E_1, E_2, E_3.

The tangent module

Proposition

M is a free module of rank 3, with basis $\{E_1, E_2, E_3\}$.

As a metric on M, we take the induced Euclidean metric:

$$h(U, V) = (U^a)^* h_{ab} V^b,$$

for $U = E_a U^a$ and $V = E_b V^b$, where

$$h_{ab} = \sum_{k=1}^{4} (E_a^k)^* E_b^k = \begin{pmatrix} |Z|^2 & 0 & 0 \\ 0 & |W|^2 & 0 \\ 0 & 0 & |Z|^2|W|^2 \end{pmatrix}.$$

Using the fact that $|Z|^2$ and $|W|^2$ are not zero divisors, one proves that h is nondegenerate, making (M, h) into a right metric module over S^3_θ.

Derivations

Can we find derivations $\partial_1, \partial_2, \partial_3$ such that

$$\partial_1(X^1, X^2, X^3, X^4) = (-X^2, X^1, 0, 0) = E_1$$
$$\partial_2(X^1, X^2, X^3, X^4) = (0, 0, -X^4, X^3) = E_2$$
$$\partial_3(X^1, X^2, X^3, X^4) = (X^1|W|^2, X^2|W|^2, -X^3|Z|^2, -X^4|Z|^2) = E_3?$$

Yes, let us formulate it in the following way:

Proposition

There exist hermitian derivations $\partial_1, \partial_2, \partial_3 \in \mathrm{Der}(S^3_\theta)$ *such that*

$$\begin{aligned} \partial_1(Z) &= iZ & \partial_1(W) &= 0 \\ \partial_2(Z) &= 0 & \partial_2(W) &= iW \\ \partial_3(Z) &= Z|W|^2 & \partial_3(W) &= -W|Z|^2, \end{aligned}$$

and $[\partial_a, \partial_b] = 0$ *for* $a, b = 1, 2, 3.$

Real metric calculus

Hence, if $\mathfrak{g}$ is the Lie algebra generated by $\partial_1, \partial_2, \partial_3$, we define $\varphi : \mathfrak{g} \to M$ as $\varphi(\partial_a) = E_a$, for $a = 1, 2, 3$, and extend it as an $\mathbb{R}$-linear map. The image of φ generates M by definition, and $h(E_a, E_b)$ is hermitian for all $a, b = 1, 2, 3$.

Thus, $(M, h, \mathfrak{g}_\varphi)$ is a real metric calculus over S^3_θ.

Via an analogue of Kozul's formula, one can find the (unique) torsionfree and metric connection on M, such that $(M, h, \mathfrak{g}_\varphi, \nabla)$ is a pseudo-Riemannian calculus?

Connection coefficients

Since the connection is torsionfree, and $[\partial_a, \partial_b] = 0$, it follows that $\nabla_{\partial_a} E_b = \nabla_{\partial_b} E_a$, which leaves the following components:

$$\begin{array}{lll} \nabla_{\partial_1} E_1 = -E_3 & \nabla_{\partial_2} E_2 = E_3 & \nabla_{\partial_3} E_3 = E_3(|W|^2 - |Z|^2) \\ \nabla_{\partial_1} E_2 = 0 & \nabla_{\partial_1} E_3 = E_1 |W|^2 & \nabla_{\partial_2} E_3 = -E_2 |Z|^2 \end{array}$$

One may easily check that this connection satisfies the condition

$$h(\nabla_{\partial_a} \nabla_{\partial_b} E_c, E_d)^* = h(\nabla_{\partial_a} \nabla_{\partial_b} E_c, E_d)$$

which implies that the corresponding curvature operator will retain all the symmetries of the classical curvature operator.

Curvature

One may proceed to compute the curvature operators

$$R(\partial_a, \partial_b)U = \nabla_{\partial_a}\nabla_{\partial_b}U - \nabla_{\partial_b}\nabla_{\partial_a}U - \nabla_{[\partial_a,\partial_b]}U$$

$$\begin{aligned} R(\partial_1, \partial_2)E_1 &= -E_2|Z|^2 & R(\partial_1, \partial_2)E_2 &= E_1|W|^2 \\ R(\partial_1, \partial_3)E_1 &= -E_3|Z|^2 & R(\partial_1, \partial_3)E_2 &= 0 \\ R(\partial_2, \partial_3)E_1 &= 0 & R(\partial_2, \partial_3)E_2 &= -E_3|W|^2 \end{aligned}$$

$$\begin{aligned} R(\partial_1, \partial_2)E_3 &= 0 \\ R(\partial_1, \partial_3)E_3 &= E_1|Z|^2|W|^2 \\ R(\partial_2, \partial_3)E_3 &= E_2|Z|^2|W|^2 \end{aligned}$$

Components of curvature

Writing $R_{abcd} = h\big(R(\partial_c, \partial_d)E_b, E_a\big)$ the only independent (nonzero) components are

$$R_{1212} = |Z|^2|W|^2 \qquad R_{1313} = |Z|^4|W|^2$$
$$R_{2323} = |Z|^2|W|^4,$$

where the others are obtained by using the symmetries.

One can also define the scalar curvature of a connection, and compute that for the 3-sphere it equals $6 \cdot \mathbb{1}$.

Summary

- We have constructed a calculus over an algebra, which involves choosing a (tangent) module with an hermitian form (M, h), and a Lie algebra of derivations $\mathfrak{g}$ together with a map $\varphi : \mathfrak{g} \to M$, associating a "vector field" to each derivation.
- In this calculus one may discuss torsionfree and metric connections on M, and prove that such a connection is unique if it exists.
- One can derive an analogue of Kozul's formula, from which one may be able to construct the connection (if it exists).
- The symmetries of the resulting curvature operator depend on hermitian properties of the connection in relation to the metric.

Summary

- Although it is clear that the above framework is too naive (and quite restrictive) it applies to a few examples of standard noncommutative manifolds: torus, 3-sphere and 4-sphere.
- We hope that these naive investigations can teach us something about what kind of properties one can expect when considering curvature of noncommutative manifolds.
- There are several things one can study in this framework: perturbations of the metric, Chern-Gauss-Bonnet theorem, Chern classes, cohomology, etc.

Thank you for your attention!