# Lie theory of vector bundles, Poisson geometry, and double structures

Henrique Bursztyn, IMPA (joint with A. Cabrera and M. del Hoyo)

Bayrischzell Workshop 2015

 $\{\text{Manifolds}\} \ \hookrightarrow \ \{\text{Lie groupoids}\}, \ G \rightrightarrows M$ 

Geometry in this context, e.g. vector bundles (and more...).

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$$\{ \text{Lie algebroids} \}, \ A \Rightarrow M$$

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## Outline:

- 1. Motivation, main results
- 2. VB-groupoids
- 3. VB-algebroids
- 4. Reformulations
- 5. Proof of main result
- 6. Application to double structures



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Adjoint repres. of Lie group 
$$G \iff TG = G \ltimes \mathfrak{g}$$

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▶ Double structures e.g. in Poisson-Lie theory Lie theory of double structures; K. Mackenzie '92, '02...

## Theorem

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- ▶ If the total algebroid of a *double Lie algebroid* is integrable, then its source-simply-connected integration is an *LA-groupoid*.

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A *VB groupoid* is... 
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## Examples:

Ordinary vector bundles...

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- Multiplicative structures

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## Role of Poisson geometry via duality:

VB-algebroids correspond to *double linear Poisson structures*:

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**Question:** Differentiation and integration of VB-structures.

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Reformulation of VB concepts...

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▶ Vertical lift map  $V_h : D \to V_h D$ ,

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Theorem: Vertical lift is diffeomorphism iff the action is regular.



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Integration of VB-structures via Lie's second theorem?

Must consider regularity...

#### **Theorem**

Consider action  $h:(\mathbb{R},\cdot) \curvearrowright (\Gamma \rightrightarrows E)$  by groupoid morphisms. Then

- ▶  $V_h\Gamma \rightrightarrows V_hE$  is VB-groupoid over  $h_0(\Gamma) \rightrightarrows h_0(E)$ ,
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#### Theorem

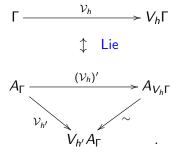
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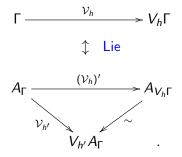


$$\Gamma \longrightarrow V_h \Gamma$$

$$\Gamma \xrightarrow{V_h} V_h \Gamma$$
 $\updownarrow$  Lie



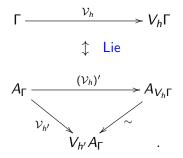
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#### Remarks:

Integrability hypothesis and obstructions...

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#### Remarks:

- Integrability hypothesis and obstructions...
- Applications: Representations up to to homotopy (Arias Abad -Schaetz), multiplicative foliations (Jotz - Ortiz, Hawkins)



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LA-groupoid:

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Lie theory?

Α



M

$$\begin{array}{ccc} A & & & (A^*,\pi) \\ \downarrow & & \downarrow \\ M & & M \end{array}$$

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#### LA-groupoids:

#### Double Lie algebroids:

By duality: Lie theory for LA-groupoids and double Lie algebroids is equivalent to Lie theory for regular actions on Poisson groupoids and Lie bialgebroids:

$$(\mathbb{R},\cdot) \curvearrowright (\Gamma \rightrightarrows E,\pi) \stackrel{Lie}{\longleftrightarrow} (\mathbb{R},\cdot) \curvearrowright (\Omega \Rightarrow E,\pi').$$

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Result follows from previous theorem for VB-structures and Poisson groupoid/Lie bialgebroid correspondence...

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#### Still missing:

From LA- to double Lie groupoids...

From LA- to multiplicative Courant algebroids... (Li-Bland)

# Thank you