Lie theory of vector bundles, Poisson geometry, and double structures

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(joint with A. Cabrera and M. del Hoyo)

Bayrischzell Workshop 2015

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Geometry in this context, e.g. vector bundles (and more...).

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Outline:

1. Motivation, main results
2. VB-groupoids
3. VB-algebroids
4. Reformulations
5. Proof of main result
6. Application to double structures

1. Motivation

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Lie theory of double structures; K. Mackenzie '92, '02...

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- If the total algebroid of a *double Lie algebroid* is integrable, then its source-simply-connected integration is an *LA-groupoid*.

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  (Crainic-Arias Abad, Mehta-Gracia Saz)
- Multiplicative structures

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Role of Poisson geometry via duality:

VB-algebroids correspond to *double linear Poisson structures*:

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Question: Differentiation and integration of VB-structures.

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Reformulation of VB concepts...

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Theorem: Vertical lift is diffeomorphism iff the action is regular.

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Must consider *regularity*...

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Consider action $h : (\mathbb{R}, \cdot) \curvearrowright (\Gamma \rightrightarrows E)$ *by groupoid morphisms. Then*

- $V_h\Gamma \rightrightarrows V_hE$ *is VB-groupoid over* $h_0(\Gamma) \rightrightarrows h_0(E)$,
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Remarks:

◇ Integrability hypothesis and obstructions...

◇ Applications: Representations up to to homotopy (Arias Abad - Schaetz), multiplicative foliations (Jotz - Ortiz, Hawkins)

5. Application to double structures (part 2 of main result)

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Lie theory?

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By duality: Lie theory for LA-groupoids and double Lie algebroids is equivalent to Lie theory for regular actions on Poisson groupoids and Lie bialgebroids:

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Still missing:

From LA- to double Lie groupoids...

From LA- to multiplicative Courant algebroids... (Li-Bland)

Thank you