Star products on graded manifolds and deformations of Courant algebroids from string theory

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Canonical momenta and winding

- Sigma model $X: \Sigma \to M = T^d$

$$S = \int_{\Sigma} h^{\alpha\beta} \partial_{\alpha} X^i \partial_{\beta} X^j G_{ij} \, d\mu_{\Sigma} + \int_{\Sigma} X^* B,$$

where $h \in \Gamma(\otimes^2 T^* \Sigma)$, $G \in \Gamma(\otimes^2 TM)$, $B \in \Gamma(\wedge^2 T^* M)$.

- Classical solutions to e.o.m. (take closed string $\Sigma = \mathbb{R} \times S^1$)

$$X^i_R = x^i_0 R + \alpha^i_0 (\tau - \sigma) + i \sum_{n \neq 0} \frac{1}{n} \alpha^i_n e^{-in(\tau - \sigma)}, \quad X^i_L = \ldots,$$

$$\alpha^i_0 = \frac{1}{\sqrt{2}} G^{ij} \left( p_j - (G_{jk} + B_{jk}) w^k \right),$$

- $p_k$: Canonical momentum zero modes
- $w^k$: Winding zero modes, $w^k := \frac{1}{2\pi} \int_0^{2\pi} \partial_{\sigma} X^k \, d\sigma$. 

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Two sets of differential operators

Siegel, Tseytlin, Hull, Zwiebach, Kugo, Hohm, Blumenhagen, Lüst, Hassler

- Two sets of momenta in $\alpha_0^i$ → differential operators:

\[ p_k \simeq \frac{1}{i} \partial_k , \quad w^k \simeq \frac{1}{i} \tilde{\partial}^k . \]

- “Level matching condition” in string theory:

\[ \partial_k \phi \tilde{\partial}^k \psi + \tilde{\partial}^k \phi \partial_k \psi = 0 , \]

for all elements $\phi, \psi$ of the algebra of observables.

Two different interpretations of observables $\phi \in C^\infty(M)$:

- $d_{dR} \phi = \partial_k \phi dx^k + \tilde{\partial}^k \phi d\tilde{x}_k$: Double configuration space, algebra of observables on it: “Double field theory”.

- Take Lie bialgebroid $(A, A^*)$ and $d_A \phi = \partial_k \phi e^k$, $d_{A^*} \phi = \tilde{\partial}^k \phi e_k^*$. Make this precise and determine its relation to physics.
Motivation: Generalized geometry

A word about notation

Hitchin, Gualtieri

- $O(d, d)$-transformations: $A \in \text{Mat}(d, d)$,

$$A \eta A^t = \eta , \quad \eta_{MN} = \begin{pmatrix} 0 & \text{id} \\ \text{id} & 0 \end{pmatrix}$$

- Generalized vectors:

$$V = X + \xi , \quad W = Y + \zeta \in \Gamma(TM \oplus T^*M) .$$

- Component notation (fundamental rep of $O(d, d)$)

$$V^M = (V^m(x), V_m(x)) \quad \text{and} \quad \partial^M = (\tilde{\partial}^m, \partial_m)$$

- Bilinear pairing:

$$\langle V, W \rangle = \iota_Y \xi + \iota_X \zeta \quad \text{i.e.} \quad V^M W_M = V^k W_k + V_k W^k .$$
Motivation: C for Courant?

The C-bracket in double field theory

Hull, Zwiebach, arXiv: 0908.1792

Double configuration space approach →: action principle + gauge symmetry. Commutator of gauge trasfos: C-bracket

\[
\left([V, W]_C\right)^M = V^K \partial_K W^M - W^K \partial_K V^M \\
- \frac{1}{2} \left( V^K \partial^M W_K - W^K \partial^M V_K \right). \tag{1}
\]

Observation for \( V = X + \xi, W = Y + \zeta \in \Gamma(TM \oplus T^*M) \):

\( \tilde{\delta}^k = 0 \): C-bracket reduces to Courant bracket.

\[
[V, W]_C = [X, Y]_L + L_X \zeta - L_Y \xi + \frac{1}{2} d_{dR}(\iota_Y \xi - \iota_X \zeta).
\]
First order $\alpha'$-deformation

Siegel, Hohm, Zwiebach

Result from string theory/double field theory:
Deformation of the pairing $\langle \cdot, \cdot \rangle$ and the C-bracket $[ \cdot, \cdot ]_C$:

$$\langle V, W \rangle_{\alpha'} = \langle V, W \rangle - \alpha' \partial_P V^Q \partial_Q W^P ,$$ (2)

$$[V, W]_C^K = [V, W]_C^K - \alpha' \left( \partial^K \partial_Q V^P \partial_P W^Q - V \leftrightarrow W \right) .$$ (3)

Remember the notation:

$$\langle V, W \rangle_{\alpha'} = V^k W_k + V_k W^k - \alpha' \left( \partial_m V^n \partial_n W^m + \partial_m V_n \tilde{\partial}^n W^m 
+ \tilde{\partial}^m V^n \partial_n W_m + \tilde{\partial}^m V_n \tilde{\partial}^n W_m \right) .$$
Two questions

- Is it possible to understand the objects $\langle, \rangle, \tilde{\partial}^k$ and $[,]_C$ using Poisson brackets on the cotangent bundle of an appropriate space?

- Can we reproduce the $\alpha'$-deformations of the last slide by using the lowest orders of a Moyal-Weyl deformation?
Lie algebroids and parity change
Mackenzie, Xu, Weinstein, Liu, Roytenberg, Voronov

Definition
A vector bundle $A \to M$ is called Lie algebroid if there exists a homological vector field $d_A$ on the supermanifold $\Pi A$, i.e. $[d_A, d_A] = 0$.

Standard examples:

- $A = TM$, basis of sections $e_i$, $[e_i, e_j]_A = f^k_{ij} e_k$ label coordinates on $\Pi A$ by $(x^i, \xi^i)$, then

$$d_A = a^i_j(x) \xi^i \partial_i - \frac{1}{2} f^{k}_{ij}(x) \xi^i \xi^j \frac{\partial}{\partial \xi^k} .$$

- $A^* = T^* M$, basis $e^i$, $[e^i, e^j]_{A^*} = Q^k_{ij} e_k$, label coordinates on $\Pi A^*$ by $(x^i, \theta_i)$, then

$$d_{A^*} = a^{ij}(x) \theta_i \partial_j - \frac{1}{2} Q^{k}_{ij}(x) \theta_i \theta_j \frac{\partial}{\partial \theta_k} .$$

The pair $(A, A^*)$ is an example of a Lie bialgebroid.
Legendre transform and Drinfel’d double

Roytenberg, arXiv:math/9910078

On cotangent bundles: $T^*\Pi A, T^*\Pi A^*$,

$$d_A \rightarrow h_{d_A} \in C^\infty( T^*\Pi A), \quad d_{A^*} \rightarrow h_{d_{A^*}} \in C^\infty( T^*\Pi A^*),$$

Relation between the two bundles: Legendre transform:

$$L : T^*\Pi A \rightarrow T^*\Pi A^*, \quad L(x^i, \xi^j, x^*_i, \xi^*_*j) = (x^i, \xi^*_j, x^*_i, \xi^j).$$

Define: $\mu := h_{d_A} + L^* h_{d_{A^*}}$.

$T^*\Pi A$: Can. graded Poisson br: $\{x^j, x^*_i\} = \delta^j_i, \{\xi^j, \xi^*_*i\} = \delta^j_i$.

Theorem

A pair of Lie algebroids $(A, A^*)$ is a Lie bialgebroid iff $\{\mu, \mu\} = 0$. Thus the following definition is justified:

Definition

The Drinfel’d double of a Lie bialgebroid $(A, A^*)$ is given by $T^*\Pi A$ together with the homological vector field $\{\mu, \cdot\}$. 
Application to double field theory

Deser, Stasheff, arXiv:1406.3601

Two sets of momenta:

\[ h_{dA} = \xi^i \left( a^i_j x^*_j - \frac{1}{2} f^{k}_{ij} \xi^j \xi^*_k \right) =: \xi^i p_i , \]

\[ L^* h_{dA*} = \xi^*_i \left( a^{ij} x^*_j + Q^k_{ij} \xi^*_j \xi^k \right) =: \xi^*_i \tilde{p}^i . \]

Thus, we get two derivative operators for \( f \in C^\infty(\mathcal{M}) \), seen as \( f \in C^\infty( T^*\Pi A) \):

\[ \partial_i f := \{ p_i , f \} , \quad \tilde{\partial}^i f := \{ \tilde{p}^i , f \} , \quad (4) \]

More general: Lift of a generalized vector field:

\[ V^m \partial_m + V_m dx^m \rightarrow V^m(x) \xi^*_m + V_m(x) \xi^m \in C^\infty( T^*\Pi A) . \]

Now, what is the C-bracket and the strong constraint?
Result 1
Deser, Stasheff, arXiv:1406.3601

Theorem
Let $V^m e_m + V_m e^m$ and $W^m e_m + W_m e^m$ be generalized vectors with corresponding lifts to $T^* \Pi A$ given by $V = V^m \xi^*_m + V_m \xi^m$ and $W = W^m \xi^*_m + W_m \xi^m$. In addition let the operation $\circ$ be defined by:

$$V \circ W = \left\{ \{\xi^i p_i + \xi^*_i \bar{p}^i, V\}, W \right\},$$

Then the C-bracket of $V$ and $W$ is given by

$$[V, W]_C = \frac{1}{2} \left( V \circ W - W \circ V \right).$$

(5)

Thus, the C-bracket can be seen as a Courant bracket, written in a form appropriate to DFT.
Result 2
Deser, Stasheff, arXiv:1406.3601

Theorem

Let $\phi(x, \tilde{x}), \psi(x, \tilde{x})$ be two double scalar fields and $D = \{ \mu, \cdot \}$ the homological vector field on $T^*\Pi A$. Then we have

$$0 = \{ D^2 \phi, \psi \} = \partial_i \phi \tilde{\partial}^i \psi + \tilde{\partial}^i \phi \partial_i \psi .$$

Thus, the strong constraint is a consequence of the condition on $T^*\Pi A$ being the Drinfel’d double of a Lie bialgebroid. Finally, it is trivial to see that $\langle V, W \rangle = \{ V, W \}$. 
Formal star products

Bayen, Flato, Fronsdal, Lichnerowicz, Sternheimer, Gerstenhaber

Definition

Let \((M, \pi)\) be a Poisson manifold and \(f, g \in C^\infty(M)\). A formal star product \(\star\) is a \(C^\infty(M)\)-bilinear map

\[
\star : C^\infty(M)[[t]] \times C^\infty(M)[[t]] \rightarrow C^\infty(M)[[t]]
\]

\[
f \star g = \sum_{k=0}^{\infty} t^k m_k(f, g),
\]

with bidifferential operators \(m_k\) such that \(\star\) has the following properties:

- \(\star\) is associative.
- \(m_0(f, g) = fg\).
- \(m_1(f, g) - m_1(g, f) = \{f, g\}\).
- \(1 \star f = f = f \star 1\).
Moyal-Weyl star product

Example

\((M, \pi)\) Poisson manifold, with constant Poisson tensor \(\pi = \frac{1}{2} \pi^i_j \partial_i \wedge \partial_j\), then

\[
f \star g = fg + \frac{t}{2} \pi^i_j \partial_i f \partial_j g + \frac{t^2}{8} \pi^i_j \pi^{mn} \partial_i \partial_m f \partial_j \partial_n g + O(t^3)
\]

and we get the Poisson bracket:

\[
m_1(f, g) - m_1(g, f) = \pi^i_j \partial_i f \partial_j g = \{f, g\}.
\]
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Remarks

► Star commutator gives deformation of the Poisson bracket:

\[ \{ f, g \}^* := \sum_{k=0}^{\infty} t^k \left( m_k(f, g) - m_k(g, f) \right) \]

\[ = \sum_{k=0}^{\infty} \left( \sum_{l,J} m^l_J \left( \partial_l f \partial_J g - \partial_l g \partial_J f \right) \right) . \]

► \( T^* \Pi A \) is a graded manifold \( \rightarrow \) take Koszul signs:

\[ \{ f, g \}^* = \sum_{k=1}^{\infty} t^k \left( \sum_{IJ} m^I_J \left( \partial_I f \partial_J g \right. \right. \]

\[ \left. \left. - \left(-1\right)^{|f||g| + |x'|(|f|-1) + |x'|(|g|-1)} \partial_I g \partial_J f \right) \right) , \]

where \( |x'| = |x_{i_1}'| + \ldots |x_{i_k}'| . \)
Idea to reproduce $\alpha'$-deformations

Deser, arXiv: 1412.5966

Recall: $\langle V, W \rangle = \{ V, W \}$

**Thm.1:** $2[V, W]_C = \{ \{ \mu, V \}, W \} - \{ \{ \mu, W \}, V \}$.

→ take star-commutators

with Moyal-Weyl star product on $T^*\Pi A$ with Poisson tensor

$$P_{T^*\Pi A} = \partial x^*_i \wedge \partial x^i + \partial \xi^*_i \wedge \partial \xi_i + \partial x^i \wedge \partial \xi^*_i + \pi^{ij} \partial x^i \wedge \partial \xi_j.$$ 

**Remark:** This means that $\tilde{\partial}^i = \pi^{ij} \partial x^j$. We restrict to this case in the following results.
Theorem

Let \( V = V^i \xi^*_i + V_i \xi^i \) and \( W = W^i \xi^*_i + W_i \xi^i \) be the lifts of two generalized vectors to \( T^* \Pi A \) and set the deformation parameter \( t = \alpha' \). Then we have

\[
\frac{1}{\alpha'} \{ V, W \}^* = \langle V, W \rangle_{\alpha'} + \mathcal{O}((\alpha')^2). 
\]

Furthermore, we have

\[
\frac{1}{2(\alpha')^2} \left( \{ \{ \mu, V \}^*, W \}^* - \{ \{ \mu, W \}^*, V \}^* \right) = [V, W]_{\alpha'} + \mathcal{O}((\alpha')^2),
\]

i.e. the deformations encountered in string theory can be understood in terms of appropriate star commutators.
Outlook

Interpreting scalars and generalized vector fields as functions on the Drinfel’d double of a Lie bialgebroid enabled us to explain deformations of a special case of the C-bracket of double field theory (where $\tilde{\partial}^k = \pi^{km} \partial_m$).

Lots of work ahead:

- General C-bracket and its deformation?
- The next order in $\alpha'$? - not known in physics up to now.
- Properties of the graded star product?
- Comparison to recent math results using the Rothstein algebra (e.g. Keller, Waldmann).
- Flux-compactification? Star-products with $R$-flux (e.g. Aschieri, Szabo et. al).