Star products on graded manifolds and deformations of Courant algebroids from string theory

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Contents

Motivation: Closed string theory

Canonical momenta and winding

- Sigma model $X: \Sigma \to M = T^d$

$$S = \int_\Sigma h^{\alpha\beta}\partial_\alpha X^i \partial_\beta X^j G_{ij}\, d\mu_\Sigma + \int_\Sigma X^* B \, ,$$

where $h \in \Gamma(\otimes^2 T^*\Sigma)$, $G \in \Gamma(\otimes^2 TM)$, $B \in \Gamma(\wedge^2 T^* M)$.

- Classical solutions to e.o.m. (take *closed* string $\Sigma = \mathbb{R} \times S^1$)

$$X^i_R = x^i_{0R} + \alpha^i_0(\tau - \sigma) + i \sum_{n \neq 0} \frac{1}{n} \alpha^i_n e^{-in(\tau-\sigma)} \, , \quad X^i_L = \dots \, ,$$

$$\alpha^i_0 = \frac{1}{\sqrt{2}} G^{ij}\Big(p_j - (G_{jk} + B_{jk}) w^k\Big) \, ,$$

  - p_k: Canonical momentum zero modes
  - w^k: *Winding* zero modes, $w^k := \frac{1}{2\pi}\int_0^{2\pi} \partial_\sigma X^k d\sigma$.

Motivation: Closed string theory

Two sets of differential operators

Siegel, Tseytlin, Hull, Zwiebach, Kugo, Hohm, Blumenhagen, Lüst, Hassler

- Two sets of momenta in $\alpha_0^i \;\; \rightarrow$ differential operators:

$$p_k \simeq \tfrac{1}{i}\partial_k \;, \quad w^k \simeq \tfrac{1}{i}\tilde{\partial}^k \;.$$

- "Level matching condition" in string theory:

$$\partial_k\phi\,\tilde{\partial}^k\psi + \tilde{\partial}^k\phi\,\partial_k\psi = 0 \;,$$

for all elements ϕ, ψ of the algebra of observables.

Two different interpretations of observables $\phi \in C^\infty(M)$:

  - $d_{dR}\phi = \partial_k\phi\,dx^k + \tilde{\partial}^k\phi\,d\tilde{x}_k$: Double configuration space, algebra of observables on it: "Double field theory".
  - Take Lie bialgebroid (A, A^*) and $d_A\phi = \partial_k\phi\,e^k$, $d_{A^*}\phi = \tilde{\partial}^k\phi\,e_k^*$. Make this precise and determine its relation to physics

Motivation: Generalized geometry

A word about notation Hitchin, Gualtieri

- $O(d,d)$-transformations: $A \in \mathrm{Mat}(d,d)$,

$$A\eta A^t = \eta\ , \quad \eta_{MN} = \begin{pmatrix} 0 & \mathrm{id} \\ \mathrm{id} & 0 \end{pmatrix}$$

- Generalized vectors:

$$V = X + \xi,\ W = Y + \zeta\ \in \Gamma(TM \oplus T^*M)\ .$$

- Component notation (fundamental rep of $O(d,d)$)

$$V^M = \ (V^m(x), V_m(x)) \quad \text{and} \quad \partial^M = (\tilde{\partial}^m, \partial_m)$$

- Bilinear pairing:

$$\langle V, W \rangle = \iota_Y \xi + \iota_X \zeta \quad \text{i.e.} \quad V^M W_M = V^k W_k + V_k W^k\ .$$

Motivation: C for Courant?

The C-bracket in double field theory

Hull, Zwiebach, arXiv: 0908.1792

Double configuration space approach $\rightarrow$: action principle + gauge symmetry. Commutator of gauge trafos: C-bracket

$$\Big([V,W]_C\Big)^M = V^K\partial_K W^M - W^K\partial_K V^M - \frac{1}{2}\Big(V^K\partial^M W_K - W^K\partial^M V_K\Big)\,. \tag{1}$$

Observation for $V = X + \xi, W = Y + \zeta \in \Gamma(TM \oplus T^*M)$:

- $\tilde{\partial}^k = 0$: C-bracket reduces to Courant bracket.

$$[V,W]_C = [X,Y]_L + L_X\zeta - L_Y\xi + \tfrac{1}{2}d_{dR}(\iota_Y\xi - \iota_X\zeta)\,.$$

First order α'-deformation

Siegel, Hohm, Zwiebach

Result from string theory/double field theory:
Deformation of the pairing $\langle , \rangle$ and the C-bracket $[\, ,\,]_C$:

$$\langle V, W\rangle_{\alpha'} = \langle V, W\rangle - \alpha' \, \partial_P V^Q \partial_Q W^P \,, \tag{2}$$

$$[V, W]^K_{\alpha'} = [V, W]^K_C - \alpha' \Big(\partial^K \partial_Q V^P \partial_P W^Q - V \leftrightarrow W \Big) \,. \tag{3}$$

Remember the notation:

$$\langle V, W\rangle_{\alpha'} = V^k W_k + V_k W^k - \alpha' \Big(\partial_m V^n \partial_n W^m + \partial_m V_n \tilde{\partial}^n W^m + \tilde{\partial}^m V^n \partial_n W_m + \tilde{\partial}^m V_n \tilde{\partial}^n W_m \Big) \,.$$

Two questions

- Is it possible to understand the objects $\langle , \rangle$,$\tilde{\partial}^k$ and $[,]_C$ using Poisson brackets on the cotangent bundle of an appropriate space?
- Can we reproduce the α'-deformations of the last slide by using the lowest orders of a Moyal-Weyl deformation?

Lie algebroids and parity change

Mackenzie, Xu, Weinstein, Liu, Roytenberg, Voronov

Definition

A vector bundle $A \to M$ is called Lie algebroid if there exists a homological vector field d_A on the supermanifold ΠA, i.e. $[d_A, d_A] = 0$.

Standard examples:

- $\underline{A = TM}$, basis of sections e_i, $[e_i, e_j]_A = f_{ij}^k e_k$ label coordinates on ΠA by (x^i, ξ^i), then

$$d_A = a_j^i(x)\xi^j \partial_i - \frac{1}{2} f_{ij}^k(x)\xi^i\xi^j \frac{\partial}{\partial \xi^k} \ .$$

- $\underline{A^* = T^*M}$, basis e^i, $[e^i, e^j]_{A^*} = Q_k^{ij} e^k$, label coordinates on ΠA^* by (x^i, θ_i), then

$$d_{A^*} = a^{ij}(x)\theta_i\partial_j - \frac{1}{2} Q_k^{ij}(x)\theta_i\theta_j \frac{\partial}{\partial \theta_k} \ .$$

The pair (A, A^*) is an example of a *Lie bialgebroid*.

Legendre transform and Drinfel'd double

Roytenberg, arXiv:math/9910078

On cotangent bundles: $T^*\Pi A$, $T^*\Pi A^*$,

$$d_A \to h_{d_A} \in C^\infty(T^*\Pi A)\,, \quad d_{A^*} \to h_{d_{A^*}} \in C^\infty(T^*\Pi A^*)\,,$$

Relation between the two bundles: *Legendre transform*:

$$L : T^*\Pi A \to T^*\Pi A^*\,,\ L(x^i, \xi^j, x^*_i, \xi^*_j) = (x^i, \xi^*_j, x^*_i, \xi^j)\,.$$

$$\text{Define:} \quad \mu := h_{d_A} + L^* h_{d_{A^*}}\,.$$

$T^*\Pi A$: Can. graded Poisson br: $\{x^j, x^*_i\} = \delta^j_i\,, \{\xi^j, \xi^*_i\} = \delta^j_i\,.$

Theorem

A pair of Lie algebroids (A, A^*) is a Lie bialgebroid iff $\{\mu, \mu\} = 0$.

Thus the following definition is justified:

Definition

The Drinfel'd double of a Lie bialgebroid (A, A^*) is given by $T^*\Pi A$ together with the homological vector field $\{\mu, \cdot\}$.

Application to double field theory

Deser, Stasheff, arXiv:1406.3601

Two sets of momenta:

$$h_{d_A} = \xi^i \left(a_i^j x_j^* - \frac{1}{2} f_{ij}^k \xi^j \xi_k^* \right) =: \xi^i p_i \ ,$$
$$L^* \, h_{d_{A^*}} = \xi_i^* \left(a^{ij} x_j^* + Q_k^{ij} \xi^k \xi_j^* \right) =: \xi_i^* \tilde{p}^i \ .$$

Thus, we get two derivative operators for $f \in \mathcal{C}^\infty(M)$, seen as $f \in \mathcal{C}^\infty(T^*\Pi A)$:

$$\partial_i f := \{p_i, f\} \ , \quad \tilde{\partial}^i f := \{\tilde{p}^i, f\} \ , \tag{4}$$

More general: Lift of a generalized vector field:

$$V^m \partial_m + V_m dx^m \to V^m(x) \xi_m^* + V_m(x) \xi^m \ \in \mathcal{C}^\infty(T^*\Pi A) \ .$$

Now, what is the C-bracket and the strong constraint?

Result 1

Deser, Stasheff, arXiv:1406.3601

Theorem

Let $V^m e_m + V_m e^m$ and $W^m e_m + W_m e^m$ be generalized vectors with corresponding lifts to $T^*\Pi A$ given by $V = V^m \xi^*_m + V_m \xi^m$ and $W = W^m \xi^*_m + W_m \xi^m$. In addition let the operation $\circ$ be defined by:

$$V \circ W = \Big\{ \{ \xi^i p_i + \xi^*_i \tilde{p}^i , V \}, W \Big\} ,$$

Then the C-bracket of V and W is given by

$$[V, W]_C = \frac{1}{2} \Big(V \circ W - W \circ V \Big) . \qquad (5)$$

Thus, the C-bracket can be seen as a Courant bracket, written in a form appropriate to DFT.

Result 2

Deser, Stasheff, arXiv:1406.3601

Theorem

Let $\phi(x,\tilde{x}), \psi(x,\tilde{x})$ be two double scalar fields and $D = \{\mu, \cdot\}$ the homological vector field on $T^*\Pi A$. Then we have

$$0 = \{D^2\phi, \psi\} = \partial_i\phi\tilde{\partial}^i\psi + \tilde{\partial}^i\phi\partial_i\psi \; . \tag{6}$$

Thus, the strong constraint is a consequence of the condition on $T^*\Pi A$ being the Drinfel'd double of a Lie bialgebroid. Finally, it is trivial to see that $\langle V, W\rangle = \{V, W\}$.

Formal star products

Bayen, Flato, Fronsdal, Lichnerowicz, Sternheimer, Gerstenhaber

Definition

Let (M, π) be a Poisson manifold and $f, g \in \mathcal{C}^\infty(M)$. A formal star product $\star$ is a $\mathcal{C}^\infty(M)$-bilinear map

$$\star : \mathcal{C}^\infty(M)[[t]] \times \mathcal{C}^\infty(M)[[t]] \to \mathcal{C}^\infty(M)[[t]]$$

$$f \star g = \sum_{k=0}^{\infty} t^k m_k(f, g) \ ,$$

with bidifferential operators m_k such that $\star$ has the following properties:

- *$\star$ is associative.*
- *$m_0(f, g) = fg$.*
- *$m_1(f, g) - m_1(g, f) = \{f, g\}$.*
- *$1 \star f = f = f \star 1$.*

Moyal-Weyl star product

Example

(M, π) Poisson manifold,
with *constant* Poisson tensor $\pi = \frac{1}{2}\pi^{ij}\partial_i \wedge \partial_j$, then

$$f \star g = fg + \frac{t}{2}\pi^{ij}\partial_i f \partial_j g + \frac{t^2}{8}\pi^{ij}\pi^{mn}\partial_i\partial_m f \partial_j\partial_n g + \mathcal{O}(t^3)$$

and we get the Poisson bracket:

$$m_1(f, g) - m_1(g, f) = \pi^{ij}\partial_i f \partial_j g = \{f, g\}\,.$$

Star commutators

Remarks

- Star commutator gives deformation of the Poisson bracket:

$$\{f,g\}^* := \sum_{k=0}^{\infty} t^k \Big(m_k(f,g) - m_k(g,f) \Big)$$
$$= \sum_{k=0}^{\infty} \Big(\sum_{I,J} m_k^{IJ} \Big(\partial_I f \partial_J g - \partial_I g \partial_J f \Big) \Big) .$$

- $T^*\Pi A$ is a graded manifold $\rightarrow$ take Koszul signs:

$$\{f,g\}^\star = \sum_{k=1}^{\infty} t^k \Big(\sum_{IJ} m_k^{IJ} \big(\partial_I f \partial_J g - (-1)^{|f||g| + |x^J|(|f|-1) + |x^I|(|g|-1)} \partial_I g \partial_J f \big) \Big) ,$$

where $|x^I| = |x^{i_1}| + \ldots |x^{i_k}|$.

Idea to reproduce α'-deformations

Deser, arXiv: 1412.5966

Recall: $\langle V, W\rangle = \{V, W\}$
Thm.1: $2[V, W]_C = \Big\{\{\mu, V\}, W\Big\} - \Big\{\{\mu, W\}, V\Big\}.$

$\rightarrow$ take star-commutators
with Moyal-Weyl star product on $T^*\Pi A$ with Poisson tensor

$$P_{T^*\Pi A} = \partial_{x_i^*} \wedge \partial_{x^i} + \partial_{\xi_i^*} \wedge \partial_{\xi^i} + \partial_{x^i} \wedge \partial_{\xi_i^*} + \pi^{ij}\partial_{x^i} \wedge \partial_{\xi^j} \ .$$

Remark: This means that $\tilde{\partial}^i = \pi^{ij}\partial_{x^j}$. We restrict to this case in the following results.

Result 3

Deser, arXiv: 1412.5966

Theorem

Let $V = V^i\xi_i^* + V_i\xi^i$ *and* $W = W^i\xi_i^* + W_i\xi^i$ *be the lifts of two generalized vectors to* $T^*\Pi A$ *and set the deformation parameter* $t = \alpha'$. *Then we have*

$$\frac{1}{\alpha'}\{V, W\}^\star = \langle V, W\rangle_{\alpha'} + \mathcal{O}((\alpha')^2) \, .$$

Furthermore, we have

$$\frac{1}{2(\alpha')^2}\Big(\{\{\mu, V\}^*, W\}^* - \{\{\mu, W\}^*, V\}^*\Big) = [V, W]_{\alpha'} + \mathcal{O}((\alpha')^2) \, ,$$

i.e. the deformations encountered in string theory can be understood in terms of appropriate star commutators.

Outlook

Interpreting scalars and generalized vector fields as functions on the Drinfel'd double of a Lie bialgebroid enabled us to explain deformations of a special case of the C-bracket of double field theory (where $\tilde{\partial}^k = \pi^{km}\partial_m$).

Lots of work ahead:

- General C-bracket and its deformation?
- The next order in α'? - not known in physics up to now.
- Properties of the graded star prduct?
- Comparison to recent math results using the Rothstein algebra (e.g. Keller, Waldmann).
- Flux-compactification? Star-products with R-flux (e.g. Aschieri, Szabo et. al).