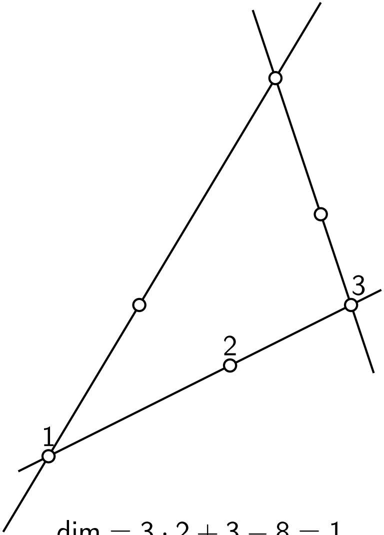
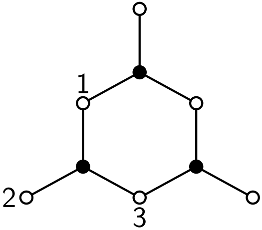
# Configuration of points and integrable systems

## Part I. Configurations of points (aka discrete geometries, aka matroids).

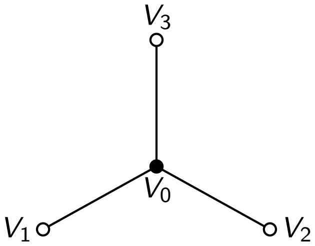
# Point configuration and the corresponding bipartite graph.




$\dim = 3 \cdot 2 + 3 - 8 = 1$

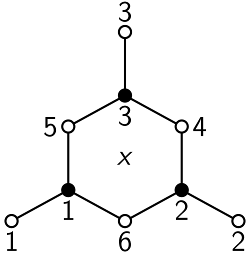


# Abelian connection on the graph



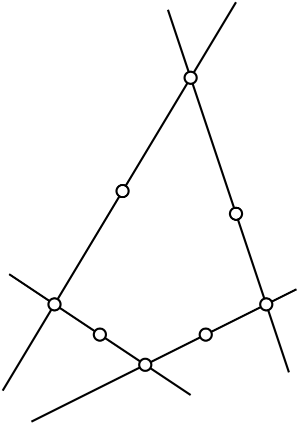

$$V_0 = \ker(V_1 \oplus V_2 \oplus V_3 \to V_1 + V_2 + V_3)$$

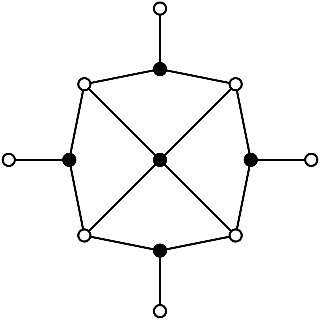
# From a graph to a configuration



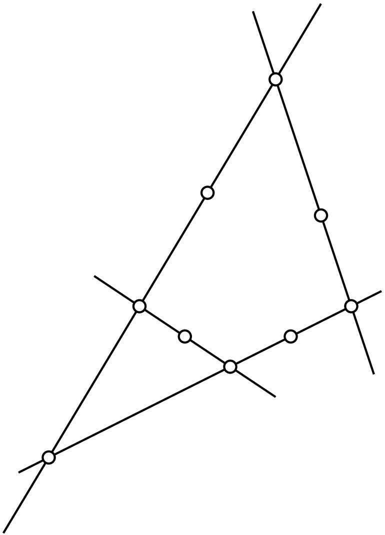

$$\operatorname{coker}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 1 & 1\\ 1 & 0 & x\\ 1 & 1 & 0\end{pmatrix}$$

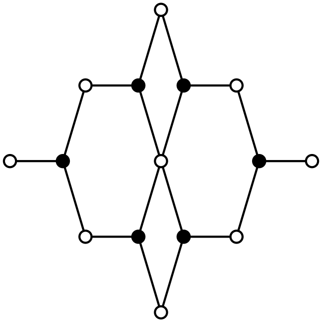
# Birational transformation





# Birational transformation





# Graph modifications

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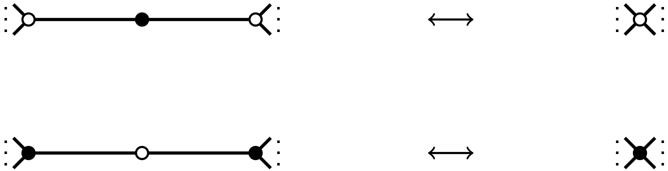
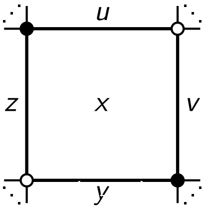


# Graph modifications



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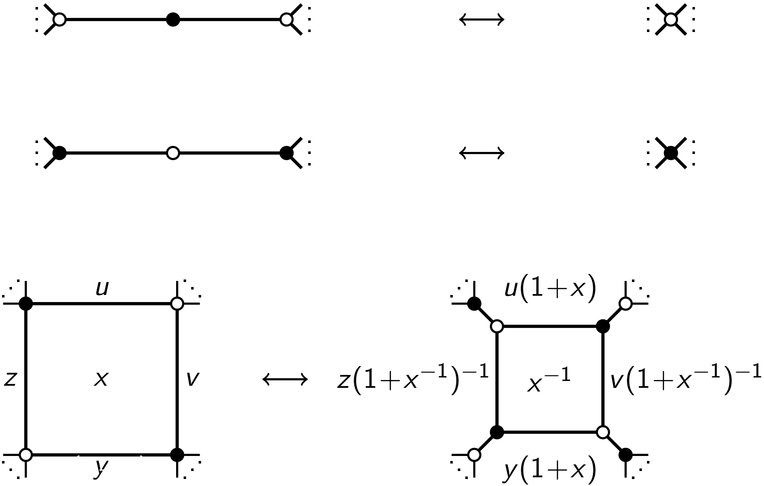


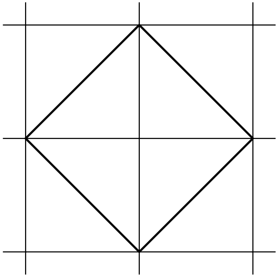
# Part II. Integrable systems.

# Integrable system.

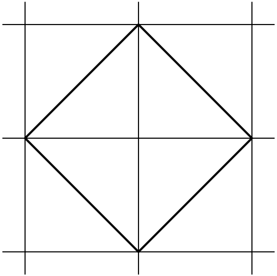
$X$-Complex Poisson variety
$F_1, \ldots, F_n : X \to \mathbb{C}$ such that $\{F_i, F_j\} = 0$ for any $i$ and $j$.
$n = \dim X - \mathrm{rk}\ X/2$.
$(F_1 = c_1, \ldots, F_n = c_n)$ is a collection of tori $\mathbb{C}^{\mathrm{rk}\ X/2}/\mathbb{Z}^{\mathrm{rk}\ X}$

In our case $X$ is the space of pairs (planar curve $\Sigma$ of genus $g$, a line bundle $\mathcal{L}$ of degree $g-1$ on it).
Planar curve $\Sigma = \{(\lambda, \mu) \in \mathbb{C}^2 | P(\lambda, \mu) = 0\}$.
$P(\lambda, \mu) = \sum_{ij \in \Delta} c_{ij} \lambda^i \mu^j$.
$\Delta$ is convex polygon on the plane.
$c_{ij}$ - commuting Hamiltonians.
Level set of $c_{ij}$ is the Picard variety of $\overline{\Sigma}$.

# Newton polygon

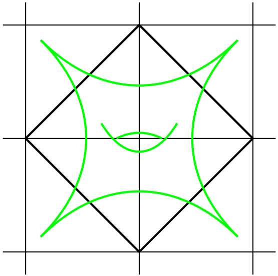


# Spectral curve



$$\lambda + \lambda^{-1} + \mu + C\mu^{-1} + H = 0$$

# Topology of the spectral curve.

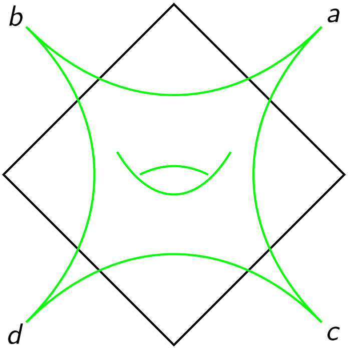


handles = integer points inside $\Delta$.

Points at infinity = segments of the boundary of $\Delta$.

# Divisors at infinity

$\lambda = \infty, \mu = 0$ $\qquad\qquad$ $\lambda = 0, \mu = 0$




$\lambda = \infty, \mu = \infty$ $\qquad\qquad$ $\lambda = 0, \mu = \infty$

$$a + c - b - d = (\lambda)$$

$$a + b - c - d = (\mu)$$

$$L(p, q, r, s) = H^0(\mathcal{L} + pa + qb + rc + sd)$$

$$\dim L(p, q, r, s) = p + q + r + s$$

If $p + q + r + s = -1$

$$L(p{+}1, q{+}1, r, s), L(p, q{+}1, r{+}1, s), L(p, q, r{+}1, s{+}1), L_(p{+}1, q, r, s{+}1)$$

$$\subset L_(p + 1, q + 1, r + 1, s + 1)$$