Geodesics on ellipsoids

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Let us start with the trivial problem of determining geodesics in $\mathbb{R}^N$, considering the length L of paths from A to B as a functional of parametrized curves $\vec{x}(t)$ connecting $A = \vec{x}(\alpha)$ and $B = \vec{x}(\beta)$:

$$L = \int_\alpha^\beta \sqrt{\dot{\vec{x}}^2}\,\mathrm{d}t, \tag{1}$$

whose stationary points satisfy

$$\ddot{\vec{x}} - \frac{\dot{\vec{x}}}{\dot{\vec{x}}^2}\left(\dot{\vec{x}} \cdot \ddot{\vec{x}}\right) = \vec{0}. \tag{2}$$

Choosing the parameter t to be the arc length, i.e. $\dot{\vec{x}}^2 = 1$, the reparametrization-invariant equation (2) reads

$$\ddot{\vec{x}} = 0, \tag{3}$$

corresponding to the Lagrangian

$$\mathcal{L}_0 := \frac{1}{2}\dot{\vec{x}}^2 \tag{4}$$

whose integral, in contrast with (1), is *not* reparametrization-invariant.

Suppose now that the motion takes place on an M dimensional hypersurface Σ, i.e. described parametrically by

$$\vec{x}\left(u^1(t),\ldots,u^M(t)\right). \tag{5}$$

As then $\dot{\vec{x}} = \dot{u}^a\partial_a\vec{x}$, hence $\dot{\vec{x}}^2 = \dot{u}^a\partial_a\vec{x}\cdot\partial_b\vec{x}\,\dot{u}^b =: \dot{u}^a g_{ab}\dot{u}_b$, the expression for the length becomes

$$L = \int_\alpha^\beta \sqrt{\dot{u}^a g_{ab}\dot{u}^b}\,\mathrm{d}t = L\left[u^a,\dot{u}^a\right], \tag{6}$$

where $g_{ab}\left(u^1,\ldots,u^M\right)$ could also be thought as intrinsically given, rather than being induced from $\mathbb{R}^N$ as $\partial_a\vec{x}\cdot\partial_b\vec{x}$.
Varying (6) gives

$$\ddot{u}^c + \gamma^c_{ab}\dot{u}^a\dot{u}^b = -\dot{u}^c\sqrt{\dot{u}^a g_{ab}\dot{u}^b}\partial_t\frac{1}{\sqrt{\dot{u}^a g_{ab}\dot{u}^b}} = -\frac{1}{2}\dot{u}^c\partial_t\ln\left(\dot{u}^a g_{ab}\dot{u}^b\right) \tag{7}$$

with

$$\gamma^c_{ab} := \frac{1}{2}g^{cd}\left(\partial_a g_{db} + \partial_b g_{ad} - \partial_d g_{ab}\right). \tag{8}$$

Again the (reparametrization-invariant) equations simplify significantly by choosing $\dot{\vec{x}}^2 = \dot{u}^a g_{ab} \dot{u}_b$ (cp. (6)) to be constant, i.e. the parameter t to be, up to constant rescaling, the arc length of the curve (making the r.h.s. of (7) vanish).
With this understanding, the coupled ODE:s

$$\ddot{u}^c + \gamma^c_{ab} \dot{u}^a \dot{u}^b = 0, \quad a, b, c = 1, \ldots, M, \tag{9}$$

are usually referred to as 'geodesic equations' for a Riemannian manifold $\mathcal{M}$ parametrized locally by parameters $u^a (a = 1, \ldots, M)$.
In case

$$\mathcal{M} = \Sigma_M(\varphi) := \left\{ \vec{x} \in \mathbb{R}^{M+1} \middle| \varphi\left(\vec{x}\right) = 0 \right\}, \tag{10}$$

one could alternatively take

$$\mathcal{L} = \frac{1}{2} \dot{\vec{x}}^2 - \lambda \varphi\left(\vec{x}\right), \tag{11}$$

with Lagrangian equations of motion

$$\ddot{\vec{x}} = -\lambda \vec{\nabla} \varphi, \quad \varphi\left(\vec{x}(t)\right) = 0, \tag{12}$$

where λ can be obtained by noting that (differentiating $\varphi\left(\vec{x}(t)\right) = 0$ twice w.r.t. t)

$$\dot{\vec{x}} \cdot \vec{\nabla}\varphi\left(\vec{x}(t)\right) = 0, \quad \ddot{\vec{x}} \cdot \vec{\nabla}\varphi + \dot{x}^i \dot{x}^j \partial^2_{ij}\varphi = 0, \tag{13}$$

the first ensuring $\dot{\vec{x}} \cdot \ddot{\vec{x}} = 0$, the second implying

$$\lambda = -\frac{\ddot{\vec{x}} \cdot \vec{\nabla}\varphi}{\left(\nabla\varphi\right)^2} = +\frac{\dot{x}^i \dot{x}^j \partial^2_{ij}\varphi}{\left(\nabla\varphi\right)^2}, \tag{14}$$

so that

$$\ddot{\vec{x}} = -\frac{\dot{x}^i \dot{x}^j \partial^2_{ij}\varphi}{\left(\nabla\varphi\right)^2}\vec{\nabla}\varphi \tag{15}$$

describes free motion on Σ_M (note that $\vec{\nabla}\varphi$ is normal to Σ_M so that there is no tangential acceleration, hence no tangential force). Before discussing how to solve (9), resp. (15), for the case of an Ellipsoid, let us (Exercise I) note that for rotationally symmetric two-dimensional surfaces,

$$\vec{x}(u,v) = \begin{pmatrix} f(u)\cos v \\ f(u)\sin v \\ h(u) \end{pmatrix}, \tag{16}$$

(9) can easily be solved by quadrature, as $(9)_{a=2}$ (calculating g_{ab} and γ^c_{ab} from (16)),

$$\ddot{v} + 2\frac{f'}{f}\dot{u}\dot{v} = 0 \tag{17}$$

integrates to

$$\dot{v} = \frac{\text{const.}}{f^2\big(u(t)\big)} =: \frac{l}{f^2}, \tag{18}$$

allowing one to eliminate v from $(9)_{a=1}$, resp. (simpler!)

$$\dot{u}^a g_{ab} \dot{u}^b = \left(f'^2 + h'^2\right)\dot{u}^2 + f^2\dot{v}^2 \stackrel{!}{=} \text{const.} =: 2E > 0. \tag{19}$$

Inserting (18) into (19) yields $u(t)$ by quadrature:

$$\pm \int \mathrm{d}u \sqrt{\frac{f'^2 + h'^2}{2E - \frac{l^2}{f^2}}} = t - t_0. \tag{20}$$

As Exercise II, note that (9) can be formulated in Hamiltonian form by considering

$$H = \frac{1}{2}\pi_a g^{ab}\pi_b = H\left[u^1, \ldots, u^M, \pi_1, \ldots, \pi_M\right] \tag{21}$$

with canonical Poisson-structure, i.e.

$$\begin{aligned}
\dot{u}^a &= \frac{\delta H}{\delta \pi_a} = g^{ab}\pi_b \\
\dot{\pi}_c &= -\frac{\delta H}{\delta u^c} = -\frac{1}{2}\pi_a \partial_c g^{ab}\pi_b = \frac{1}{2}\pi_a g^{a'a}\partial_c g_{a'b'} g^{b'b}\pi_b = \frac{1}{2}\dot{u}^a \left(\partial_c g_{ab}\right)\dot{u}^b.
\end{aligned} \tag{22}$$

One way of stating Jacobi's seminal result is that for an Ellipsoid, (21) separates in elliptic coordinates – which Jacobi originally [1838] defined (for $M = 2$) as angles φ and ψ in

$$\begin{aligned}
x_1 &= \sqrt{\frac{\alpha_1}{\alpha_3 - \alpha_1}} \sin\varphi \sqrt{\alpha_2 \cos^2\psi + \alpha_3 \sin^2\psi - \alpha_1} \\
x_2 &= \sqrt{\alpha_2} \cos\varphi \sin\psi \\
x_3 &= \sqrt{\frac{\alpha_3}{\alpha_3 - \alpha_1}} \cos\psi \sqrt{\alpha_3 - \alpha_1 \cos^2\varphi - \alpha_2 \sin^2\varphi}
\end{aligned} \tag{23}$$

and then, for general M, as (apart from $u^0 = 0$) the zeros of

$$f(u) := \sum_{i=1}^{N} \frac{x_i^2}{\alpha_i - u} - 1 =: -\frac{\prod_{A=0}^{M} \left(u^A - u \right)}{\prod_{i=1}^{M+1} (\alpha_i - u)}; \qquad (24)$$

that f fully factorizes into real factors, with

$$\alpha_1 < u^1 < \alpha_2 < \ldots < u^M < \alpha^{M+1=N} \qquad (25)$$

is easily seen by noting that

$$f'(u) = + \sum_{i=1}^{N} \frac{x_i^2}{(\alpha_i - u)^2} > 0. \qquad (26)$$

The (elliptic coordinates) u^a ($a = 1, \ldots, M$) coordinatize the M-dimensional Ellipsoid

$$\mathbb{E}_M := \left\{ \vec{x} \in \mathbb{R}^{M+1} \middle| \sum_{i=1}^{M+1=N} \frac{x_i^2}{\alpha_i} = 1 \right\}. \qquad (27)$$

By a simple residue-argument

$$x_i^2 = \frac{\prod_A \left(\alpha_i - u^A\right)}{\prod_{j\neq i} \left(\alpha_i - \alpha_j\right)}, \tag{28}$$

hence

$$\begin{aligned} 4\,\mathrm{d}\vec{x}^2 &= \sum_i x_i^2 \left(\frac{2\,\mathrm{d}x_i}{x_i}\right)^2 = \sum_i x_i^2 \left(-\sum_A \frac{\mathrm{d}u^A}{\alpha_i - u^A}\right)^2 \\ &= \sum_{i,A,B} \frac{\mathrm{d}u^A\,\mathrm{d}u^B}{\left(\alpha_i - u^A\right)\left(\alpha_i - u^B\right)} \frac{\prod_C \left(\alpha_i - u_C\right)}{\prod_{j\neq i}\left(\alpha_i - \alpha_j\right)} =: 4g_{AB}\,\mathrm{d}u^A\,\mathrm{d}u^B. \end{aligned} \tag{29}$$

Jacobi then used (four times!) that for any distinct numbers $z_1, \ldots, z_{J>1}$

$$\sum_{j=1}^{J} \frac{z_j^s}{\prod_{k(\neq j)} z_j - z_k} = \begin{cases} 0 & \text{for } s = 0, \ldots, J-2, \\ 1 & \text{for } s = J-1, \\ \sum \alpha_j & \text{for } s = J; \end{cases} \tag{30}$$

firstly (easy!) showing that the u^A are orthogonal coordinates, i.e. $g_{A\neq B} = 0$ (the factors $\alpha_i - u^A$ and $\left(\alpha_i - u^B\right)$ can then be cancelled in (29), leaving in the numerator a polynomial of degree $N-2$); secondly (writing, for $A = B$, each factor $\left(\alpha_i - u^{C\neq A}\right)$ as $\left(\alpha_i - u^A\right) + \left(u^A - u^C\right)$ and then having to always pick the second term, in order to avoid getting zero according to $(30)_{z_i=\alpha_i}$) to show that

$$g_{AA} = \frac{1}{4}\sum_i \frac{\prod_{C\neq A}\left(u^A - u^C\right)}{\left(\alpha_i - u^A\right)\prod_j'\left(\alpha_i - \alpha_j\right)}; \tag{31}$$

thirdly (with $J = N+1, z_i = \alpha_i, z_{N+1} = u^A$) to conclude that

$$4g_{AA} = -\frac{\prod_{C\neq A}\left(u^A - u^C\right)}{\prod_i\left(u^A - \alpha_i\right)} \overset{(A=a\neq 0)}{=} -u^a\frac{\prod_{c(\neq a)}'\left(u^a - u^c\right)}{\prod_i\left(u^a - \alpha_i\right)}. \tag{32}$$

Hence

$$H=-2\sum_{a=1}^{M}\pi_a\frac{q\left(u^a\right)}{\prod_{c\neq a}\left(u^a-u^c\right)}\pi_a$$

with

$$q(u):=\prod_{i=1}^{N}\frac{\left(u-\alpha_i\right)}{u} \tag{33}$$

describes geodesics on $\mathbb{E}_M$; the simplest non-trivial case being $N=3$, resp.

$$H=2\frac{\pi_1^2 q\left(u^1\right)}{u^2-u^1}-2\frac{\pi_2^2 q\left(u^2\right)}{u^2-u^1} \tag{34}$$

(note that $q\left(u^1\right)>0$, while $q\left(u^2\right)<0$).
The celebrated Hamilton-Jacobi method then solves the problem by first replacing the π_a by $\frac{\partial S}{\partial u^a}$ (transforming $H=E$ into a PDE) and making the separation Ansatz $S=\sum_{a=1}^{N-1}S_a\left(u^a\right)$, which indeed will produce solutions S depending on $N-1$ free constants $\beta_1,\ldots,\beta_{N-3},\beta_{N-2}=\beta,\beta_{N-1}=E$, provided the S_a satisfy

$$2S_a'\left(u^a\right)q\left(u^a\right) = E\left(\beta + \beta_1 u^a + \ldots + \beta_{N-3}\left(u^a\right)^{N-3} + (-)^N\left(u^a\right)^{N-2}\right)$$
$$=: T_{N-2}\left(u^a; \beta_1, \ldots, \beta_{N-3}, \beta_{N-2} = \beta, \beta_{N-1} = E\right); \tag{35}$$

resp.

$$\pm\, \mathrm{d}S_a = \mathrm{d}u^a \sqrt{\frac{T_{N-2}\left(u^a\right)}{2q\left(u^a\right)}} \overset{(N=3)}{=} \sqrt{\frac{E}{2}} \sqrt{\frac{\left(\beta - u^a\right)u^a}{\left(u^a - \alpha_1\right)\left(u^a - \alpha_2\right)\left(u^a - \alpha_3\right)}}\, \mathrm{d}u^a \tag{36}$$

hence

$$S = \sqrt{\frac{E}{2}} \sum_{a=1}^{N-1} \pm \int^{u^a} \sqrt{\frac{\frac{1}{E}\, T_{N-2}(u)}{q(u)}}\, \mathrm{d}u; \tag{37}$$

$\frac{\partial S}{\partial \beta} = \text{const.}$ (in accordance with action-angle coordinates) and $(N = 3)$

$$u^1 = \alpha_1 \cos^2\varphi + \alpha_2 \sin^2\varphi, \quad u^2 = \alpha_3 \sin^2\psi + \alpha_2 \cos^2\psi \tag{38}$$

give Jacobi's celebrated solution [1] (note that his β is $\alpha_2 - \beta$ here).

A simple and slightly more direct derivation (including relatively explicit formulae for the x_i as ratios of elliptic θ-functions) was presented by Weierstrass [3] (introducing conserved quantities that were discovered again 100 years later [5]). He noted that, as a consequence of the equations of motion (cp. (15))

$$\ddot{x}_i = -\frac{\sum_k \frac{\dot{x}_k^2}{\alpha_k}}{\sum_l \frac{x_l^2}{\alpha_l}} \frac{x_i}{\alpha_i} \tag{39}$$

$$\left(1 + \sum_i \frac{x_i^2}{u-\alpha_i}\right)\left(\sum_k \frac{\dot{x}_k^2}{u-\alpha_k}\right) - \left(\sum_l \frac{x_l \dot{x}_l}{u-\alpha_l}\right)^2 = \sum_i \frac{H_i}{u-\alpha_i} = \frac{S(}{Q(} \tag{40}$$

will be time-independent, hence defining $N-1$ constants of the motion via

$$S(u) = cu \prod_{\alpha=1}^{N-2} (u - \delta_\alpha), \tag{41}$$

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In accordance with (cp. (24))

$$P(u) := \left(1 + \sum_i \frac{x_i^2}{u - \alpha_i}\right) \prod_i (u - \alpha_i) =: u \prod_{a=1}^{N-1} (u - u^a)\,, \tag{42}$$

$$\dot{P}\Big|_{u=u^a} = -u^a \dot{u}^a \prod_c{}' (u^a - u^c)\,, \tag{43}$$

while (40), being of the form

$$\frac{P}{Q} \sum_k \frac{\dot{x}_k^2}{u - \alpha_k} - \frac{1}{4}\frac{\dot{P}^2}{Q^2} = \frac{S}{Q},$$

implying

$$\dot{P}\,(u^a) = \pm 2\sqrt{-QS}\,(u^a) =: \pm 2\sqrt{R}, \tag{44}$$

one deduces that

$$\mp\frac{u^a\,\mathrm{d}u^a}{2\sqrt{-QS}} = \frac{\mathrm{d}t}{\prod_c' (u^a - u^c)}, \tag{45}$$

hence (multiplying with $(u^a)^{s-1}$, and using (30))

$$\sum_{a=1}^{N-1} \mp \int^{u^a(t)} \frac{u^s}{\sqrt{R(u)}}\,\mathrm{d}u = \begin{cases} 0 & \text{for } s = 1, 2, \ldots, N-2, \\ 2(t-t_0) & \text{for } s = N-1, \end{cases} \tag{46}$$

with $R(u) = -cu \prod_{i=1}^{N} (u-\alpha_i) \prod_{\alpha=1}^{N-2}(u-\delta_\alpha)$ being a time-independent polynomial of degree $2N-1$.
Note that for $N = 3$ $(c > 0, u^1 - \delta_1 < 0)$ the integrability also follows from the (once observed [14] 'trivial') time-independence of

$$I = \sum_{i=1}^{N} \frac{x_i^2}{\alpha_i^2} \sum_{k=1}^{N} \frac{\dot{x}_k^2}{\alpha_k}. \tag{47}$$

Among Hamiltonian treatments using the constrained embedding coordinates $x^i(t)$ rather than the intrinsic $u^a(t)$, let me first mention the one using Dirac's theory of constraints: consider

$$\begin{aligned}\varphi &:= \frac{1}{2}\left(\sum_i \frac{x_i^2}{\alpha_i} - 1\right) =: \varphi_1, \quad \pi := \sum_i \frac{x_i p_i}{\alpha_i} =: \varphi_2, \\ \{\varphi, \pi\} &= \sum_i \frac{x_i^2}{\alpha_i^2} =: J,\end{aligned} \tag{48}$$

leading to the Dirac-bracket

$$\begin{aligned}\{f, g\}_D &:= \{f, g\} - \{f, \varphi_a\}\, \chi^{ab}\, \{\varphi_b, g\} \\ &= \{f, g\} + \{f, \varphi\}\, \frac{1}{J}\, \{\pi, g\} - \{f, \pi\}\, \frac{1}{J}\, \{\varphi, g\},\end{aligned} \tag{49}$$

as the inverse of the constraint-matrix

$$\Big(\chi_{ab} := \{\varphi_a, \varphi_b\}\Big) = \begin{pmatrix} 0 & J \\ -J & 0 \end{pmatrix} \text{ is } \frac{1}{J}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Exercise III (cp. [12]):

$$\left\{x_i, x_j\right\}_D = 0, \quad \left\{x_i, p_j\right\}_D = \delta_{ij} - \frac{1}{J}\frac{x_i x_j}{\alpha_i \alpha_j}, \quad \left\{p_i, p_j\right\}_D = -\frac{L_{ij}}{\alpha_i \alpha_j J},$$

$$L_{ij} := x_i p_j - x_j p_i.$$

Instead of using (50) to (tediously) show the Dirac-Poisson-commutativity (.i.e. on the constrained phase-space) of the

$$F_i = p_i^2 + {\sum_j}' \frac{L_{ij}^2}{\alpha_i - \alpha_j} \tag{51}$$

it is much simpler to first show (Exercise IV)

$$\{F_i, F_j\} = 0 \tag{52}$$

and then note that due to $\{F_i, \varphi\} \approx 0$

$$\{F_i, F_j\}_D = 0 \tag{53}$$

trivially follows.
Let me finish this excursion with a Hamiltonian description communicated to me by Martin Bordemann [5]: Let π be the projection-operator onto the normal of $\mathbb{E}$, resp.

$$Q_{ij} = \delta_{ij} - \frac{\dfrac{x_i x_j}{\alpha_i \alpha_j}}{\displaystyle\sum_l \frac{x_l^2}{\alpha_l^2}} \tag{54}$$

the projection onto the tangent-space of the ellipsoid. To verify that

$$H = \frac{1}{2}\langle \vec{p}, Q\vec{p} \rangle = \frac{1}{2}\langle \vec{p}, \vec{p} \rangle - \frac{1}{2}\frac{\langle \vec{p}, A\vec{x} \rangle^2}{\langle \vec{x}, A^2\vec{x} \rangle}, \quad A_{ij} := \delta_{ij}\frac{1}{\alpha_i}, \tag{55}$$

describes geodesic motion on $\mathbb{E}$ one can either prove that

$$\dot{\vec{x}} = Q\vec{p}, \quad \dot{\vec{p}} = -\frac{1}{2}\langle \vec{p}, \vec{\nabla} Q\vec{p} \rangle \tag{56}$$

implies $Q\ddot{\vec{x}} = \vec{0}$ (for this one can prove that for the general case of several constraints [5] $\varphi_1(\vec{x}) = 0, \ldots, \varphi_k(\vec{x}) = 0$ defining a submanifold, $h_{\alpha\beta} = \vec{\nabla}\varphi_\alpha \vec{\nabla}\varphi_\beta$ pos. def.,

$$\pi_{ij} := h^{\alpha\beta}\partial_i\varphi_\alpha\partial_j\varphi_\beta, \tag{57}$$

that $Q_{mi}\left(\partial_i Q_{kj}\right) Q_{jn}$ is symmetric in $(m \leftrightarrow n)$); or (Exercise V) explicitly calculate $\ddot{\vec{x}}$ from

$$\begin{aligned} \dot{\vec{x}} &= \vec{p} - \frac{\langle \vec{p}, A\vec{x} \rangle}{\langle \vec{x}, A^2\vec{x} \rangle} A\vec{x} = \vec{p} - \gamma A\vec{x}, \\ \dot{\vec{p}} &= \frac{\langle \vec{p}, A\vec{x} \rangle}{\langle \vec{x}, A^2\vec{x} \rangle} A\vec{p} - \frac{\langle \vec{p}, A\vec{x} \rangle^2}{\langle \vec{x}, A^2\vec{x} \rangle^2} A^2\vec{x} = \gamma A\vec{p} - \gamma^2 A^2\vec{x}. \end{aligned} \tag{58}$$

With many terms canceling, one arrives at

$$\ddot{\vec{x}} = \left(-\frac{\langle \vec{p}, A\vec{p} \rangle}{\langle \vec{x}, A^2\vec{x} \rangle} + 2\frac{\langle \vec{p}, A\vec{x} \rangle \langle \vec{x}, A^2\vec{p} \rangle}{\langle \vec{x}, A^2\vec{x} \rangle^2} - \frac{\langle \vec{p}, A\vec{x} \rangle^2}{\langle \vec{x}, A^2\vec{x} \rangle} \langle \vec{x}, A^3\vec{x} \rangle \right) A\vec{x} = -\dot{\gamma} A\vec{x} \tag{59}$$

Inserting $\dot{\vec{x}} = \vec{p} - \gamma A\vec{x}$ (cp. (58)) into $\langle \dot{\vec{x}}, A\dot{\vec{x}} \rangle$, (39) becomes (59). To then show the integrability of (55), a canonical transformation

$$\tilde{\vec{x}} = \sqrt{A}\vec{x}, \quad p = \sqrt{A}\tilde{\vec{p}}$$

is made in [5], with

$$
\begin{aligned}
H(\vec{x},\vec{p}) = \tilde{H}\big(\tilde{\vec{x}},\tilde{\vec{p}}\big) &= \frac{1}{2}\frac{\big\langle\tilde{\vec{p}},A\tilde{\vec{p}}\big\rangle\big\langle\tilde{\vec{x}},A\tilde{\vec{x}}\big\rangle - \big\langle\tilde{\vec{p}},A\tilde{\vec{x}}\big\rangle^2}{\big\langle\tilde{\vec{x}},A\tilde{\vec{x}}\big\rangle} \\
&= \frac{1}{2}\frac{\check{H}(\tilde{\vec{x}},\tilde{\vec{p}}) - 2E\big\langle\tilde{\vec{x}},A\tilde{\vec{x}}\big\rangle}{\big\langle\tilde{\vec{x}},A\tilde{\vec{x}}\big\rangle} + E = \hat{H}\big(\tilde{\vec{x}},\tilde{\vec{p}}\big) + E,
\end{aligned} \tag{60}
$$

so that $H(\vec{x},\vec{p}) = E = \tilde{H}(\tilde{\vec{x}},\tilde{\vec{p}})$ corresponds to $\hat{H} = 0$, and then note [5] that generally

$$H(\vec{x},\vec{p}) = \frac{G(\vec{x},\vec{p})}{Q(\vec{x},\vec{p})} \text{ on } G = 0 = H$$

for positive Q generates the same dynamics as G.
Finally,

$$G(\vec{x},\vec{p}) = \left\langle \vec{p}, A\vec{x} \right\rangle \left\langle \vec{x}, A\vec{x} \right\rangle - \left\langle \vec{p}, A\vec{x} \right\rangle^2 - 2E\left\langle \vec{x}, A\vec{x} \right\rangle = -\sum_i \frac{G_i}{\alpha_i}, \tag{61}$$

with Poisson-commuting

$$G_i := 2Ex_i^2 + \sum_j{}' \frac{L_{ij}^2}{\alpha_i - \alpha_j} \tag{62}$$

is Liouville-integrable.

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