Toward the classification of differential calculi on κ-Minkowski space and related field theories

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Content

- κ-Minkowski spacetime and the classification of differential calculi
- NC differential calculus over κ-Minkowski space
- NC field theory

κ-Minkowski space

κ-Minkowski space is a Lie-algebraic deformation of the usual Minkowski space, where $\kappa \propto \frac{1}{a_0}$ is the deformation parameter, usually interpreted as the Planck mass or the new scale of quantum gravity

$$[\hat{x}_i, \hat{x}_j] = 0, \quad [\hat{x}_0, \hat{x}_i] = i a_0 \hat{x}_i.$$

Or more generally, we can introduce a deformation vector a_μ such that

$$[\hat{x}_\mu, \hat{x}_\nu] = i C_{\mu\nu}{}^{\lambda} \hat{x}_\lambda = i(a_\mu \hat{x}_\nu - a_\nu \hat{x}_\mu).$$

Differential calculus of classical dimension

- We want to construct the most general algebra of differential one-forms $\hat{\xi}_\mu \equiv \hat{\mathrm{d}}\hat{x}_\mu \in \hat{\Omega}^1$ compatible with κ-Minkowski spacetime.
- We will impose that the differential algebra is closed in differential forms, i.e. the differential calculus is of classical dimension

$$[\hat{\xi}_\mu, \hat{x}_\nu] = iK_{\mu\nu}{}^{\alpha}\hat{\xi}_\alpha, \quad K_{\mu\nu}{}^{\alpha} \in \mathbb{R}, \tag{1}$$

- We claim that for all the solutions for $K_{\mu\nu}{}^{\alpha}$, we can find suitable Hopf algebras, so that the bicovariance condition is fulfilled.

Exterior derivative

We also introduce the exterior derivative $\hat{\mathrm{d}} \equiv [\hat{\eta}, \cdot]$ in a natural way

$$\hat{\mathrm{d}}\hat{x}_\mu = [\hat{\eta}, \hat{x}_\mu] = \hat{\xi}_\mu, \quad \hat{\eta}^2 = 0, \quad \hat{\eta} \in \hat{\mathcal{SH}}. \tag{2}$$

When we apply $\hat{\mathrm{d}} = [\hat{\eta}, \cdot]$ on (4) we get

$$[\hat{\xi}_\mu, \hat{x}_\nu] - [\hat{\xi}_\nu, \hat{x}_\mu] = iC_{\mu\nu}{}^\lambda \hat{\xi}_\lambda \longrightarrow K_{\mu\nu}{}^\alpha - K_{\nu\mu}{}^\alpha = C_{\mu\nu}{}^\alpha. \tag{3}$$

We call eq. (3) the *consistency condition*.

super-Jacobi identities

The only super-Jacobi identity that gives a constraint on $K_{\mu\nu}{}^{\alpha}$ is $\left[\hat{x}_\mu, [\hat{x}_\nu, \hat{\xi}_\rho]\right] + \left[\hat{x}_\nu, [\hat{\xi}_\rho, \hat{x}_\mu]\right] + \left[\hat{\xi}_\rho, [\hat{x}_\mu, \hat{x}_\nu]\right] = 0$ and it leads to

$$K_{\lambda\mu}{}^{\alpha} K_{\alpha\nu}{}^{\rho} - K_{\lambda\nu}{}^{\alpha} K_{\alpha\mu}{}^{\rho} = C_{\mu\nu}{}^{\beta} K_{\lambda\beta}{}^{\rho}. \tag{4}$$

Eq. (4) is valid for general Lie algebraic deformations of spacetime.

Ansatz for ${K_{\mu\nu}}^{\alpha}$

Classify differential algebras $\Leftrightarrow$ solve (4) and (3) for ${K_{\lambda\mu}}^{\alpha}$
For ${K_{\mu\nu}}^{\alpha} \in \mathbb{R}$ we demand

- that in the limit $a_\mu \to 0$ the problem reduces to commutative case i.e. $\lim_{a_\mu \to 0} {K_{\mu\nu}}^{\alpha} = 0$
- ${K_{\mu\nu}}^{\alpha}$ has the dimension of length.

Therefore, it follows that the most general ansatz (for $n > 2$) is given only in terms of $\eta_{\mu\nu}$ and a_μ via

$$K_{\mu\nu\alpha} = A_0 a_\mu a_\nu a_\alpha + A_1 \eta_{\mu\nu} a_\alpha + A_2 \eta_{\mu\alpha} a_\nu + A_3 \eta_{\nu\alpha} a_\mu, \qquad (5)$$

where $A_1, A_2, A_3 \in \mathbb{R}$ are dimensionless parameters and A_0 is of dimension (lenght)$^{-2}$, hence $A_0 = \frac{c}{a^2}$, $c \in \mathbb{R}$ for $a^2 \neq 0$ and $A_0 = 0$ for $a^2 = 0$.

System of equations for parameters A_0, A_1, A_2, A_3

After we impose (3) we get

$$A_3 = 1 + A_2. \tag{6}$$

Equation (4) gives

$$A_3(a^2 A_0 + A_3 - 1) = 0, \tag{7}$$
$$A_1(a^2 A_0 + A_1 + 1) = 0, \tag{8}$$
$$A_1 A_3 a^2 = 0. \tag{9}$$

Four families of solution for A_0, A_1, A_2, A_3

1. $A_1 = 0, \quad A_2 = -1, \quad A_3 = 0, \quad a^2 A_0 = c$
2. $A_1 = 0, \quad A_2 = -c, \quad A_3 = 1 - c, \quad a^2 A_0 = c$
3. $A_1 = -1 - c, \quad A_2 = -1, \quad A_3 = 0, \quad a^2 A_0 = c$
4. $A_1 = -1, \quad A_2 = 0, \quad A_3 = 1, \quad a^2 = A_0 = 0$

where $c \in \mathbb{R}$ is a free parameter.

Differential algebras $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$ and $\mathcal{C}_4$

$$
\begin{aligned}
\mathcal{C}_1 : \quad [\hat{\xi}_\mu, \hat{x}_\nu] &= \begin{cases} i\frac{c}{a^2} a_\mu a_\nu (a\hat{\xi}) - i a_\nu \hat{\xi}_\mu, & \text{if } a^2 \neq 0 \\ -i a_\nu \hat{\xi}_\mu, & \text{if } a^2 = 0 \end{cases} \\
\mathcal{C}_2 : \quad [\hat{\xi}_\mu, \hat{x}_\nu] &= \begin{cases} i\frac{c}{a^2} a_\mu a_\nu (a\hat{\xi}) - i c a_\nu \hat{\xi}_\mu + i(1-c) a_\mu \hat{\xi}_\nu, & \text{if } a^2 \neq 0 \\ i a_\mu \hat{\xi}_\nu, & \text{if } a^2 = 0 \end{cases} \\
\mathcal{C}_3 : \quad [\hat{\xi}_\mu, \hat{x}_\nu] &= \begin{cases} i\frac{c}{a^2} a_\mu a_\nu (a\hat{\xi}) - i(1+c)\eta_{\mu\nu}(a\hat{\xi}) - i a_\nu \hat{\xi}_\mu, & \text{if } a^2 \neq 0 \\ -i\eta_{\mu\nu}(a\hat{\xi}) - i a_\nu \hat{\xi}_\mu, & \text{if } a^2 = 0 \end{cases} \\
\mathcal{C}_4 : \quad [\hat{\xi}_\mu, \hat{x}_\nu] &= -i\eta_{\mu\nu}(a\hat{\xi}) + i a_\mu \hat{\xi}_\nu, \quad a^2 = 0
\end{aligned}
\tag{10}
$$

where we used $a\hat{\xi} \equiv a_\alpha \hat{\xi}^\alpha$ and $\mathcal{C}$ stands for *covariant*.

- It is important to note that the first three solutions $\mathcal{C}_{1,2,3}$ are valid for all $a^2 \in \mathbb{R}$.
- There are two cases: when $a^2 = 0$ then $A_0 = c = 0$, and when $a^2 \neq 0$ then $A_0 = c/a^2$.
- The fourth solution $\mathcal{C}_4$ is only valid in the light-like case $a^2 = 0$.

Explicitly for the tensor $K_{\mu\nu\alpha}$ we have

$$
\begin{aligned}
\mathcal{C}_1 : \quad K_{\mu\nu\alpha} &= \begin{cases} \frac{c}{a^2} a_\mu a_\nu a_\alpha - \eta_{\mu\alpha} a_\nu , & \text{if } a^2 \neq 0 \\ -\eta_{\mu\alpha} a_\nu , & \text{if } a^2 = 0. \end{cases} \\
\mathcal{C}_2 : \quad K_{\mu\nu\alpha} &= \begin{cases} \frac{c}{a^2} a_\mu a_\nu a_\alpha - c\eta_{\mu\alpha} a_\nu + (1-c)\eta_{\nu\alpha} a_\mu , & \text{if } a^2 \neq 0 \\ \eta_{\nu\alpha} a_\mu , & \text{if } a^2 = 0. \end{cases} \\
\mathcal{C}_3 : \quad K_{\mu\nu\alpha} &= \begin{cases} \frac{c}{a^2} a_\mu a_\nu a_\alpha - (1+c)\eta_{\mu\nu} a_\alpha - \eta_{\mu\alpha} a_\nu , & \text{if } a^2 \neq 0 \\ -\eta_{\mu\nu} a_\alpha - \eta_{\mu\alpha} a_\nu , & \text{if } a^2 = 0, \end{cases} \\
\mathcal{C}_4 : \quad K_{\mu\nu\alpha} &= -\eta_{\mu\nu} a_\alpha + \eta_{\nu\alpha} a_\mu , \quad \text{only for } a^2 = 0.
\end{aligned} \tag{11}
$$

Some special cases

If $c = 0$ and $A_0 = 0$ in (11) $\Rightarrow$ three special cases[1] $\mathcal{S}_1$, $\mathcal{S}_2$ and $\mathcal{S}_3$

$$\begin{aligned} \mathcal{S}_1 : \quad & [\hat{\xi}_\mu, \hat{x}_\nu] = -ia_\nu \hat{\xi}_\mu, \quad a^2 \in \mathbb{R} \\ \mathcal{S}_2 : \quad & [\hat{\xi}_\mu, \hat{x}_\nu] = ia_\mu \hat{\xi}_\nu, \quad a^2 \in \mathbb{R} \\ \mathcal{S}_3 : \quad & [\hat{\xi}_\mu, \hat{x}_\nu] = -i\eta_{\mu\nu}(a\hat{\xi}) - ia_\nu \hat{\xi}_\mu, \quad a^2 \in \mathbb{R} \end{aligned} \tag{12}$$

- $\mathcal{S}_1 \Rightarrow$ *right covariant* realization
- $\mathcal{S}_2 \Rightarrow$ *left covariant* realization
- $\mathcal{S}_3 \Rightarrow$ *Magueijo-Smolin* realization
- $\mathcal{C}_4 \Rightarrow$ *natural* realization

[1] Where $\mathcal{S}$ stands for *special*.

If we take $a^2 \neq 0$ we get three families of algebras which we denote by[2] $\mathcal{D}_1$, $\mathcal{D}_2$ and $\mathcal{D}_3$:

$$\mathcal{D}_1 : \quad [\hat{\xi}_\mu, \hat{x}_\nu] = i\frac{c}{a^2} a_\mu a_\nu (a\hat{\xi}) - i a_\nu \hat{\xi}_\mu \tag{13}$$

$$\mathcal{D}_2 : \quad [\hat{\xi}_\mu, \hat{x}_\nu] = i\frac{c}{a^2} a_\mu a_\nu (a\hat{\xi}) - ic a_\nu \hat{\xi}_\mu + i(1-c) a_\mu \hat{\xi}_\nu \tag{14}$$

$$\mathcal{D}_3 : \quad [\hat{\xi}_\mu, \hat{x}_\nu] = i\frac{c}{a^2} a_\mu a_\nu (a\hat{\xi}) - i(1+c)\eta_{\mu\nu}(a\hat{\xi}) - i a_\nu \hat{\xi}_\mu \tag{15}$$

For $a_\mu = (a_0, \vec{0})$ algebras $\mathcal{D}_1$ and $\mathcal{D}_2$ can be found in
S. Meljanac, S. Kresic-Juric ,R. Strajn, Int. J. Mod. Phys. A27 (2012) 1250057.

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[2] Where $\mathcal{D}$ stands for *differential algebra* and was already used in S. Meljanac, S. Kresic-Juric ,R. Strajn, Int. J. Mod. Phys. A27 (2012) 1250057.

Some remarks

- Solutions $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$ are new and all of them are valid for time-like, light-like and space-like deformation parameter a_μ.
- For time-like deformation $a_\mu = (a_0, 0, 0, ...)$, differential algebras $\mathcal{D}_1$, $\mathcal{D}_2$ were constructed in

  S. Meljanac, S. Kresic-Juric ,R. Strajn, Int. J. Mod. Phys. A27 (2012) 1250057,
- $\mathcal{D}_3$ obtained from $\mathcal{C}_3$ is a new solution.
- For $c = 1 \Rightarrow \mathcal{D}_1^{c=1} = \mathcal{D}_2^{c=1}$. This case was in detail investigated in
  T J S. Meljanac, R. Štrajn, Eur. Phys. J. C (2013) 73: 2472, arXiv:1211.6612 [hep-th]
- In R. Oeckl, J. Math. Phys. 40, 3588-3604, 1999 (see Corollary 5.1.) the cases $\mathcal{D}_1^{c=0}$ and $\mathcal{D}_2^{c=0}$ were obtained from a different construction.
- For light-like deformation $a^2 = 0$, we have also found three new solutions.

Usual differential calculus in the algebraic language

- Algebra of functions $\Rightarrow$ unital Abelian algebra $\mathcal{A}$ generated by x_μ
- Algebra of forms $\Rightarrow \Omega^p \subset \mathcal{S}\mathcal{A}$, $p \in \{0, 1, \dots, n\}$, generated by p-forms $\omega = \omega_{\alpha_1 \dots \alpha_p} \xi^{\alpha_1} \dots \xi^{\alpha_p}$, where $\omega_{\alpha_1 \dots \alpha_p} \in \mathcal{A}$ and $\Omega^0 \equiv \mathcal{A}$.
- Exterior derivative $\mathrm{d} : \Omega^p \to \Omega^{p+1}$ satisfies the Leibniz rule

$$\mathrm{d}(fg) = (\mathrm{d}f)g + f(\mathrm{d}g), \tag{16}$$

  where $f, g \in \mathcal{A}$
- Simply realized by $\mathrm{d} \equiv [\eta, \cdot]$ and $\eta \equiv \xi^\alpha \partial_\alpha$. Of course, since $[x_\mu, \xi_\nu] = 0$ we have that one forms are given by

$$\mathrm{d}f = \xi^\alpha (\partial_\alpha \triangleright f) = (\partial_\alpha \triangleright f) \xi^\alpha, \tag{17}$$

  where $\partial_\alpha \triangleright f = \frac{\partial f}{\partial x^\alpha}$.
- Usually one forgets about $\triangleright$ and simply writes $\mathrm{d}f = \frac{\partial f}{\partial x^\alpha} \mathrm{d}x^\alpha$.

NC case

- $\mathcal{A} \Rightarrow \hat{\mathcal{A}}$ is an unital algebra generated by $\hat{x}_\mu$
- $\Omega^p \subset \mathcal{S}\mathcal{A} \Rightarrow \hat{\Omega}^p \subset \hat{\mathcal{S}\mathcal{A}}$ is an algebra generated by NC p-forms $\hat{\omega} = \hat{\omega}_{\alpha_1 \dots \alpha_p} \hat{\xi}^{\alpha_1} \dots \hat{\xi}^{\alpha_p}$, where $\hat{\omega}_{\alpha_1 \dots \alpha_p} \in \hat{\mathcal{A}}$ and $\hat{\Omega}^0 \equiv \hat{\mathcal{A}}$.
- $\mathrm{d} \Rightarrow \hat{\mathrm{d}}$ is a map $\hat{\mathrm{d}} : \hat{\Omega}^p \to \hat{\Omega}^{p+1}$ that satisfies the Leibniz rule

$$\hat{\mathrm{d}}(\hat{f}\hat{g}) = (\hat{\mathrm{d}}\hat{f})\hat{g} + \hat{f}(\hat{\mathrm{d}}\hat{g}), \qquad (18)$$

  where $\hat{f}$, $\hat{g} \in \hat{\mathcal{A}}$.
- Eq. (18) is fulfilled by choosing $\hat{\mathrm{d}} \equiv [\hat{\eta}, \cdot]$ and $\hat{\eta} \equiv \hat{\xi}^\alpha \partial_\alpha$.

The commutation relations between differential forms $\hat{\xi}_\mu$ and an arbitrary element of $\hat{\mathcal{A}}$ can be written as

$$\hat{\xi}_\mu \hat{f} = (\Lambda_{\mu\alpha} \blacktriangleright \hat{f})\hat{\xi}^\alpha, \quad \hat{f}\hat{\xi}_\mu = \hat{\xi}^\alpha(\Lambda^{-1}_{\mu\alpha} \blacktriangleright \hat{f}), \tag{19}$$

where $\Lambda_{\mu\nu}$ is expressed in terms of derivatives ∂_μ and $\Lambda^{-1}_{\mu\nu} \equiv (\Lambda^{-1})_{\mu\nu}$ denotes the inverse matrix, i.e. $\Lambda^{-1}_{\mu\alpha}\Lambda^\alpha{}_\nu = \eta_{\mu\nu}$. Since $\hat{\mathrm{d}}\hat{f} = [\hat{\eta}, \hat{f}] \in \hat{\Omega}^1 \subset \hat{\mathcal{S}\mathcal{A}}$ it follows

$$\hat{\mathrm{d}}\hat{f} = [\hat{\eta}, \hat{f}] \blacktriangleright 1 = \hat{\eta}\hat{f} \blacktriangleright 1 = \hat{\eta} \blacktriangleright \hat{f} = \hat{\xi}^\alpha(\partial_\alpha \blacktriangleright \hat{f}). \tag{20}$$

.

That is we have

$$\hat{\mathrm{d}}\hat{f} = \hat{\xi}^{\alpha}(\partial_{\alpha} \blacktriangleright \hat{f}) = (\partial_{\beta}\Lambda^{\beta}{}_{\alpha} \blacktriangleright \hat{f})\hat{\xi}^{\alpha}. \tag{21}$$

Furthermore, eq. (21) and Leibniz rule (18) imply

$$\begin{aligned} \hat{\mathrm{d}}(\hat{f}\hat{g}) &\overset{(18)}{\equiv} (\hat{\mathrm{d}}\hat{f})\hat{g} + \hat{f}(\hat{\mathrm{d}}\hat{g}) \overset{(21)}{\equiv} \hat{\xi}^{\alpha}\partial_{\alpha} \blacktriangleright (\hat{f}\hat{g}) \\ &\overset{(21)}{=} \hat{\xi}^{\alpha}(\partial_{\alpha} \blacktriangleright \hat{f})\hat{g} + \hat{f}\hat{\xi}^{\alpha}(\partial_{\alpha} \blacktriangleright \hat{g}) \\ &\overset{(19)}{=} \hat{\xi}^{\alpha}[(\partial_{\alpha} \blacktriangleright \hat{f})\hat{g} + (\Lambda^{-1}_{\beta\alpha} \blacktriangleright \hat{f})(\partial^{\beta} \blacktriangleright \hat{g})] \end{aligned} \tag{22}$$

and by comparing the first and last line in (22) we get the Leibniz rule for ∂_{α}

$$\partial_{\alpha} \blacktriangleright (\hat{f}\hat{g}) = (\partial_{\alpha} \blacktriangleright \hat{f})\hat{g} + (\Lambda^{-1}_{\beta\alpha} \blacktriangleright \hat{f})(\partial^{\beta} \blacktriangleright \hat{g}). \tag{23}$$

The Leibniz rule and coproduct are related via $\partial_\alpha \blacktriangleright (\hat{f}\hat{g}) = m[\Delta\partial_\alpha \blacktriangleright (\hat{f} \otimes \hat{g})]$, where $m(a \otimes b) = ab$, so that the coproduct for ∂_μ is given by

$$\Delta\partial_\mu = \partial_\mu \otimes 1 + \Lambda^{-1}_{\alpha\mu} \otimes \partial^\alpha, \tag{24}$$

and since ∂_μ generates a Hopf algebra of translations it follows that its antipode and counit are

$$S(\partial_\mu) = -\partial_\alpha \Lambda^\alpha{}_\mu, \quad \epsilon(\partial_\mu) = 0. \tag{25}$$

The associativity of the product between forms and elements of $\hat{\mathcal{A}}$ gives

$$\begin{aligned}\hat{\xi}_\mu(\hat{f}\hat{g}) &= (\hat{\xi}_\mu\hat{f})\hat{g} \\ \overset{(19)}{\Longrightarrow} [\Lambda_{\mu\alpha} \blacktriangleright (\hat{f}\hat{g})]\hat{\xi}^\alpha &= (\Lambda_{\mu\alpha} \blacktriangleright \hat{f})\hat{\xi}^\alpha\hat{g} \overset{(19)}{\Longleftarrow} \\ &\overset{(19)}{=} (\Lambda_{\mu\alpha} \blacktriangleright \hat{f})(\Lambda^\alpha_\beta \blacktriangleright \hat{g})\hat{\xi}^\beta.\end{aligned} \tag{26}$$

We can read out the Leibniz rule for $\Lambda_{\mu\alpha}$ as

$$\Lambda_{\mu\alpha} \blacktriangleright (\hat{f}\hat{g}) = (\Lambda_{\mu\beta} \blacktriangleright \hat{f})(\Lambda^\beta{}_\alpha \blacktriangleright \hat{g}), \tag{27}$$

and extract the following coproduct

$$\Delta\Lambda_{\mu\nu} = \Lambda_{\mu\alpha} \otimes \Lambda^\alpha{}_\nu. \tag{28}$$

Additionally we have

$$S(\Lambda_{\mu\nu}) = \Lambda^{-1}_{\mu\nu}, \quad \epsilon(\Lambda_{\mu\nu}) = \eta_{\mu\nu}. \tag{29}$$

Construction of higher forms $\Rightarrow$ all products of higher forms are again differential forms regardless of the way they are multiplied. Example: consider two different forms: $\hat{\omega} \in \hat{\Omega}^q$ and $\hat{\theta} \in \hat{\Omega}^p$. Written in basis we have

$$\begin{aligned} \hat{\omega} &= \hat{\omega}_{\mu_1 \dots \mu_q} \hat{\xi}^{\mu_1} \dots \hat{\xi}^{\mu_q} = \hat{\xi}^{\mu_1} \dots \hat{\xi}^{\mu_q} \hat{\tilde{\omega}}_{\mu_1 \dots \mu_q}, \\ \hat{\theta} &= \hat{\theta}_{\mu_1 \dots \mu_p} \hat{\xi}^{\mu_1} \dots \hat{\xi}^{\mu_p} = \hat{\xi}^{\mu_1} \dots \hat{\xi}^{\mu_p} \hat{\tilde{\theta}}_{\mu_1 \dots \mu_p}, \end{aligned} \tag{30}$$

where $\hat{\omega}_{\mu_1 \dots \mu_q}$, $\hat{\tilde{\omega}}_{\mu_1 \dots \mu_q}$, $\hat{\theta}_{\mu_1 \dots \mu_p}$ and $\hat{\tilde{\theta}}_{\mu_1 \dots \mu_p} \in \hat{\mathcal{A}}$. There is a relation between $\hat{\omega}_{\mu_1 \dots \mu_q}$ and $\hat{\tilde{\omega}}_{\mu_1 \dots \mu_q}$ via (19) (same for $\hat{\theta}_{\mu_1 \dots \mu_p}$ and $\hat{\tilde{\theta}}_{\mu_1 \dots \mu_p}$).

Of course if we multiply $\hat{\omega}$ with $\hat{\theta}$ it is easy to see that $\hat{\omega}\hat{\theta} \neq \hat{\theta}\hat{\omega}$, but we have

$$\begin{aligned} \hat{\omega}\hat{\theta} &= \hat{\alpha}_{\mu_1 \dots \mu_{q+p}} \hat{\xi}^{\mu_1} \dots \hat{\xi}^{\mu_{q+p}} = \hat{\xi}^{\mu_1} \dots \hat{\xi}^{\mu_{q+p}} \hat{\tilde{\alpha}}_{\mu_1 \dots \mu_{q+p}} \in \hat{\Omega}^{q+p} \\ \hat{\theta}\hat{\omega} &= \hat{\beta}_{\mu_1 \dots \mu_{q+p}} \hat{\xi}^{\mu_1} \dots \hat{\xi}^{\mu_{q+p}} = \hat{\xi}^{\mu_1} \dots \hat{\xi}^{\mu_{q+p}} \hat{\tilde{\beta}}_{\mu_1 \dots \mu_{q+p}} \in \hat{\Omega}^{q+p} \end{aligned} \tag{31}$$

where $\hat{\alpha}_{\mu_1 \dots \mu_{q+p}}$, $\hat{\tilde{\alpha}}_{\mu_1 \dots \mu_{q+p}}$, $\hat{\beta}_{\mu_1 \dots \mu_{q+p}}$ and $\hat{\tilde{\beta}}_{\mu_1 \dots \mu_{q+p}} \in \hat{\mathcal{A}}$ and they are interrelated with $\hat{\omega}_{\mu_1 \dots \mu_q}$ and $\hat{\theta}_{\mu_1 \dots \mu_p}$ by using (19). We can define all the higher forms in a consistent way (as illustrated above).

In order to use this differential calculus and do practical calculations it is important to know all the commutation rules for (and between) the elements of $\hat{\mathcal{A}}$ and $\hat{\Omega}^p$.

- The commutation rule between $\hat{\xi}_\mu$ and an arbitrary element of $\hat{\mathcal{A}}$ is determined by $\Lambda_{\mu\nu}$.
- The commutation rule between $\hat{x}_\mu$ and an arbitrary element of $\hat{\mathcal{A}}$ is determined by $O_{\mu\nu}$.

$$\hat{x}_\mu \hat{f} = (O_{\mu\alpha} \blacktriangleright \hat{f})\hat{x}^\alpha, \quad \hat{f}\hat{x}_\mu = \hat{x}^\alpha([O^{-1}]_{\mu\alpha} \blacktriangleright \hat{f}). \tag{32}$$

$$O_{\mu\nu} = \left(e^{\mathcal{C}}\right)_{\mu\nu}, \qquad \mathcal{C}_{\mu\nu} = iC_{\mu\alpha\nu}(\partial^W)^\alpha, \tag{33}$$

and similarly $\Lambda_{\mu\nu}$ is given by

$$\Lambda_{\mu\nu} = \left(e^{\mathcal{K}}\right)_{\mu\nu}, \qquad \mathcal{K}_{\mu\nu} = iK_{\mu\alpha\nu}(\partial^W)^\alpha, \tag{34}$$

where ∂^W is the derivative corresponding to the Weyl ordering:

$$[\partial^W_\mu, \hat{x}_\nu] = \eta_{\mu\nu}\frac{ia\partial^W}{e^{ia\partial^W} - 1} + \frac{ia_\nu \partial^W_\mu}{ia\partial^W}\left(1 - \frac{ia\partial^W}{e^{ia\partial^W} - 1}\right), \tag{35}$$

where we used $a\partial^W \equiv a^\alpha \partial^W_\alpha$.

Symmetries of differential algebras and bicovarinace

- Covariance of differential calculus under a certain symmetry algebra $\mathcal{G} \subset \hat{\mathcal{SH}}$ generated by $g_i \in \mathcal{G}$ is defined in the following way

$$\begin{aligned} g_i \blacktriangleright (\hat{x}_\mu \hat{\xi}_\nu) &= m\left(\Delta g_i (\blacktriangleright \otimes \blacktriangleright)(\hat{x}_\mu \otimes \hat{\xi}_\nu)\right), \\ g_i \blacktriangleright (\hat{\xi}_\nu \hat{x}_\mu) &= m\left(\Delta g_i (\blacktriangleright \otimes \blacktriangleright)(\hat{\xi}_\nu \otimes \hat{x}_\mu)\right). \end{aligned} \tag{36}$$

- Combined with

$$[\hat{\xi}_\mu, \hat{x}_\nu] = iK_{\mu\nu}{}^{\alpha}\hat{\xi}_\alpha, \tag{37}$$

we can show that all the requirements for bicovariance are satisfied $\Leftrightarrow$ The differential calculi that we developed so far are bicovariant.

- $\mathcal{C}_{1,2,3,4}$ are covariant under certain κ-deformation of the $\mathfrak{igl}(n)$-Hopf algebra, but in the special case of $\mathcal{S}_1$ we have Poincaré -Weyl, and in the case of $\mathcal{C}_4$ κ- Poincaré covariance.

- The study of field theory over κ-Minkowski space may provide some physical manifestations of quantum aspects of gravity.
- There is fairly large literature on κ-deformed field theory **[a lot of ref.!]**, but all of these theories are special in a sense that they can be related to a specific realization or they are using the differential calculus with one extra form.
- Our goal is to give a framework for constructing field theory with differential calculus of classical dimension.
- In order to do this we need to introduce higher-degree forms, the Hodge-$*$ operation and an integral to define an action for the fields.

- The higher degree forms in NC case are defined via

$$\hat{\boldsymbol{\omega}} = \hat{\omega}_{\alpha_1 \dots \alpha_p} \hat{\xi}^{\alpha_1} \dots \hat{\xi}^{\alpha_p} \in \hat{\Omega}^p. \tag{38}$$

- The Hodge-$\hat{*}$ operation is defined as a mapping $\hat{*} : \hat{\Omega}^p \to \hat{\Omega}^{n-p}$ by

$$\hat{\boldsymbol{\alpha}} \ (\hat{\boldsymbol{\beta}})^{\hat{*}} = (\hat{\boldsymbol{\alpha}})^{\hat{*}} \ \hat{\boldsymbol{\beta}} \equiv \hat{\alpha}_{\mu_1 \dots \mu_k} \hat{\beta}^{\mu_1 \dots \mu_k} \ \hat{\text{vol}}, \tag{39}$$

where $\hat{\boldsymbol{\alpha}}, \hat{\boldsymbol{\beta}} \in \hat{\Omega}^k$ and $\hat{\textbf{vol}} = \xi^0 \dots \xi^{n-1}$ is the volume form.

- For $n = 4$ we have

$$\begin{aligned} &(1)^{\hat{*}} = \hat{\textbf{vol}} = \hat{\xi}^0 \hat{\xi}^1 \hat{\xi}^2 \hat{\xi}^3 = \frac{1}{4!} \epsilon_{\mu\nu\rho\sigma} \hat{\xi}^\mu \hat{\xi}^\nu \hat{\xi}^\rho \hat{\xi}^\sigma, \\ &(\hat{\xi}^\mu)^{\hat{*}} = \frac{1}{3!} \epsilon^\mu{}_{\alpha_1\alpha_2\alpha_3} \hat{\xi}^{\alpha_1} \hat{\xi}^{\alpha_2} \hat{\xi}^{\alpha_3}, \quad (\hat{\xi}^\mu \hat{\xi}^\nu)^{\hat{*}} = \frac{1}{2!} \epsilon^{\mu\nu}{}_{\alpha_1\alpha_2} \hat{\xi}^{\alpha_1} \hat{\xi}^{\alpha_2}, \\ &(\hat{\xi}^\mu \hat{\xi}^\nu \hat{\xi}^\rho)^{\hat{*}} = \epsilon^{\mu\nu\rho}{}_\alpha \hat{\xi}^\alpha, \quad (\hat{\xi}^\mu \hat{\xi}^\nu \hat{\xi}^\rho \hat{\xi}^\sigma)^{\hat{*}} = \epsilon^{\mu\nu\rho\sigma}. \end{aligned} \tag{40}$$

- The integral is defined as a linear map

$$\int : \hat{\Omega}^n \to \mathbb{C}. \tag{41}$$

- Integral is closed in the sense that

$$\int \hat{\mathrm{d}}\hat{\boldsymbol{\omega}} = 0, \quad \forall \hat{\boldsymbol{\omega}} \in \hat{\Omega}^n. \tag{42}$$

At this level, the integral defined here is just a formal notation. However, in the $\mathcal{C}_4$-case the integral is invariant under the action of κ-Poincaré algebra, so that the integral introduced here is the standard Lebesque integral applied to the functions which give a realization of the κ-Poincaré algebra through the $\star$-product.

Now, we are ready to write an action $\hat{\mathbf{S}}$ for a real NC scalar field $\hat{\phi} \in \hat{\mathcal{A}}$. We have

$$\hat{\mathbf{S}} = \int \hat{\mathrm{d}}\hat{\phi} \ \ (\hat{\mathrm{d}}\hat{\phi})^{\hat{*}} + m^2 \hat{\phi} \ \ (\hat{\phi})^{\hat{*}}. \tag{43}$$

Since $\hat{\mathrm{d}}\hat{\phi} = (\partial_\beta \Lambda^\beta{}_\alpha \blacktriangleright \hat{\phi})\hat{\xi}^\alpha$ and using (39) we have

$$\hat{\mathbf{S}} = \int \left((\partial_\beta \Lambda^\beta{}_\alpha \blacktriangleright \hat{\phi})(\partial_\rho \Lambda^{\rho\alpha} \blacktriangleright \hat{\phi}) + m^2 \hat{\phi}\hat{\phi} \right) \hat{\mathbf{vol}}. \tag{44}$$

To find the equation of motion we impose $\delta\hat{\mathbf{S}} = 0$, that is

$$\delta\hat{\mathbf{S}} = \int \hat{\mathrm{d}}\delta\hat{\phi}\ \ (\hat{\mathrm{d}}\hat{\phi})^{\hat{*}} + \hat{\mathrm{d}}\hat{\phi}\ \ (\hat{\mathrm{d}}\delta\hat{\phi})^{\hat{*}} + m^2\delta\hat{\phi}\ \ (\hat{\phi})^{\hat{*}} + m^2\hat{\phi}\ \ (\delta\hat{\phi})^{\hat{*}}$$
$$= \int \delta\hat{\phi}\ \ \left[-\hat{\mathrm{d}}(\hat{\mathrm{d}}\hat{\phi})^{\hat{*}} + m^2(\hat{\phi})^{\hat{*}}\right] + \left[-\hat{\mathrm{d}}(\hat{\mathrm{d}}\hat{\phi})^{\hat{*}} + m^2(\hat{\phi})^{\hat{*}}\right]\ \ \delta\hat{\phi} \tag{45}$$

which leads to

$$\left[\hat{\mathrm{d}}(\hat{\mathrm{d}}\hat{\phi})^{\hat{*}}\right]^{\hat{*}} = m^2\hat{\phi} \tag{46}$$

where we used

$$\int \hat{\mathrm{d}}[\delta\hat{\phi}\ \ (\hat{\mathrm{d}}\hat{\phi})^{*}] = 0 = \int \hat{\mathrm{d}}\delta\hat{\phi}\ \ (\hat{\mathrm{d}}\hat{\phi})^{\hat{*}} + \int \delta\hat{\phi}\ \ [\hat{\mathrm{d}}(\hat{\mathrm{d}}\hat{\phi})^{\hat{*}}],$$
$$\int \hat{\mathrm{d}}[\hat{\mathrm{d}}\hat{\phi}\ \ (\delta\hat{\phi})^{\hat{*}}] = 0 = \int \hat{\mathrm{d}}(\hat{\mathrm{d}}\hat{\phi})^{\hat{*}}\ \ \delta\hat{\phi} + \int \hat{\mathrm{d}}\hat{\phi}\ \ (\hat{\mathrm{d}}\delta\hat{\phi})^{\hat{*}}, \tag{47}$$
$$((\hat{\phi})^{\hat{*}})^{\hat{*}} = \hat{\phi}.$$

Eq. (46) represents the NC generalization of the Klein-Gordon equation. Let us investigate the l.h.s. of eq.(46). We have

$$\begin{aligned} \left[\hat{\mathrm{d}}(\hat{\mathrm{d}}\hat{\phi})^{\hat{*}}\right]^{\hat{*}} &= \left[\hat{\mathrm{d}}\left((\partial_\beta \Lambda^\beta{}_\alpha \blacktriangleright \hat{\phi})\hat{\xi}^\alpha\right)^{\hat{*}}\right]^{\hat{*}} \\ &= \left[\hat{\mathrm{d}}\left((\partial_\beta \Lambda^\beta{}_\alpha \blacktriangleright \hat{\phi})\frac{1}{3!}\epsilon^\alpha{}_{\rho_1\rho_2\rho_3}\hat{\xi}^{\rho_1}\hat{\xi}^{\rho_2}\hat{\xi}^{\rho_3}\right)\right]^{\hat{*}} \\ &\dots \\ &= \partial_\gamma \Lambda^\gamma{}_\delta \partial_\beta \Lambda^{\beta\delta} \blacktriangleright \hat{\phi}. \end{aligned} \tag{48}$$

So, for the equation of motion we have

$$\partial_\alpha \partial_\beta \Lambda^\alpha{}_\sigma \Lambda^{\beta\sigma} \blacktriangleright \hat{\phi} - m^2 \hat{\phi} = 0. \tag{49}$$

Dispersion relations

- $$\mathcal{S}_1 : \quad (\partial^R)^2 Z^{-2} \blacktriangleright \hat{\phi} - m^2 \hat{\phi} = 0, \tag{50}$$
  $$\mathcal{S}_2 : \quad (\partial^L)^2 Z^{2} \blacktriangleright \hat{\phi} - m^2 \hat{\phi} = 0. \tag{51}$$
- $\Rightarrow$ the main new NC feature is the modification of dispersion relations
  $$\begin{aligned} \mathcal{S}_1 : &\quad E^2 - \vec{p}^2 = (mZ)^2, \quad Z = 1 - ap \\ \mathcal{S}_2 : &\quad E^2 - \vec{p}^2 = \left(\frac{m}{Z}\right)^2, \quad Z = \frac{1}{1 + ap} \end{aligned} \tag{52}$$
- $$\mathcal{C}_4 : \quad D^2 \blacktriangleright \hat{\phi} - m^2 \hat{\phi} = 0. \tag{53}$$
- For $\mathcal{C}_4$ we get $D^2 = \square$ $(a^2 = 0)$, that is the Casimir operator of the Poincaré algebra. This was expected, since $\mathcal{C}_4$ is compatible with the Poincaré algebra, and only the coalgebraic sector is deformed.

Future prospects

- Formulate κ-deformed electrodynamics via

$$\hat{\mathbf{S}} = -\frac{1}{4}\int \hat{\mathbf{F}} \ \ (\hat{\mathbf{F}})^{\hat{*}}, \tag{54}$$

where $\hat{\mathbf{F}} = \hat{\mathrm{d}}\hat{\mathbf{A}}$. The equations of motion are given by $\delta\hat{\mathbf{S}} = 0$, that is

$$\hat{\mathrm{d}}(\hat{\mathrm{d}}\hat{\mathbf{A}})^{\hat{*}} = 0, \quad \Leftrightarrow \quad \hat{\mathrm{d}}(\hat{\mathbf{F}})^{\hat{*}} = 0. \tag{55}$$

The NC version of Bianchi identity also holds
$\hat{\mathrm{d}}\hat{\mathbf{F}} = \hat{\mathrm{d}}(\hat{\mathrm{d}}\hat{\mathbf{A}}) = \hat{\eta}^2 \blacktriangleright \hat{\mathbf{A}} = 0$.

Future prospects

- So far we have analyzed the NC version of the free classical field theory $\Rightarrow$ modification of dispersion relations.
- Interacting classical field theory $\Rightarrow$ adding " $\hat{\phi}^n$ "
- NC quantum field theory $\Rightarrow$ R-matrix will modify the quantization procedure $\Rightarrow$ modification of the algebra of creation and annihilation operators

$$\phi(x) \otimes \phi(y) - R\phi(y) \otimes \phi(x) = 0 \tag{56}$$

- R-matrix is defined by the twist operator $R = \tilde{\mathcal{F}}\mathcal{F}^{-1}$.
  -R-matrix $\Rightarrow$ particle statistics
  -twist operator $\Rightarrow$ star-product $\Rightarrow$ action in terms of commutative fields
  $\Rightarrow$ Feynman rules $\Rightarrow$ NC correction to the propagator and vertex.
- What to expect? $\Rightarrow$ modification of the usual spin-statistics relations of free bosons at Planck scale.

Thank you for your attention!