Deformation quantizations of symplectic Lie groups and associated PDEs hierarchies

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The terminology of « quantization » is used to allude to the expression at a quantum level of facts related to a classical system

$\rightsquigarrow$ different mathematical tools :

	classical	quantum
states :	symplectic manifold	Hilbert space
observables :	smooth fonctions	linear operators
	($\rightsquigarrow$ commutative)	($\rightsquigarrow$ noncommutative)

Many methods exist to approach this problem ...

Here : deformation quantization

> $\rightsquigarrow$ « quantization be understood as a deformation of the structure of the algebra of classical observables, rather than as a radical change in the nature of the observables »
>
> (Bayen - Flato - Fronsdal - Lichnerowicz - Sternheimer, 1978)

Formal and non-formal deformation quantization

on the symplectic manifold (M, ω) :

? $K_\hbar(-,-,-)$? explicit 3-point kernel such that the formula

$$(f *_\hbar g)(x) = \int_{M\times M} K_\hbar(x,y,z)\; f(y)\; g(z)\; \underset{\swarrow}{dy}\, \underset{\swarrow}{dz}$$

(Liouville)

- defines an associative product on an « interesting » space of functions $(\ni f, g)$;
- admits an asymptotic expansion :

$$f *_\hbar g \sim fg + \hbar C_1(f,g) + o\left(\hbar^2\right).$$

$\rightsquigarrow$ star-product

Example on the phase space $\left(\mathbb{R}^2 = \{q, p\}, \omega = dq \wedge dp\right)$:

- the formula

$$\begin{aligned} &\left(f_1 *_\hbar^W f_2\right)(q_0, p_0) \\ &= \left(\frac{1}{2\pi\hbar}\right)^2 \int_{\mathbb{R}^2 \times \mathbb{R}^2} \exp\left[\frac{2i}{\hbar}\left(q_0 p - p_0 q + q p' - p q' + q' p_0 - p' q_0\right)\right] \\ &\qquad f_1(q, p)\ f_2(q', p')\ dp\, dq\, dp'\, dq'. \end{aligned}$$

  defines an associative (noncommutative) product on $\mathcal{S}\left(\mathbb{R}^2\right)$;

- formal asymptotic expansion of $f_1 *_\hbar^W f_2$ in $\hbar$:

$$\begin{aligned} f_1 *_\nu^0 f_2 \quad := \quad & f_1 f_2 + \nu \{f_1, f_2\} \\ & + \sum_{k=2}^{+\infty} \frac{\nu^k}{k!} \sum_{\substack{1 \le i_1, \dots, i_k \le 2 \\ 1 \le j_1, \dots, j_k \le 2}} \omega^{i_1 j_1} \dots \omega^{i_k j_k}\, \partial_{i_1 \dots i_k}(f_1)\, \partial_{j_1 \dots j_k}(f_2) \end{aligned}$$

  where $2i\nu := \hbar$, $\omega^{11} = \omega^{22} = 0$ and $\omega^{12} = -\omega^{21} = 1$.

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For $S \subset \mathrm{Symp}(M,\omega)$:
S-invariance $\leftrightsquigarrow$ $s^\star(f *_\hbar g) = s^\star f *_\hbar s^\star g$ for each $s \in S$

Let $\mathbb{D}$ be a homogeneous complex bounded domain in $\mathbb{C}^n$.

Motivation & general problem

Can we determine explicitly all $\mathrm{Aut}\,(\mathbb{D})$-invariant deformation quantizations on $\mathbb{D}$?

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Structural point

(Pyatetskii-Shapiro theory)

- $\exists\,\tilde{\mathbb{S}} \subset \mathrm{Aut}\,(\mathbb{D})$ solvable Lie group acting simply transitively on $\mathbb{D}$;
- $\tilde{\mathbb{S}} = (\ldots(\mathbb{S}_N \ltimes \mathbb{S}_{N-1}) \ltimes \ldots \ltimes \mathbb{S}_2) \ltimes \mathbb{S}_1$ where :
  (1) $\mathbb{S}_j$ is the Iwasawa group of $G_j = SU\,(1, n_j)$,
  (2) $\mathbb{S}_j$ acts simply transitively on the complex unit ball in $\mathbb{C}^{n_j}$.

Explicit resolution

The resolution can be associated with the determination of a $\tilde{\mathbb{S}}$-equivariant convolution operator that intertwines $\tilde{\mathbb{S}}$-invariant deformation theory (P. Bieliavsky, V. Gayral, ...) with the $\mathrm{Aut}\,(\mathbb{D})$-invariant one.

$$\begin{tikzcd}
\mathrm{Aut}\left(\text{other space} \simeq \tilde{\mathbb{S}}\right) \arrow[d] & \tilde{\mathbb{S}} \arrow[l, hook'] \arrow[r, hook] & \mathrm{Aut}\,(\mathbb{D}) \arrow[d] \\
(\tilde{\mathbb{S}}\text{-invariant DQ theory}) \arrow[r, dashed, no head] & ? \arrow[r, dashed] \arrow[d] & (\mathrm{Aut}\,(\mathbb{D})\text{-invariant DQ theory ?}) \\
 & \tilde{\mathbb{S}}\text{-equivariant convolution operator} &
\end{tikzcd}$$

$\rightsquigarrow$ description of the kernel of this operator through a PDEs hierarchy

These PDE's were explicitly written

(1) for the Poincaré disk : Bieliavsky, Detournay, Spindel (2009),

(2) for the unit ball in $\mathbb{D}_n \subset \mathbb{C}^n$, $n > 1$: Bieliavsky, K. (2013),

but ...

... it was not so easy ...

Here is one of the equation for $n > 1$: $\Box_{(a,\vec{v},\xi)}\, \vartheta \;=\; i\,\xi\, e^{-2a}\, \vartheta$ where

$$
\begin{aligned}
\Box_{(a,\vec{v},\xi)} \quad = \quad & \frac{i\,\xi\, e^{2a}}{4}\left[\left[\left(1+\sqrt{1-\nu^2\xi^2}\right)(\,\vec{v}\,|\,\vec{v}\,)\,+\,2\right]^2 +\,4\;(n+3)\;\nu^2\right]\,\mathrm{Id} \\
& +4\,i\,\nu^2\,\xi\,e^{2a}\,\partial_a \\
& -3\,i\,\nu^2\,\xi\,e^{2a}\,\Theta\,+\;e^{2a}\left[\left[1+\sqrt{1-\nu^2\xi^2}-\nu^2\xi^2\right](\,\vec{v}\,|\,\vec{v}\,)\,+\,2\sqrt{1-\nu^2\xi^2}\right]\,\Xi \\
& -4\,i\,e^{2a}\left[2-3\,\nu^2\xi^2\right]\,\partial_\xi \\
& +i\,\nu^2\,\xi\,e^{2a}\,\partial_a^2 \\
& +\frac{i\,e^{2a}}{2\,\xi}\left[\nu^2\xi^2\;(\,\vec{v}\,|\,\vec{v}\,)\,-\,2\left(-1+\sqrt{1-\nu^2\xi^2}\right)\right]\,\Delta \\
& +\frac{i}{\xi}\,e^{2a}\left[2\left(-1+\sqrt{1-\nu^2\xi^2}\right)+\nu^2\xi^2\right]\left(\Theta^2-\Theta\right)\,+\,i\,\nu^2\,\xi\,e^{2a}\left(\Xi^2+\Theta\right) \\
& -4\,i\,\xi\,e^{2a}\left[1-\nu^2\xi^2\right]\,\partial_\xi^2 \\
& -\frac{2\,i}{\xi}\,e^{2a}\left[-1+\sqrt{1-\nu^2\xi^2}+\nu^2\xi^2\right]\,\Theta\,\partial_a \\
& -4\,i\,e^{2a}\left[1-\nu^2\xi^2\right]\,\partial_a\,\partial_\xi \\
& -4\,i\,e^{2a}\left[-1+\sqrt{1-\nu^2\xi^2}+\nu^2\xi^2\right]\,\Theta\,\partial_\xi \\
& +\frac{1}{\xi^2}\,e^{2a}\left[-1+\sqrt{1-\nu^2\xi^2}+\nu^2\xi^2\right]\,\Xi\,\Delta \\
& -\frac{i}{4\,\xi^3}\,e^{2a}\left[2\left(-1+\sqrt{1-\nu^2\xi^2}\right)+\nu^2\xi^2\right]\,\Delta^2
\end{aligned}
$$

The PDE's were explicitly written and solved

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(2) for the unit ball in $\mathbb{D}_n \subset \mathbb{C}^n$, $n > 1$:

Theorem [Bieliavsky - K., 2013]

For each $SU(1,n)$-invariant deformation theory on $\mathbb{D}_n$, there exists $g \in \mathcal{D}'(\mathbb{R})[[\nu]]$ (with a possible reparameterization of ν), such that the convolution operator with kernel

$$\begin{aligned}\mathcal{V}(a,r,z) \quad = \quad & \int_{-\infty}^{+\infty} d\xi \; \nu^2 \; \mathrm{sign}(\xi) \, e^{-2a + i\xi z} \int_{-\infty}^{+\infty} d\gamma \;\; \left(\gamma^2+1\right)^{\frac{n-3}{2}} \\ & g\left(\frac{-4\,\nu^2 \; \mathrm{sign}(\xi)\, e^{-2a}}{\gamma^2+1}\left(1 - \cosh^2\left(\frac{\mathrm{arcsinh}(i\nu\xi)}{2}\right)\left(\gamma^2+1\right)\right)\right) \\ & \exp\left(-\frac{\mathrm{arccotan}(\gamma)}{\nu} + \frac{\gamma}{\nu}\left(\frac{e^{-2a}}{\gamma^2+1} + \cosh^2\left(\frac{\mathrm{arcsinh}(i\nu\xi)}{2}\right) r^2\right)\right)\end{aligned}$$

is an intertwiner with the $\mathbb{S}$-invariant deformation theory.

Complementary questions

- Analysis of these solutions ? Typical solutions ? (e.g. Berezin, Fedosov, ... ?)
- Determination of an underlying C^*-algebra for each parameter of deformation ? Continuity of this field of C^*-algebras ?
- Generalization for an arbitrary homogeneous complex bounded domain in $\mathbb{C}^n$?
- ...

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- Analysis of these solutions ? Typical solutions ? (e.g. Berezin, Fedosov, ... ?)
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- Generalization for an arbitrary homogeneous complex bounded domain in $\mathbb{C}^n$?
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Thanks for listening !
