Noncommutative principal bundles via twist deformation

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Noncommutative principal bundle?

space X	$\longleftrightarrow$	algebra A ($+$ possibly some extra structure)
group G	$\longleftrightarrow$	Hopf algebra H with (Δ, ε, S)
action $\mu : G \times V \to V$	$\longleftrightarrow$	*coaction* $\delta^V : V \to V \otimes H$, (V is a H-comodule) linear map $+$ 'dual of group action axioms':
$g(hv) = (gh)v$; $ev = v$		$(id_V \otimes \Delta) \circ \delta^V = (\delta^V \otimes id_H) \circ \delta^V$; $(\mathrm{id}_V \otimes \varepsilon)\delta^V = id_V$
G-space	$\longleftrightarrow$	H-comodule algebra A: algebra A with coaction δ^A which is an algebra morphism
principal bundle	$\longleftrightarrow$	Hopf-Galois extension

Hopf-Galois extensions [Kreimer, Takeuchi 1981]

Definition

Let H be a Hopf algebra, A an H-comodule algebra via δ^A and consider the subalgebra of coinvariants $B := A^{co(H)} := \{a \in A | \delta^A(a) = a \otimes \mathbb{1}_H\}$.
The algebra extension $B \subseteq A$ is Hopf-Galois if the canonical map χ is bijective

$$\begin{aligned} \chi = (m_A \otimes id)(id \otimes_B \delta^A) : A \otimes_B A &\rightarrow A \otimes H \\ a \otimes_B a' &\mapsto a \cdot \delta^A(a') \end{aligned}$$

Hopf-Galois extensions [Kreimer, Takeuchi 1981]

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- Generalization of Galois field extensions: E, F fields, $G \subset Aut(E)$.
  $F \subset E$ is Galois, $Gal(E/F) = G \Leftrightarrow F = E^{co((kG)^*)} \subset E$ is Hopf-Galois.

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- Principal group action: X a G-space with action $\mu : (x, g) \mapsto xg$. Let

  $$\alpha : X \times G \to X \times_{X/G} X, \quad (x, g) \mapsto (x, xg).$$

  Then α bijective $\Leftrightarrow$ action free and transitive.

  Dualize to the algebras of functions $\longleftrightarrow$ $\mu^* := \delta^A : A \to A \otimes H$ and $\alpha^* = \chi$ is the canonical map.

Proposition

The canonical map $\chi = (m \otimes id) \circ (id \otimes_B \delta^A) : A \otimes_B A \to A \otimes H$ *is a morphism of relative Hopf* ${}_A\mathcal{M}_A^H$*-modules.*

- $\mathcal{M}^H$ = right H-comodules ($V, \delta^V : v \mapsto v_{(0)} \otimes v_{(1)}$) and H-comodule morphisms.
- $\mathcal{A}^H$ = right H-comodule algebras: $(A, \delta^A) \in \mathcal{M}^H$ s.t. A algebra and δ^A algebra map.

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- ${}_A\mathcal{M}^H$: For $A \in \mathcal{A}^H$, a **relative module** is a $V \in \mathcal{M}^H$ with a compatible left A-module structure, i.e. the action $\triangleright_V : A \otimes V \to V$ is a morphism of H-comodules:

$$\begin{array}{ccc} A \otimes V & \xrightarrow{\delta^{A \otimes V}} & A \otimes V \otimes H \\ \downarrow{\scriptstyle \triangleright_V} & & \downarrow{\scriptstyle \triangleright_V \otimes \mathrm{id}} \\ V & \xrightarrow{\delta^V} & V \otimes H \end{array}$$

  $Mor({}_A\mathcal{M}^H)$: morphisms of left A-modules $\in Mor(\mathcal{M}^H)$.
- Analogously introduce: $\mathcal{M}_A^H$, ${}_A\mathcal{M}_A^H, \ldots$

EXAMPLES:

- The Hopf bundle over S^4.

$$\begin{array}{ccc} S^7 & & A = A(S^7) \\ {\scriptstyle SU(2)}\downarrow & \rightsquigarrow & \uparrow {\scriptstyle H=A(SU(2))} \\ S^4 & & B \simeq A(S^4) \end{array} \qquad \text{(coordinate functions)}$$

with $A(S^4) = A(S^7)^{co\,A(SU(2))} \subset A(S^7)$ Hopf-Galois for $H = A(SU(2))$.

- The quantum Hopf bundle over S^4_θ: ex

$$A = A(S^7_\theta), \quad H = A(SU(2)), \quad B \simeq A(S^4_\theta), \quad \theta \in \mathbb{R}$$

The algebra extension $A(S^4_\theta) \subset A(S^7_\theta)$ is Hopf-Galois [Landi-van Suijlekom '05]: .

EXAMPLES:

- The Hopf bundle over S^4.

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The algebra extension $A(S^4_\theta) \subset A(S^7_\theta)$ is Hopf-Galois [Landi-van Suijlekom '05]: .

Obs.: $A(S^n_\theta)$ are twist deformations of their classical counterparts $A(S^n)$.

Q.: Can we use Drinfeld twists to deform principal bundles into new principal bundles?

GOAL: use the theory of Drinfel'd to deform algebra extensions

$$\begin{array}{ccc} A & & \tilde{A} \\ H \uparrow & \overset{\text{twist deformation}}{\rightsquigarrow} & \tilde{H} \uparrow \\ B = A^{coH} & & \tilde{B} = \tilde{A}^{co\,(\tilde{H})} \end{array}$$

into new algebra extensions in such a way to preserve the 'principality condition', i.e. the invertibility of the canonical map

$$Mor({}_A\mathcal{M}_A^H) \ni \chi : A \otimes_B A \quad \to \quad A \otimes H$$

$\longrightarrow$ deform classical principal bundles into noncommutative principal bundles.

▸ case 1 ▸ case 2

Drinfel'd theory of twists

Definition

A linear map $\gamma : H \otimes H \to \mathbb{K}$ *is called a* **2-cocycle** *on* H *provided*

$$\gamma\left(g_{(1)} \otimes h_{(1)}\right) \gamma\left(g_{(2)} h_{(2)} \otimes k\right) = \gamma\left(h_{(1)} \otimes k_{(1)}\right) \gamma\left(g \otimes h_{(2)} k_{(2)}\right)$$

for all $g, h, k \in H$. *A cocycle is said to be* **unital** *if*

$$\gamma\left(h \otimes \mathbb{1}_H\right) = \varepsilon(h) = \gamma\left(\mathbb{1}_H \otimes h\right)$$

Twisting Hopf-algebras:

Let γ be a convolution invertible 2-cocycle on (H, Δ, ε) with inverse $\bar{\gamma}$. Then

$$m_\gamma(h \otimes k) := h \cdot_\gamma k := \gamma\left(h_{(1)} \otimes k_{(1)}\right) h_{(2)} k_{(2)} \bar{\gamma}\left(h_{(3)} \otimes k_{(3)}\right)$$

defines a new associative product on (the $\mathbb{K}$-module underlying) H.

The resulting algebra $H_\gamma := (H, m_\gamma, \mathbb{1}_H)$ with unchanged coproduct Δ and counit ε and twisted antipode $S_\gamma := u_\gamma * S * \bar{u}_\gamma$ is a Hopf algebra.

OBS.: $H \overset{id}{\simeq} H_\gamma$ as coalgebras $\Rightarrow$ if $(V, \delta^V : v \to v_{(0)} \otimes v_{(1)}) \in \mathcal{M}^H$, then $V =: V_\gamma$ with same coaction, but now thought to as a map $\delta^{V_\gamma} : V \to V \otimes H_\gamma$, is an H_γ-comodule.

Theorem

There exists an equivalence of categories

$$(\mathcal{M}^H, \otimes) \simeq (\mathcal{M}^{H_\gamma}, \otimes^\gamma)$$

consisting of the functor Γ *defined as*

$$\Gamma : (V, \delta^V) \in \mathcal{M}^H \mapsto (V, \delta^V) =: (V_\gamma, \delta^{V_\gamma}) \in \mathcal{M}^{H_\gamma}$$

on objects and as the identity on morphisms, and the natural isomorphisms

$$\begin{aligned} \varphi_{V,W} : V_\gamma \otimes^\gamma W_\gamma &\to (V \otimes W)_\gamma \\ v \otimes^\gamma w &\mapsto v_{(0)} \otimes w_{(0)}\ \bar{\gamma}\,(v_{(1)} \otimes w_{(1)}) \end{aligned}$$

relating the monoidal structures.

proof 1

If $(A, \delta^A) \in \mathcal{A}^H$ is a right H-comodule algebra ($\delta^A : A \to A \otimes H$ alg. morph.)

$$A \mapsto \Gamma(A) = A_\gamma \in \mathcal{M}^{H_\gamma}$$

moreover

$$\begin{array}{ccc} A_\gamma \otimes^\gamma A_\gamma & \xrightarrow{\quad m_\gamma \quad} & A_\gamma \\ \Big\downarrow{\scriptstyle \varphi_{A,A}} & \nearrow{\scriptstyle \Gamma(m)} & \\ (A \otimes A)_\gamma & & \end{array}$$

defines a (deformed) associative algebra structure on A_γ such that $(A_\gamma, \delta^{A_\gamma}) \in \mathcal{A}^{H_\gamma}$. Explicitly

$$a \bullet_\gamma a' := a_{(0)} a'_{(0)} \bar{\gamma}\left(a_{(1)} \otimes a'_{(1)}\right)$$

$\cdots \rightsquigarrow$ similarly, by deforming left and right A-module structures we obtain functors

$$\Gamma : {}_A\mathcal{M}^H \to {}_{A_\gamma}\mathcal{M}^{H_\gamma}, \ldots$$

$$\Gamma : {}_A\mathcal{M}_A{}^H \to {}_{A_\gamma}\mathcal{M}_{A_\gamma}{}^{H_\gamma}$$

Twisting of Hopf-Galois extensions

[Aschieri, Bieliavsky, P., Schenkel, 2015]

Case 1: cocycle γ on the 'structure group' H

- deform H into the Hopf algebra H_γ
- use $\Gamma : \mathcal{A}^H \to \mathcal{A}^{H_\gamma}$ to deform the total space: $A \in \mathcal{A}^H \rightsquigarrow A_\gamma \in \mathcal{A}^{H_\gamma}$ with multiplication

$$a \bullet_\gamma a' := a_{(0)} a'_{(0)} \bar{\gamma}\left(a_{(1)} \otimes a'_{(1)}\right)$$

- Notice: $\delta^A = \delta^{A_\gamma}$ as linear maps $\Rightarrow B = B_\gamma$ as v.s. but also as algebras: the 'base space' $B = \{a \in A / \delta(a) = a_{(0)} \otimes a_{(1)} = a \otimes 1_H\}$ is NOT twisted

$$\begin{array}{ccc} A & & A_\gamma \\ {\scriptstyle H}\uparrow & \overset{\text{twisting}}{\underset{\gamma \text{ on } H}{\rightsquigarrow\quad\rightsquigarrow}} & {\scriptstyle H_\gamma}\uparrow \\ B = A^{coH} & & B = A_\gamma^{coH_\gamma} \end{array}$$

QUESTION: Can we relate the "twisted" canonical map

$$\chi_\gamma : A_\gamma \otimes_B^\gamma A_\gamma \to A_\gamma \otimes^\gamma H_\gamma \in Mor\left({}_{A_\gamma}\mathcal{M}_{A_\gamma}^{H_\gamma}\right)$$

with the original one $\chi : A \otimes_B A \to A \otimes H$ via bijective maps

$$\begin{array}{ccc} A_\gamma \otimes_B A_\gamma & \xrightarrow{\chi_\gamma} & A_\gamma \otimes H_\gamma \\ {\scriptstyle ?}\,\Big\downarrow & & \Big\downarrow\,{\scriptstyle ?} \\ A \otimes_B A & \xrightarrow{\chi} & A \otimes H \end{array}$$

in a canonical way, i.e. via morphisms of relative Hopf modules?

IF YES:

Corollary

The extension $B = A^{coH} \subset A$ is Hopf-Galois if and only if the extension $B \simeq A_\gamma^{coH_\gamma} \subset A_\gamma$ is Hopf-Galois.

Theorem

The following diagram in ${}_{A_\gamma}\mathcal{M}_{A_\gamma}^{H_\gamma}$ commutes:

φ

$$\begin{array}{ccc} A_\gamma \otimes_B^\gamma A_\gamma & \xrightarrow{\ \chi_\gamma\ } & A_\gamma \otimes^\gamma \underline{(H_\gamma)} \\ & & \Big\downarrow{\scriptstyle \mathrm{id}\otimes^\gamma \mathcal{Q}} \\ {\scriptstyle \varphi_{A,A}}\Big\downarrow & & A_\gamma \otimes^\gamma (\underline{H})_\gamma \\ & & \Big\downarrow{\scriptstyle \varphi_{A,H}} \\ (A \otimes_B A)_\gamma & \xrightarrow{\ \Gamma(\chi)=\chi\ } & (A \otimes \underline{H})_\gamma \end{array}$$

Corollary

The extension $B = A^{coH} \subset A$ is Hopf-Galois if and only if the extension $B \simeq A_\gamma^{coH_\gamma} \subset A_\gamma$ is Hopf-Galois.

given γ on H we can:

• first deform H as an Hopf algebra and then consider the twisted Hopf algebra H_γ as an H_γ-comodule coalgebra $\underline{(H_\gamma)}$ via the adjoint coaction

$$Ad^\gamma : h \mapsto h_{(2)} \otimes S_\gamma(h_{(1)}) \cdot_\gamma h_{(3)} \ .$$

• see H as an H-comodule coalgebra: $\underline{H} \in \mathcal{M}^H$ with coaction $Ad : h \mapsto h_{(2)} \otimes S(h_{(1)})\, h_{(3)}$ and deform it as comodule coalgebra to $(\underline{H})_\gamma$ with twisted coproduct

$$\Delta_\gamma(h) = h_{(3)} \otimes h_{(7)} \bar{\gamma}\left(S(h_{(2)}) \otimes h_{(4)}\right) \; u_\gamma(h_{(5)}) \; \bar{\gamma}\left(S(h_{(6)}) \otimes h_{(8)}\right) \; \gamma\left(S(h_{(1)}) \otimes h_{(9)}\right)$$

Theorem

The map

$$\mathcal{Q} : \underline{(H_\gamma)} \to \left(\underline{H}\right)_\gamma \qquad h \mapsto h_{(3)} u_\gamma(h_{(1)}) \bar{\gamma}\left(S(h_{(2)}) \otimes h_{(4)}\right)$$

is an isomorphism of H_γ-comodule coalgebras.

EXAMPLE. Let $A = A(SO(2n+1,\mathbb{R}))$ and $H = A(SO(2n,\mathbb{R})) = A(SO(2n+1,\mathbb{R}))\backslash I$ with right coaction given by $\delta^A := (\mathrm{id} \otimes \pi)\Delta_A$ (quantum homog. space).

$$\begin{array}{ccc} SO(2n+1,\mathbb{R}) & & A = A(SO(2n+1,\mathbb{R})) \\ {\scriptstyle SO(2n,\mathbb{R})}\big\downarrow & \rightsquigarrow & {\scriptstyle A(SO(2n,\mathbb{R}))}\big\uparrow \\ S^{2n} & & B = A(S^{2n}) \end{array}$$

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- Take the 2-cocycle $\gamma(t_j \otimes t_k) = \exp\left(i\pi\theta_{jk}\right)$ on $\mathcal{A}(\mathbb{T}^n) \subset A(SO(2n,\mathbb{R}))$, $\theta_{ij} = -\theta_{ji} \in \mathbb{R}$:

$$H = A(SO(2n,\mathbb{R})) \rightsquigarrow H_\theta = A(SO_\theta(2n,\mathbb{R})) \text{ Hopf}$$

$$A = A(SO(2n+1,\mathbb{R})) \rightsquigarrow A_\theta = \tilde{A}(SO_\theta(2n+1,\mathbb{R})) \text{ comodule algebra}$$

$$\Downarrow$$

$$A(S^{2n}) = \tilde{A}_\theta(SO(2n+1,\mathbb{R}))^{co(H_\gamma)} \subset \tilde{A}_\theta(SO(2n+1,\mathbb{R})) \text{ principal bundle}$$

Remark: - the base space is "classical": not twisted
- no longer homogeneous space

Case 2: cocycle σ on an external Hopf algebra of symmetries

Let K be a Hopf algebra.

Suppose that the total space A carries an additional structure of left K-comodule algebra $A \in {}^K\mathcal{A}$ s.t. the coaction $\rho^A : A \to K \otimes A$ is H-equivariant:

$$(\rho^A \otimes id)\delta^A = (\mathrm{id} \otimes \delta^A)\rho^A$$

i.e. $A \in {}^K\mathcal{M}^H$.

- Let σ be a 2-cocycle on K: $K \rightsquigarrow {}_\sigma K$, and there exists an equivalence of categories

$$(\Sigma, \varphi^\ell) : ({}^K\mathcal{M}, \otimes) \longrightarrow ({}^{\sigma K}\mathcal{M}, \otimes^\sigma)$$

- We can use it to deform $A \in {}^K\mathcal{A}$ into ${}_\sigma A \in {}^{\sigma K}\mathcal{A}$ with twisted algebra structure

- equivariant condition $\Rightarrow {}_\sigma A$ still carries the coaction of H and ${}_\sigma A \in \mathcal{A}^H$

$$
\begin{array}{ccc}
A & & {}_\sigma A \\
{\scriptstyle H}\big\uparrow & \underset{\sigma \;\text{ on }\; K}{\rightsquigarrow^{\textit{twisting}}\rightsquigarrow} & {\scriptstyle H}\big\uparrow \\
B = A^{coH} & & {}_\sigma B \simeq ({}_\sigma A)^{co(H)}
\end{array}
$$

Notice: the 'base space' B is twisted! H undeformed.

Theorem

The following diagram in ${}^{K}_{\sigma A}\mathcal{M}^{H}_{\sigma A}$ commutes:

$$\begin{array}{ccc} {}_\sigma A \otimes^{\sigma}_{B_\sigma} {}_\sigma A & \xrightarrow{\chi_\sigma} & {}_\sigma A \otimes^{\sigma} H \\ \downarrow{\scriptstyle \varphi^{\ell}_{A,A}} & & \downarrow{\scriptstyle \varphi^{\ell}_{A,H}=id} \\ {}_\sigma(A \otimes_B A) & \xrightarrow{\Sigma(\chi)=\chi} & {}_\sigma(A \otimes H) \end{array}$$

Corollary

$B \subseteq A$ is Hopf-Galois if and only if ${}_\sigma B \subseteq {}_\sigma A$ is Hopf-Galois.

EXAMPLE. The quantum Hopf bundle on the Connes-Landi sphere S^4_θ

ex

- Let $K = \mathcal{A}(\mathbb{T}^2)$ be the (commutative) algebra of functions on the 2-torus $\mathbb{T}^2$, $\exists$ a left coaction of $\mathcal{A}(\mathbb{T}^2)$ on the algebra $\mathcal{A}(S^7)$:

$$\rho : \mathcal{A}(S^7) \to \mathcal{A}(\mathbb{T}^2) \otimes \mathcal{A}(S^7)\,, \qquad z_i \mapsto \tau_i \otimes z_i$$

which is $\mathcal{A}(SU(2))$-equivariant.

- Let σ be the exponential 2-cocycle on $\mathcal{A}(\mathbb{T}^2)$ determined by setting

$$\sigma\,(t_j \otimes t_k) = exp(i\pi\Theta_{jk}); \qquad \Theta = \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix}; \qquad \theta \in \mathbb{R}$$

$$\begin{array}{ccc} \mathcal{A}(S^7) & & \mathcal{A}(S^7_\theta) \\ SU(2)\uparrow & \rightsquigarrow_{\sigma \text{ on } K}^{twisting}\rightsquigarrow & SU(2)\uparrow \\ \mathcal{A}(S^4) & & \mathcal{A}(S^4_\theta) \end{array}$$

The resulting bundle is the quantum Hopf bundle on the Connes-Landi sphere $\mathcal{A}(S^4_\theta)$ [Landi, van Suijlekom, 2005].

Remark: Its principality follows from the theory and doesn't need to be proved!

Case 3: combination of deformations

Case 1. γ on H: $(A, H, B) \rightsquigarrow (A_\gamma, H_\gamma, B)$

Case 2. σ on K: $(A, H, B) \rightsquigarrow ({}_\sigma A, H, {}_\sigma B)$

- Let as before A be a right H- comodule algebra with an equivariant left coaction of K
- Let γ a 2-cocycle on H and σ a 2-cocycle on K

$$\begin{array}{c} A \\ {\scriptstyle H}\uparrow \\ B \end{array} \quad \rightsquigarrow \textit{double twisting} \rightsquigarrow \quad \begin{array}{c} {}_\sigma A_\gamma \\ {\scriptstyle H_\gamma}\uparrow \\ {}_\sigma B \end{array}$$

Theorem

$B \subseteq A$ is H-Hopf Galois if and only if ${}_\sigma B \subseteq {}_\sigma A_\gamma$ is H_γ-Hopf Galois.

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- Let as before A be a right H- comodule algebra with an equivariant left coaction of K
- Let γ a 2-cocycle on H and σ a 2-cocycle on K

$$\begin{array}{ccc} A & & {}_\sigma A_\gamma \\ H \uparrow & \rightsquigarrow^{\textit{double}}_{\textit{twisting}} \rightsquigarrow & H_\gamma \uparrow \\ B & & {}_\sigma B \end{array}$$

Theorem

$B \subseteq A$ is H-Hopf Galois if and only if ${}_\sigma B \subseteq {}_\sigma A_\gamma$ is H_γ-Hopf Galois.

EXAMPLE/Application: bundles over quantum homogeneous spaces, e.g. θ-spheres as quantum homogeneous spaces of $\mathcal{A}_\theta(SO(n, \mathbb{R}))$

Conclusions

Summary: We can use the theory of twist deformation and deform a principal bundle $B = A^{co(H)} \hookrightarrow A$ into another principal bundle where base, total space and structure group are all deformed.

- the construction extends to sheaves of Hopf-Galois extensions (Pflaum 1994, Cirio-P. 2014);
- from algebraic to C^*